Numerical Methods

Completely Based on Syllabus of Pokhara University

- **For BE Civil, Civil and Rural, Computer, Software, Electrical, IT, Electronics & Communications, Civil for Diploma Holders and BCA.**

SALIENT FEATURES

1. Comprehensive Coverage of the Syllabus
2. Complete Board Exam Question Solutions from 2013 to 2020
3. Solve out 68 Examples and 49 Additional Questions
4. Step by Step Numerical Solutions With Procedure to iterate in Programmable Calculator
5. Also Effective for TU, PU, MWU and KU
6. Programs in C and C++ Languages

Er. Sushant Bakhunchhe
B.E Electrical

Er. Sanjaya Chauwal
B.E Electrical & Electronics

Scanned with CamScanner

Numerical Methods (3-1-3)

Evaluation:

	Theory	Practical	Total
Sessional	30	20	50
Final	50	-	50
Total	80	20	100

Course Objectives:

1. To introduce numerical methods for interpolation, regressions, and root finding to the solution of problems.
2. To solve elementary matrix arithmetic problems analytically and numerically
3. To find the solution of ordinary and partial differential equations.
4. To proved knowledge of relevant high level programming language for computing, implementing, solving, and testing of algorithms.

Course Contents:

1. Solution of Nonlinear Equations **[10 hrs]**

1.1 Review of calculus and Taylor's theorem
1.2 Errors in numerical calculations
1.3 Bracketing methods for locating a root, initial approximation and convergence criteria
1.4 False position method, secant method and their convergence, Newton's method and fixed point iteration and their convergence.

2. Interpolation and Approximation **[7 hrs]**

2.1 Lagrangian's polynomials
2.2 Newton's interpolation using difference and divided differences
2.3 Cubic spline interpolation
2.4 Curve fitting: Least square lines for linear and nonlinear data

3. Numerical Differentiation and Integration **[5 hrs]**

3.1 Newton's differentiation formulas
3.2 Newton-Cote's, Quadrature formulas
3.3 Trapezoidal and Simpson's Rules
3.4 Gaussian integration algorithm
3.5 Romberg integration formulas

4. Solution of Linear Algebraic Equations **[10 hrs]**

4.1 Matrices and their properties
4.2 Elimination methods, Gauss Jordan method, pivoting
4.3 Method of factorization: Dolittle, Crout's and Cholesky's methods
4.4 The inverse of matrix
4.5 Ill-conditional system
4.6 Iterative methods: Gauss Jacobi, Gauss Seidel, Relaxation methods
4.7 Power method

Scanned with CamScanner

5. **Solution of Ordinary Differential Equations** **[8 hrs]**
   - 5.1 Overview of initial and boundary value problems
   - 5.2 The Taylor's series method
   - 5.3 The Euler method and its modifications
   - 5.4 Huen's method
   - 5.5 Runge-Kutta methods
   - 5.6 Solution of higher order equations
   - 5.7 Boundary value problems: Shooting method
6. **Solution of Partial Differential Equations** **[5 hrs]**
   - 6.1 Review of partial differential equations
   - 6.2 Elliptical equations, parabolic equations, hyperbolic equations and their relevant examples

Laboratory:

Use of Matlab/Math-CAD/C/C++ or any other relevant high level programming language for applied numerical analysis. The laboratory experiments will consists of program development and testing of:

1. Solution of nonlinear equations
2. Interpolation, extrapolation, and regression
3. Differentiation and integration
4. Linear systems of equations
5. Ordinary differential equations (ODEs)
6. Partial differential equations (PDEs)

Text Books:

1. Gerald, C.F. and Wheatly, P.O., '*Applied Numerical Analysis*', (7[th] Edition), New York: Addison Wesley Publishing Company.
2. Guha, S. and Srivastava, R., '*Numerical Methods: For Engineers and Scientists*', Oxford University Press.
3. Grewal, B. S. and Grewal, J. S. '*Numerical Methods in Engineering and Science*', (8[th] Edition), New Delhi: Khanna Publishers, 2010.
4. Balagurusamy, E., '*Numerical Methods*', New Delhi: Tata McGraw Holl, 2010.

References:

1. Moin, Parviz. *Fundamentals of Engineering Numerical Analysis*. Cambridge University Press, 2001.
2. Lindfield, G.R. & Penny, J.E.T. *Numerical Methods: Using MATLAB*. Academic Press, 1012.
3. Schilling, J. & Harris, S.L. *Applied Numerical Methods for Engineers using MATLAB and C*. Thomson Publishers, 2005.
4. Sastry, S.S. Introductory Methods of Numerical Analysis (3[rd] Edition). New Dehli: Prentice Hall of India, 2002.
5. Rao, S.B. and Shantha, C.K. *Numerical Methods with Programs in Basic, Fortran and Pascal*. Hyderabad: Universities Press 2000.
6. Pratap, Rudra. Getting Started with MATLAB. Oxford University Press, 2010.
7. Vedamurthy, V.N. & Lyengar, N. *Numerical Methods*. Noida: Vikash Publication House, 2009

Scanned with CamScanner

CONTENTS

Scanned with CamScanner

Scanned with CamScanner

Scanned with CamScanner

Chapter

1

SOLUTION OF NON-LINEAR EQUATIONS

▲▲▲

1.1 INTRODUCTION

Mathematical models for a wide variety of problems in engineering can be formulated into equations of the form,

$$f(x) = 0 \qquad \text{..... (1)}$$

where, x and f(x) may be real, complex or vector quantities. The solution process often involves finding the values are called the roots of the equation. Since the function f(x) becomes zero at these values, they are also known as the zeros of the function f(x). Equation (1) may belong to one of the following types of equations:

a) Algebraic equations.
b) Polynomial equations.
c) Transcendental equations.

Any function of one variable which does not graph as a straight line in two dimensions or any function of two variables which does not graph as a plane in three dimensions, can be said to be non-linear.

Consider the function, $y = f(x)$. f(x) is a linear function, if the dependent variable y changes in direct proportion to the change in independent variable x. For example, $y = 6x + 10$ is a linear function.

Scanned with CamScanner

On the other hand, f(x) is said to be non-linear if the response of the dependent variable y is not in direct or exact proportion to the changes in the independent variable x. For example, $y = 2x^2 + 3$ is a non-linear function.

a) Algebraic equations

An equation of type y = f(x) is said to be algebraic if it can be expressed in the form,

$$f_n y_n + f_{n-1} y_{n-1} + \cdots + f_1 y_1 + f_0 = 0 \quad \text{..... (1)}$$

where, f_i is an i^{th} order polynomial in x. Equation (1) can be thought of as having a general form

$$f(x, y) = 0 \quad \text{..... (2)}$$

This implies that equation (2) describes a dependence between the variables x and y.

Some examples of algebraic equations are,

i) $4x - 6y - 24 = 0$ (linear)

ii) $3x + 4xy - 30 = 0$ (non-linear)

These equations have an infinite number of pairs of values of x and y which satisfy them.

b) Polynomial Equations

Polynomial equations are a simple class of algebraic equations that are represented as follows;

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \quad \text{..... (1)}$$

This is called nth degree polynomial and has n roots. The roots may be

i) Real and different

ii) Real and repeated

iii) Complex numbers

Since complex roots appear in pairs, if n is odd, then the polynomial has atleast one real root. Some examples of polynomial equations are

i) $6x^5 - x^3 + 2x^2 = 0$

ii) $3x^2 - 4x + 8 = 0$

c) Transcendental equations

A non-algebraic equation is called a transcendental equation. These include trigonometric, exponential and logarithmic functions. Some example of transcendental equations are,

i) $3 \cos x - x = 0$

ii) $\log x - 2 = 0$

iii) $e^x \sin x - \frac{1}{4} x = 0$

A transcendental equation may have a finite or an infinite number of real roots or may not have real root at all.

Scanned with CamScanner

1.2 ACCURACY OF NUMBERS

a) Approximate Numbers

There are two types of numbers *i.e.*, exact and approximate. Exact numbers are 1, 2, 4, 9, 13, $\frac{8}{3}$, 7.78, 14.20, ... etc. But there are numbers such as $\frac{5}{3}$ (=1.6666666......), $\sqrt{5}$ (= 2.23606......) and π (=3.141592.....) which cannot be expressed by a finite number of digits. These may be approximated by numbers 1.6666, 2.2360 and 3.1415 respectively. Such numbers which represent the given numbers to a certain degree of accuracy are called approximate numbers.

b) Significant Figure

The digits used to express a number are called significant digits (figures). Thus each of the numbers 3467, 4.689, 0.3692 contains four significant figures while the numbers 0.00468, 0.000236 contain only three significant figure since zero only help to fix the position of the decimal point. Similarly the numbers 65000 and 8400.00 have two significant figures only.

c) Rounding Off

There are numbers with large number of digits, for example: $\frac{27}{7}$ = 3.857142857. In practice, it is desirable to limit such numbers to a manageable number off digits such as 3.85 or 3.857. This process of dropping unwanted digits is called rounding off.

Rules to Round off a Number to n Significant Figures

i) Discard all digits to the right of the n^{th} digit.

ii) If this discarded number is,

   a) Less than half a unit in the n^{th} place, leave the n^{th} digit unchanged.

   b) Greater than half a unit in the n^{th} place, increase the n^{th} digit by unity.

   c) Exactly half a unit in the n^{th} place, increase the n^{th} digit by unity if it is odd otherwise leave it unchanged.

For instance, the following numbers rounded off to three significant figures are;

6.893 to 6.89
3.678 to 3.68
11.765 to 11.8
6.8254 to 6.82
84767 to 84800

Also the numbers 6.284359, 9.864651, 12.464762 rounded off to four places of decimal are 6.2844, 9.8646 and 12.4648 respectively.

NOTE:
The numbers thus rounded off to n-significant figure (or n decimal places) are said to be correct to n significant figures (or n decimal places).

Scanned with CamScanner

1.3 ERRORS IN NUMERICAL CALCULATIONS

Approximation and errors are an integral part of human life. They are unavoidable. Errors come in a variety of forms and sizes; some are avoidable, some are not. For example, data conversion and round off errors cannot be avoided but a human error can be eliminated. Although certain errors cannot be eliminated completely, we must at least know the bounds of these errors to make use of our final solution. It is therefore essential to know how errors arise, how they grow during the numerical process and how they affect the accuracy of a solution.

In any numerical computation, we come across the following types of errors.

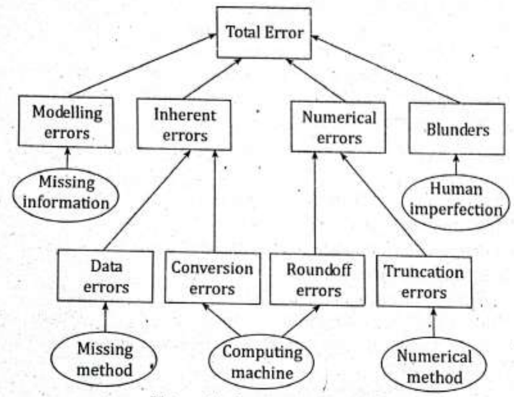


Figure 1.1: Taxonomy of errors

NOTE:

a) Inherent Errors

Errors which are already present in the statement of a problem before its solution, are called inherent errors. Such errors arise either due to the given data being approximate or due to the limitations of mathematical tables, calculators or the digital computer. Inherent errors can be minimized by taking better data or by using high precision computing aids.

b) Rounding Errors

Rounding errors arise from the process of rounding off the numbers during the computation. Such errors are unavoidable in most of the calculations due to the limitations of the computing aids. Rounding errors can, however, be reduced.

i) By changing the calculation procedure so as to avoid subtraction of nearly equal numbers or division by a small number.

Scanned with CamScanner

OR,

ii) By retaining at least one more significant figure at each step than that given in the data and rounding off at the last step.

c) **Truncation errors**

Truncation errors are caused by using approximate result or on replacing an infinite process by a finite one. If we are using a decimal computer having a fixed word length of 4 digits, rounding off of 13.658 gives 13.66 whereas truncation gives 13.65. Truncation error is a type of algorithm error.

d) **Absolute, relative and percentage errors**

If X is the true value of a quantity and X' is its approximate value then, |X − X'| i.e., |Error| is called the absolute error, E_a.

The relative error is defined by $E_r = \left|\frac{X - X'}{X}\right| = \frac{|\text{Error}|}{|\text{True value}|}$

and, The percentage error is, $E_p = 100\ E_r = 100\ \left|\frac{X - X'}{X}\right|$

If $\bar{X}$ be such a number that $|X - X'| \le \bar{X}$, then $\bar{X}$ is an upper limit on the magnitude of absolute error and measures the absolute accuracy.

NOTE:

1. The relative and percentage errors are independent of the units used while absolute error is expressed in terms of these units.
2. If a number is correct to n decimal places, then the error $= \frac{10^{-n}}{2}$. For example, if the number is 3.1416 correct to 4 decimal places, then the error $= \frac{10^{-4}}{2} = 0.0005$.

1. Inherent Errors

Inherent errors are those types of error that are present in the data supplied to the model. Inherent errors (also known as input efforts) contain two components, namely, data errors and conversion errors.

A. Data errors or empirical errors

Data errors arises when data for a problem are obtained by some experimental means and are, therefore, of limited accuracy and precision. This may be due to some limitations in instrumentation and reading and therefore may be unavoidable. A physical measurement, such as voltage, time period, current, distance cannot be exact. It is therefore important to remember that there is no use in performing arithmetic operations to, say, four decimal places when the original data themselves are only correct to two decimal places.

B. Conversion errors or representation errors

Conversion errors arises due to the limitations of the computer to store the data exactly. Many numbers cannot be represented exactly in a given number of decimal digits. In some cases, a decimal number 0.1 has a non

Scanned with CamScanner

terminating binary form like 0.00011001100110011........ but the computer retains only a specified number of bits. Thus, if we add 10 such numbers in a computer, the result will not be exactly 1.0 because of round off error during the conversion of 0.1 to binary form.

2. Numerical Errors

Numerical errors are introduced during the process of implementation of a numerical method. The total numerical error is the summation of round off errors and truncation errors. The total errors can be reduced by devising suitable techniques for implementing the solution.

A. Round off errors

Round off errors occurs when a fixed number of digits are used to represent exact numbers. Since the numbers are stored at every stage of computation, round off error is introduced at end of every arithmetic operation. hence, individual round off error could be very small, but cumulative effect of a series of computations can be very significant.

Rounding a number can be done in two ways. One is known as chopping and other is known as symmetric rounding.

i) Chopping

In chopping, the extra digits are dropped. This is called truncating the number. Suppose we are using a computer with a fixed word length of four as 32.45687 and the digits 687 will be dropped.

ii) Symmetric error

In the symmetric round off method, the last retained significant digit is "rounded up" by 1 if the first discarded digit is larger or equal to 5; otherwise, the retained digit is unchanged. For example, the number 32.45687 would become 32.46 and the number 33.2342 would become 33.23.

B. Truncation errors

Truncation errors arise from using an approximation in place of an exact mathematical procedure. Typically, it is the error resulting from the truncation of the numerical process. Many of the iterative procedures used in numerical computing are infinite and, hence, knowledge of this error is important. Truncation error can be reduced by using a better numerical model which usually increases the number of arithmetic operations. For example; in numerical integration, the truncation error can be reduced by increasing the number of points at which the function is integrated. But care should be exercised to see that the round off error which is bound to increase due to increase in arithmetic operations does not off-set the reduction in truncation error.

Scanned with CamScanner

3. Modelling Errors

Mathematical models are the basis for numerical solutions. They are formulated to represent physical processes using certain parameters involved in the situations. In many situations, it is impractical or impossible to include all of the real problem and hence certain simplifying assumptions are made. Since a model is a basic input to the numerical process, no numerical method will provide adequate result if the model is erroneously conceived and formulated. We can reduce these types of errors by refining or enlarging the models by incorporating more features. But the enhancement may make the model more difficult to solve or may take more time to implement the solution process. It is also not always true that an enhanced model will provide better results. We must note that modelling, data quality and computation go hand in hand. An overly refined model with inaccurate data or an inadequate computer may not be meaningful. On the other hand, an oversimplified model may produce a result that is unacceptable. It is, therefore, necessary to strike a balance between the level of accuracy and the complexity of the model.

4. Blunders

Blunders are errors that are caused due to human imperfection. As the name indicates, such errors may cause a very serious disaster in the result. Since these errors are due to human mistakes, it should be possible to avoid them to a large extent by acquiring a sound knowledge of all aspects of the problem as well as the numerical process.

Human errors can occur at any stage of the numerical processing cycle. Some common types of errors are;

i) Lack of understanding of the problem
ii) Wrong assumptions
iii) Overlooking of some basic assumptions required for formulating the model.
iv) Errors in deriving the mathematical equation or using a model that does not describe adequately the physical system under study.
v) Selecting a wrong numerical method for solving the mathematical model.
vi) Selecting a wrong algorithm for implementing the numerical method.
vii) Making mistakes in the computer program such as testing a real number for zero and using < symbol in place of > symbol.
viii) Mistakes of data input such as misprints, giving values column-wise instead of row-wise to a matrix, forgetting a negative sign etc.
ix) Wrong guessing of initial values

All these mistakes can be avoided through a reasonable understanding of the problem and the numerical solution methods, and use of good programming techniques and tools.

Scanned with CamScanner

Example 1.1

Round off the numbers 865250 and 37.46235 to four significant figures and compute E_a, E_r, E_p in each case.

Solution:

i) Number rounded off to four significant figures = 865200

$\therefore$ $E_a = |X - X'| = |865250 - 865200| = 50$

$$E_r = \left|\frac{X - X'}{X}\right| = \frac{50}{865250} = 6.71 \times 10^{-5}$$

$$E_p = E_r \times 100 = 6.71 \times 10^{-3}$$

ii) Number rounded off to four significant figures = 37.46

$\therefore$ $E_a = |X - X'| = |37.46235 - 37.46000| = 0.00235$

$$E_r = \left|\frac{X - X'}{X}\right| = \frac{0.00235}{37.46235} = 6.27 \times 10^{-5}$$

$$E_p = E_r \times 100 = 6.27 \times 10^{-3}$$

Example 1.2

Find the absolute error and relative error in $\sqrt{6} + \sqrt{7} + \sqrt{8}$ correct to 4 significant digits.

Solution:

Given that;

$$\sqrt{6} = 2.449$$
$$\sqrt{7} = 2.646$$
$$\sqrt{8} = 2.828$$

$\therefore$ $S = \sqrt{6} + \sqrt{7} + \sqrt{8} = 2.449 + 2.646 + 2.828 = 7.923$

Then the absolute error E_a in S is,

$$E_a = 0.0005 + 0.0007 + 0.0004 = 0.0016$$

This shows that S is correct to 3 significant digits only. Hence, we take S = 7.92.

Then the relative error,

$$E_r = \frac{E_a}{S} = \frac{0.0016}{7.92} = 0.0002$$

Example 1.3

The function $f(x) = \tan^{-1} x$ can be expanded as, $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \ldots\ldots$. Find n such that the series determine $\tan^{-1} x$ correct to eight significant digits at x = 1.

Solution:

If we retain n terms in the expansion of $\tan^{-1} x$, then $(n + 1)^{th}$ term,

$$= (-1)^n \frac{x^{2n+1}}{2n+1} = \frac{(-1)^n}{2n+1} \text{ for } x = 1$$

Scanned with CamScanner

To determine $\tan^{-1}(1)$ correct to eight significant digits accuracy

$$\left|\frac{(-1)^n}{2n+1}\right| < \frac{1}{2} \times 10^{-8}$$

i.e., $2n + 1 > 2 \times 10^8$ or $n > 10^8 - \frac{1}{2}$

Hence, value of $n = 10^8 + 1$

Example 1.4

Which of the following numbers has the greatest precision.

- **a)** **4.3201**
- **b)** **4.32**
- **c)** **4.320106**

Solution:

a) 4.3201 has a precision of 10^{-4}

b) 4.32 has a precision of 10^{-2}

c) 4.320106 has a precision of 10^{-6}

The last number (4.320106) has the greatest precision

Example 1.5

What is the accuracy of the following numbers?

- **a)** **95.763**
- **b)** **0.008472**
- **c)** **0.0456000**
- **d)** **36**
- **e)** **3600**
- **f)** **3600.00**

Solution:

a) 95.763
Ans: 95.763 has five significant digits.

b) 0.008472
Ans: 0.008472 has four significant digits. The leading or higher order zeros are only place holders.

c) 0.0456000
Ans: 0.0456000 has six significant digits.

d) 36
Ans: 36 has two significant digits.

e) 3600
Ans: Accuracy is not fixed.

f) 3600.00
Ans: 3600.00 has six significant digits. Note that the zeros were made significant by writing .00 after 3600.

Scanned with CamScanner

Example 1.6

Find the absolute error if the number X = 0.00545828 is,

i) Truncated to three decimal digits.

ii) Rounded off to three decimal digits.

Solution:

Given that;

$X = 0.00545828 = 0.545828 \times 10^{-2}$

i) After truncation to three decimal places, its approximate value

$X' = 0.545 \times 10^{-2}$

$\therefore$ Absolute error $= |X - X'| = 0.000828 \times 10^{-2}$

$= 0.828 \times 10^{-5}$

ii) After rounding off to three decimal places, its approximate value

$X' = 0.546 \times 10^{-2}$

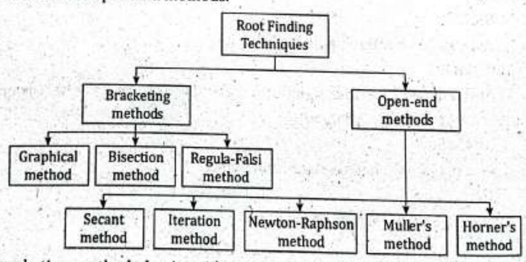
$\therefore$ Absolute error $= |X - X'|$

$= |0.545828 - 0.546| \times 10^{-2}$

$= 0.000172 \times 10^{-2} = 0.172 \times 10^{-5}$

1.4 ITERATIVE METHODS

An iterative method begins with an approximate value of the root which is generally obtained with the help of intermediate value property of the equation. This initial approximation is then successively improved iteration by iteration and this process stops when the desired level of accuracy is achieved. The various iterative methods begin their process with one or more initial approximations. Based on the number of initial approximations used, these iterative methods are divided into two categories. Bracketing methods and open-end methods.



Bracketing methods begin with two initial approximations which bracket the root. Then the width of this bracket is systematically reduced until the root is reached to desired accuracy. The commonly used methods in this category are;

Scanned with CamScanner

1. Graphical method
2. Bisection method
3. Method of false position

Open-end methods are used on formula which require a single starting value or two starting values which do not necessarily bracket the root. Open end methods may diverge as the computation progress but when they do converge they usually do so much faster than bracketing method. The following methods fall under this category.

1. Secant method
2. Iteration method
3. Newton-Raphson method
4. Muller's method
5. Horner's method
6. Lin-Bairstow method

It may be noted that the bracketing method require to find sign changes in the function during every iteration. Open end methods do not require this.

1.4.1 Starting and Stopping an Iterative Process

A. Starting the Process

Before an iterative process is initiated, we have to determine either an approximate value of root or a "search" interval that contains a root. One simple method of guessing starting points is to plot the curve of f(x) and to identify a search interval near the root of interest. Graphical representation of a function cannot only provide us rough estimates of the roots but also help us in understanding the properties of the function, there by identifying possible problems in numerical computing. In case of polynomials, many theoretical relationships between roots and coefficients are available.

B. Largest Possible Root

For a polynomial represented by,

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots\cdots + a_1 x + a_0$$

The largest possible root is given by,

$$x_1^* = \frac{a_{n-1}}{a_n}$$

This value is taken as the initial representation when no other value is suggested by the knowledge of the problem at hand.

C. Search Bracket

Another relationship that might be useful for determining the search intervals that contain the real roots of a polynomial is,

$$|x^*| \le \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)}$$

where, x is the root of the polynomial. Then, the maximum absolute value of

Scanned with CamScanner

the root is,

$$|x^*_{max}| = \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)}$$

This means that no root exceeds x_{max} in absolute magnitude and thus, all real roots lie within the interval $(-|x^*_{max}|, |x^*_{max}|)$.

There is yet another relationship that suggests an interval for roots. All real roots x satisfy the inequality.

$$|x^*| \le 1 + \frac{1}{|a_n|} \max\{|a_0|, |a_1|, \ldots, |a_{n-1}|\}$$

where, the 'max' denotes the maximum of the absolute values $|a_0|$, $|a_1|$, ... $|a_{n-1}|$.

D. Stopping Criterion

An iterative process must be terminated at some stage. When? We must have an objective criterion for deciding when to stop the process. We may use one (or combination) of the following tests, depending on the behaviour of the function, to terminate the process.

i) $|x_{i+1} - x_i| \le E_a$ (absolute error in x)

ii) $\left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \le E_r$ (Relative error in x), $x \ne 0$

iii) $|f(x_{i+1})| \le E$ (value of function at root)

iv) $|f(x_{i+1}) - f(x_i)| \le E$ (difference in function values)

v) $|f(x)| \le F_{max}$ (large function value)

vi) $|x_i| \le XL$ (large value of x)

Here, x_i represents the estimate of the root at i^{th} iteration and $f(x_i)$ is the value of the function at x_i.

There may be situations where these tests may fail when used alone. Sometimes even a combination of two tests may fail. A practical convergence test should use a combination of these tests. In cases where we do not know whether the process converges or not, we must have a limit on the number of iterations, like

Iterations $\ge$ N (limit on iterations)

1.5 BISECTION METHOD OR BINARY CHOPPING METHOD OR BOLZANO OR HALF INTERVAL OR BINARY SEARCH METHOD

The bisection method is one of the simplest and most reliable of iterative methods for the solution of non-linear equations. This method, also known as Binary chopping or half-interval method, relies on the fact that if f(x) is real and continuous in the interval a < x < b and f(a) and f(b) are of opposite sings, that is f(a) f(b) < 0, then, there is at least one real root in the interval between a and b. (There may be more than one root in the interval).

Scanned with CamScanner

Let $x_1 = a$ and $x_2 = b$. Let us also define another point x_0 to be the midpoint between a and b. That is,

$$x_0 = \frac{x_1 + x_2}{2}$$

Now, there exists the following three conditions,

a) If $f(x_0) = 0$, we have a root at x_0

b) If $f(x_0)\, f(x_1) < 0$, there is a root between x_0 and x_1

c) If $f(x_0)\, f(x_2) < 0$, there is a root between x_0 and x_2

It follows that by testing the sign of the function at midpoint, we can deduce which part of the interval contains the root. This is illustrated in figure 1.2 which shows that, sine $f(x_0)$ and $f(x_2)$ are of opposite sign, a root lies between x_0 and x_2. We can further divide this subinterval into two halves to locate a new subinterval containing the halves to locate a new subinterval containing the root. This process can be repeated until the interval containing the root is as small as we desire.

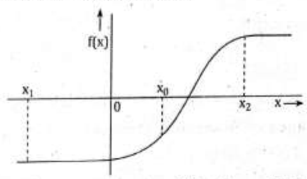


Figure 1.2: Illustration of bisection method

NOTE:
Since the new interval containing the root, is exactly half the length of the previous one, the interval width is reduced by a factor of $\frac{1}{2}$ at each step. At the end of the n^{th} step, the new interval will therefore be of length $\frac{(b-a)}{2^n}$. If on repeating this process n times, the latest interval is as small as given E, then $\frac{(b-a)}{2^n} \le E$.

or, $n \ge \frac{[\log(b-a) - \log E}{\text{Log}2}$

This gives the number of iterations required for achieving an accuracy E.
In particular, the minimum number of iterations required for converging to a root in the interval (0, 1) for a E are as under:

E :	10^{-2}	10^{-3}	10^{-4}
n :	7	10	14

b) As the error decreases with each step by a factor of $\frac{1}{2}\left(i.e., \frac{E_{n+1}}{E_n} = \frac{1}{2}\right)$, the convergence in the bisection method is linear.

Scanned with CamScanner

1.5.1 Algorithm for Bisection Method

1. Start.
2. Decide initial values for x_1 and x_2 and stopping criterion, E.
3. Compute $f_1 = f(x_1)$ and $f_2 = f(x_2)$.
4. If $f_1 \times f_2 > 0$, x_1 and x_2 do not bracket any root and go to step 8; otherwise continue.
5. Compute $x_0 = (x_1 + x_2)/2$ and compute $f_0 = f(x_0)$.
6. If $f_1 \times f_0 < 0$, then
   set $x_2 = x_0$
   else
   set $x_1 = x_0$.
   set $f_1 = f_0$
7. If absolute value of $(x_2 - x_1)/x_2$ is less than error E_1 then,
   root = $(x_1 + x_2)/2$
   write the value of root
   go to step 8
   else
   go to step 5
8. Stop.

1.5.2 Advantages of Bisection Method

i) Convergent is guaranteed
   Bisection method is bracketing method and it is always convergent.
ii) Error can be controlled
   In Bisection method, increasing number of iteration always yields more accurate root.
iii) Does not involve complex calculations
   Bisection method does not require any complex calculations. To perform this method, all we need is to calculate average of two numbers.
iv) Guaranteed error bound
   In this method, there is guarantee error bound and it decreases with each successive iteration. The error bound decreases by $\frac{1}{2}$ with each iteration.
v) Bisection method is fast in case of multiple roots.
vii) The function does not have to be differentiable.

1.5.3 Disadvantages of Bisection Method

i) Slow rate of convergence
   Although convergence of bisection method is guaranteed, it is generally slow.

Scanned with CamScanner

ii) Choosing one guess close to root has no advantage choosing one guess close to the root may result in requiring many iterations to converge.

iii) Cannot find root of some equations. For example, $f(x) = x^2$ as there are no bracketing values.

iv) It has linear rate of convergence.

v) It fails to determine complex roots.

vi) It cannot be applied if there are discontinuities in the guess interval.

Example 1.7

Find the root of the equation $\cos x = xe^x$ using the Bisection method correct to four decimal places.

Solution:

Let, $f(x) = \cos x - xe^x$

Since, $f(0) = 1$

$f(1) = -2.18$

so, a root lies between 0 and 1.

$\therefore$ First approximation, $x_1 = \frac{1}{2}(0 + 1) = 0.5$

Now,

$f(x_1) = 0.05$ and $f(1) = -2.18$

Hence, the root lies between 1 and $x_1 = 0.5$

$\therefore$ Second approximation, $x_2 = \frac{1}{2}(0.5 + 1) = 0.75$

Now,

$f(x_2) = -0.86$ and $f(0.5) = 0.05$

Hence, the root lies between 0.5 and 0.75

$\therefore$ Third approximation, $x_3 = \frac{1}{2}(0.5 + 0.75) = 0.625$

Now,

$f(x_3) = -0.36$ and $f(0.5) = 0.05$

Hence, the root lies between 0.5 and 0.625

$\therefore$ Fourth approximation, $x_4 = \frac{1}{2}(0.5 + 0.625) = 0.5625$

Now,

$f(x_4) = -0.14$ and $f(0.5) = 0.05$

Hence, the root lies between 0.5 and 0.5625

$\therefore$ Fifth approximation, $x_5 = \frac{1}{2}(0.5 + 0.5625) = 0.5312$

Now,

$f(x_5) = -0.04$ and $f(0.5) = 0.05$

Hence, the root lies between 0.5 and 0.5312

$\therefore$ Sixth approximation, $x_6 = \frac{1}{2}(0.5 + 0.5312) = 0.5156$

Scanned with CamScanner

Now,

$f(fx_6) = 0.00655$ and $f(1) = -2.18$

Hence, the root lies between 1 and 0.5156

$\therefore$ Seventh approximation, $x_7 = \frac{1}{2}(0.5156 + 1) = 0.7178$

Now,

$f(x_7) = -0.7182$ and $f(0.5) = 0.05$

Hence, the root lies between 0.5 and 0.7178

$\therefore$ Eight approximation, $x_8 = \frac{1}{2}(0.5 + 0.7178) = 0.6089$

Now,

$f(x_8) = -0.2991$ and $f(0.5) = 0.05$

Hence, the root lies between 0.5 and 0.6089

$\therefore$ Ninth approximation, $x_9 = \frac{1}{2}(0.5 + 0.6089) = 0.5544$

Now,

$f(x_9) = -0.1149$ and $f(0.5) = 0.05$

Hence, the root lies between 0.5 and 0.5544

$\therefore$ Tenth approximation, $x_{10} = \frac{1}{2}(0.5 + 0.5544) = 0.5272$

Now,

$f(x_{10}) = -0.02896$ and $f(0.5) = 0.05$

Hence, the root lies between 0.5 and 0.5272

$\therefore$ 11^{th} approximation, $x_{11} = \frac{1}{2}(0.5 + 0.5272) = 0.5136$

Now,

$f(x_{11}) = 0.0126$ and $f(1) = -2.18$

Hence, the desired approximation to the root is 0.5136

Alternative method

Let, $f(x) = xe^x - \cos x$

The initial guess be

$x_0 = 0, \quad f(0) = 0e^0 - \cos(0) = -1 < 0$

$x_1 = 1, \quad f(1) = 1\,e^1 - \cos(1) = 2.177 > 0$

i.e., root lies between 0 and 1.

$\therefore$ $x_L = 0$ and $x_U = 1$

Now, first approximated root using bisection method,

$$x_N = \frac{x_L + x_U}{2} = \frac{0 + 1}{2} = 0.5$$

$\therefore$ $f(x_N) = 0.5 \times e^{0.5} - \cos(0.5) = -0.053 < 0$

so, now root lies between 0.5 and 1.

Scanned with CamScanner

Remaining iterations are solved in Tabular form.

Iteration	x_L	$f(x_L)$	x_U	$f(x_U)$	x_N	$f(x_N)$
1	0	−1	1	2.177	0.5	−0.053
2	0.5	−0.053	1	2.177	0.75	0.8560
3	0.5	−0.053	0.75	0.8560	0.625	0.3566
4	0.5	−0.053	0.625	0.3566	0.5625	0.1412
5	0.5	−0.053	0.5625	0.1412	0.5312	0.0413
6	0.5	−0.053	0.5312	0.0413	0.5156	−0.0065
7	0.5156	−0.0065	0.5312	0.0413	0.5234	0.0172
8	0.5156	−0.0065	0.5234	0.0172	0.5195	0.0053
9	0.5156	−0.0065	0.5195	0.0053	0.5175	−0.0007
10	0.5175	−0.0007	0.5195	0.0053	0.5185	0.0022
11	0.5175	−0.0007	0.5185	0.0022	0.5180	0.0007
12	0.5175	−0.0007	0.5180	0.0007	0.5177	−0.0001
13	0.5177	−0.0001	0.5180	0.0007	0.5178	0.0001
14	0.5177	−0.0001	0.5178	0.0001	0.5177	−0.0001
15	0.5177	−0.001	0.5178	0.0001	0.5177	−0.0001

Here, the value of x_N do not change up to 4 decimal places, so required root of given function is 0.5177.

1.6 FALSE POSITION OR REGULA-FALSI OR INTERPOLATION METHOD

This is the oldest method of finding the real roots of an equation $f(x) = 0$ and closely resembles the bisection method.

Here, we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs *i.e.,* the graph of $y = f(x)$ crosses the x-axis between these points. This indicates that a root lies between x_0 and x_1 and consequently $f(x_0)\, f(x_1) < 0$.

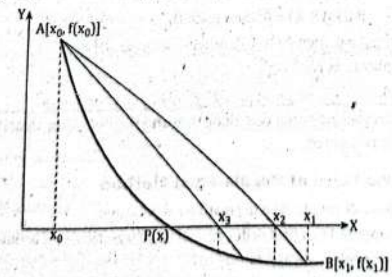


Figure 1.3

Scanned with CamScanner

Equation of the chord joining the points A $[x_0, f(x_0)]$ and B $[x_1, f(x_1)]$ is,

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

This method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with the x-axis as an approximation to the root. So the abscissa of the point where the x-axis (y = 0) is given by,

$$x_2 - x_0 = \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \quad \text{..... (1)}$$

which is an approximation to the root.

If now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2. So replacing x_1 by x_2 in (1), we obtain the next approximation x_3. The root could as well lie between x_1 and x_2 and we would obtain x_3 accordingly. This procedure is repeated until the root is found to the desired accuracy. The iteration process based on (1) is known as the method of false position. This method has linear rate of convergence which is faster than that of the bisection method.

1.6.1 Algorithm for False Position Method

1.	Start.
2.	Define function f(x)
3.	Choose initial guesses x_0 and x_1 such that $f(x_0)\,f(x_1) < 0$
4.	Choose pre-specified tolerance error
5.	Calculate new approximated root as $x_2 = x_0 - ((x_0 - x_1) \times f(x_0))/(f(x_0) - f(x_1))$
6.	Calculate $f(x_0)\,f(x_2)$ a) If $f(x_0)\,f(x_2) < 0$, then $x_0 = x_0$ and $x_1 = x_2$ b) If $f(x_0)\,f(x_2) > 0$, then $x_0 = x_2$ and $x_1 = x_1$ c) If $f(x_0)\,f(x_2) = 0$, then go to (8)
7.	If $\lvert f(x_2)\rvert > e$, then go to (5), otherwise go to (8)
8.	Display x_2 as root.
9.	Stop.

A major difference between this algorithm and the bisection algorithm is the way x_2 is computed.

1.6.2 Advantages of Regula-Falsi Method

i) It does not require the derivative calculations.

ii) This method has first order rate of convergence *i.e.*, it is linearly convergent. It always converges.

iii) It is a quick method.

Scanned with CamScanner

1.6.3 Disadvantages of Regula-Falsi Method

i) It is used to calculate only a single unknown in the equation.
ii) As it is trial and error method, in some cases it may take large time span to calculate the correct root and there by slowing down the process.
iii) It can't predict number of iterations to reach a given precision.
iv) It can be less precise than bisection method.

Example 1.8

Find the root of the equation $\cos x = xe^x$ using the regular-falsi method correct to four decimal places.

Solution:

Let, $f(x) = \cos x - x e^x - 0$

Here, $f(0) = \cos 0 - 0\, xe^0 = 1$

$f(1) = \cos 1 - e = 2.17798$

i.e., the root lies between 0 and 1

i) Taking $x_0 = 0$, $x_1 = 1$, $f(x_0) = 1$ and $f(x_2) = -2.17798$

In the regular-falsi method, we get,

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 0 + \frac{1}{3.17798} \times 1 = 0.31467$$

Now,

$f(0.31467) = 0.51987$ *i.e.,* the root lies between 0.31467 and 1

ii) Taking $x_0 = 0.31467$, $x_1 = 1$, $f(x_0) = 0.51987$ $f(x_1) = -2.17798$

$$\therefore \quad x_3 = 0.31467 + \frac{0.68533}{2.69785} \times 0.51987 = 0.44673$$

Now,

$f(0.44673) = 0.20356$ *i.e.,* the root lies between 0.44673 and 1

iii) Taking $x_0 = 0.44673$, $x_1 = 1$, $f(x_0) = 0.20356$, $f(x_1) = -2.17798$

$$\therefore \quad x_4 = 0.44673 + \frac{0.55327}{2.38154} \times 0.20356 = 0.49402$$

Repeating this process, the successive approximations are,

$x_5 = 0.50995$, $x_6 = 0.51520$, $x_7 = 0.51692$

$x_8 = 0.51748$, $x_9 = 0.51767$, $x_{10} = 0.51775$

Hence, the root is 0.5177 correct to four decimal places

Alternative method

Let, $f(x) = x e^x - \cos x$

The initial guess be,

$x_l = x_0 = 0, \quad f(x_0) = 0e^0 - \cos(0) = -1 < 0$

$x_u = x_1 = 1, \quad f(x_1) = 1\, e^1 - \cos(1) = 2.177 > 0$

i.e., root lies between 0 and 1.

Scanned with CamScanner

Using false position method,

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0) = 0 - \frac{(1-0)}{2.177+1} \times (-1) = 0.3147$$

$\therefore \quad f(x_2) = -0.5197 < 0$

Now root lies between 0.3147 and 1.

Solving other iterations in tabular form as follow,

Iteration	x_L	$f(x_L)$	x_U	$f(x_U)$	$x_N = x_L - \frac{f(x_L)(x_U - x_L)}{f(x_U) - f(x_L)}$	$f(x_N)$
1	0	-1	1	2.177	0.3147	-0.5197
2	0.3147	-0.5197	1	2.177	0.4467	-0.2036
3	0.4467	-0.2036	1	2.177	0.4940	-0.0708
4	0.4940	-0.0708	1	2.177	0.5099	-0.0237
5	0.5099	-0.0237	1	2.177	0.5151	-0.0080
6	0.5151	-0.0080	1	2.177	0.5168	-0.0029
7	0.5168	-0.0029	1	2.177	0.5174	-0.0010
8	0.5174	-0.0010	1	2.177	0.5177	-0.0004
9	0.5177	-0.0004	1	2.177	0.5177	-0.0002

Here, the value of x_N does not change up to 4 decimal places. Hence, the root of given equation is 0.5177.

1.7 SECANT METHOD

This method is an important over the method of false position as it does not require the condition $f(x_0)\, f(x_1) < 0$ of that method.

Here, also the graph of the function $y = f(x)$ is approximated by a secant line but at each iteration, two most recent approximations to the root are used to find out the next approximation. Also, it is not necessary that the interval must contain the root.

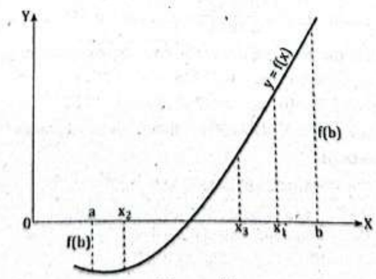


Figure: 1.4

Scanned with CamScanner

Taking x_0, x_1 as the initial limits of the interval, we write the equation of the chord joining these as,

$$y - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1)$$

Then the abscissa of the point where it crosses the x-axis ($y = 0$) is given by,

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

which is an approximation to the root. The general formula for successive approximation is, therefore, given by,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n),\ n \geq 1$$

If at any integration $f(x_n) = f(x_{n-1})$, this method fails and shows that it does not converge necessarily. This is a drawback of secant method over false position which always converges. But if the secant method once converges, its rate of convergence is 1.6 which is faster than that of the method of false position.

1.7.1 Algorithm for Secant Method

1. Start.
2. Decide two initial points x_0 and x_1, accuracy level required, E.
3. Compute $f_0 = f(x_0)$ and $f_1 = f(x_0)$ and $f_1 = f(x_1)$
4. Compute $x_2 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$
5. Test for accuracy of x_2.

   If $\left|\frac{x_2 - x_1}{x_3}\right| > E_1$ then

   set $x_0 = x_1$ and $f_0 = f_1$

   set $x_1 = x_2$ and $f_1 = f(x_2)$

   go to step 4

   otherwise

   set root = x_2

   print results
6. Stop.

1.7.2 Advantages of Secant Method

i) It converges at faster than a linear rate, so that it is more rapidly convergent than the bisection method.

ii) It requires only one function evaluation per iteration, as compared with Newton's method which requires two.

iii) It does not require the use of derivative of the function, something that is not available in a number of applications.

Scanned with CamScanner

1.7.3 Disadvantages of Second Method

i) It may not converge *i.e.*, may diverge.

ii) There is no guaranteed error bound for the computed iterates

Example 1.9

Find the root of the equation $xe^x = \cos x$ using secant method correct to four decimal place.

Solution:

Let $f(x) = xe^x - \cos x$

$x_0 = 0$ and $x_1 = 1$ be the initial guesses

$f(x_0) = 0e^0 - \cos(0) = -1$

$f(x_1) = 1e^1 - \cos(1) = 2.1779$

Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = 1 - \frac{2.1779(1-0)}{2.1779-(-1)} = 0.3146$$

$f(x_2) = 0.3146\, e^{0.3146} - \cos(0.3146) = -0.5200$

Now, solving other iterations in tabular form as follows

Iteration	x_{n-1}	$f(x_{n-1})$	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	0	-1	1	2.1779	0.3146	-0.5200
2	1	2.1779	0.3146	-0.5200	0.4467	-0.2036
3	0.3146	-0.5200	0.4467	-0.2036	0.5317	0.0429
4	0.4467	-0.2036	0.5317	0.0429	0.5169	-2.60×10^{-3}
5	0.5317	0.0429	0.5169	-2.60×10^{-3}	0.5177	-1.74×10^{-4}
6	0.5169	-2.60×10^{-3}	0.5177	-1.74×10^{-4}	0.5177	4.47×10^{-8}

Here, the value of x_{n+1} do not change up to four decimal places.

Hence, the root of the equation is 0.5177.

1.8 NEWTON-RAPHSON METHOD

Let x_0 be an approximate root of the equation $f(x) = 0$. If $x_1 = x_0 + h$ be the exact root, then $f(x_1) = 0$.

∴ Expanding $f(x_0 + h)$ by Taylor's series

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots = (0)$$

Since h is small, neglecting h^2 and higher powers of h_1 we get,

$$f(x_0) + hf'(x_0) = 0$$

or, $$h = -\frac{f(x_0)}{f'(x_0)}$$

∴ A closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Scanned with CamScanner

Similarly starting with x_1, a still better approximation x_2 is given by,

$$x_2 = x_1 - \frac{f(x_0)}{f'(x)}$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (n = 0, 1, 2, \ldots\ldots)$$

which is known as the Newton Raphson formula or Newton's iteration formula.

NOTE:

1. Newton's method is useful in cases of large values of $f'(x)$ *i.e.*, when the graph of $f(x)$ while crossing the x-axis is nearly vertical.
   If $f'(x)$ is small in the vicinity of the root, then by (1), h will be large and the computation of the root is slow or may not be possible. Thus this method is not suitable in those cases where the graph of $f(x)$ is nearly horizontal while crossing the x-axis.
2. Newton's method is generally used to improve the result obtained by other methods. It is applicable to the solution of both algebraic and transcendental equations.

Newton's formula converges provided the initial approximation x_0 is chosen sufficiently close to the root. If it is not near the root, the procedure may lead to an endless cycle. A bad initial choice will lead one astray. Thus a proper choice of the initial guess is very important for the success of Newton's method.

We have,

$$\phi(x_n) = x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In general,

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

which gives

$$\phi'(x) = \frac{f(x)\, f''(x)}{[f'(x)]^2}$$

Since the iteration method converges if $|\phi'(x)| < 1$. So the Newton's formula will converge if, $|f(x)\, f''(x)| < |f'(x)|^2$ in the interval considered. Assuming $f(x)$, $f'(x)$ and $f''(x)$ to be continuous, we can select a small interval in the vicinity of the root α, in which the above condition is satisfied. Hence, the result.

Newton's method converges conditionally while the regular-Falsi method always converges. However when the Newton-Raphson method converges it converges faster and is preferred. The Newton-Raphson method has second order convergence.

Scanned with CamScanner

1.8.1 Algorithm for Newton-Raphson Method

1. Start.
2. Assign an initial value to x_1 say x_0.
3. Evaluate $f(x_0)$ and $f'(x_0)$.
4. Find the improved estimate of x_0.
   $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.
5. Check for accuracy of the latest estimate. Compare relative error to a predefined value E. If $\left|\frac{x_1 - x_0}{x_1}\right| \le E$, stop. Otherwise, continue.
6. Replace x_0 by x_1 and repeat steps 4 and 5.

Example 1.10

Find the root of the equation $xe^x = \cos x$ using Newton Raphson method correct to four decimal places.

Solution:

Let $f(x) = xe^x - \cos x$ ..... (1)

Differentiating equation (1) with respect to x

$f'(x) = x\,e^x + e^x + \sin x$ ..... (2)

From equation (1)

Let the initial guess be

$x_0 = 0$

$f(x_0) = 0e^0 - \cos(0) = -1$

$f'(x_0) = 0e^0 + e^0 + \sin(0) = 1$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-1)}{1} = 1$$

$f(x_1) = 2.1779$

Now, continuing process in tabular form

Iteration	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	0	-1	1	2.1779
2	1	2.1779	0.6530	0.4603
3	0.6530	0.4603	0.5313	0.0416
4	0.5313	0.0416	0.5179	4.33×10^{-4}
5	0.5179	4.33×10^{-4}	0.5177	-1.74×10^{-4}
6	0.5177	-1.74×10^{-4}	0.5177	-4.90×10^{-7}

Here, the value of x_{n+1} do not change up to 4 decimal places.

Hence, the desired root is 0.5177 of the equation.

Scanned with CamScanner

1.8.2 Some Deductions from Newton-Raphson Formula

We can derive the following result from the Newton's iteration formula:
Iterative formula to find,

a) $\frac{1}{N}$ is $x_{n+1} = x_n (2 - Nx_n)$

b) $\sqrt{N}$ is $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$

c) $\frac{1}{\sqrt{N}}$ is $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{Nx_n}\right)$

d) $\sqrt[k]{N}$ is $x_{n+1} = \frac{1}{k}\left[(k-1)\, x_n + \frac{N}{x_n^{k-1}}\right]$

1.8.3 Advantages of Newton-Raphson Method

i) It converges fast if it converges, *i.e.*, in most case we get root in less number of steps.
ii) It requires only one guess.
iii) It has simple formula so it is easy to program.
iv) Formulation of this method is simple. So, it is very easy to apply.
v) Can be used to 'polish' a root found by other methods.
vi) Easy to convert to multiple dimensions.
vii) It is suitable for large size system.
viii) It is faster, reliable and the results are accurate.

1.8.4 Disadvantages of Newton-Raphson Method

i) Division by zero problem can occur.
ii) Inflection point issue might occur.
iii) In case of multiple roots, this method converges slowly.
iv) Near local maxima and local minima, due to oscillation, its convergence is slow.
v) Root jumping might take place thereby not getting intended solution.
vi) More complicated to code, particularly when implementing sparse matrix algorithms.
vii) Requires more memory.
viii) Must find the derivative

1.9 FIXED POINT ITERATION METHOD

Any function in the form of,

$$f(x) = 0 \quad \text{..... (1)}$$

Scanned with CamScanner

can be manipulated such that x is on the left-hand side of the equation as shown below

$$x = g(x) \qquad \text{..... (2)}$$

Equation (1) and (2) are equivalent and therefore, a root of equation (2) is also a root of equation (1). The root of equation (2) is given the point of intersection of the curves y = x and y = g(x). This intersection point is known as the fixed point of g(x).

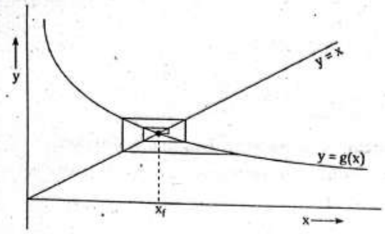


Figure 1.5: Fixed point iteration method

The above transformation can be obtained either by algebraic manipulation of the given equation or by simply adding x to both sides of the equation for example,

$$x^2 - x + 2 = 0$$

can be written as

$$x = x^2 + 2$$

or, $$x = x^2 + x + 2 + x = x^2 + 2x + 2$$

Adding of x to both sides is normally done in situations where the original equation is not amenable to algebraic manipulations.

For example, tan x = 0

Would be put into the form of equation (2) by adding x to both sides. That is, x = tan x + x.

The equation x = g(x) is known as the fixed point equation. If provides a convenient form for predicting the value of x as a function of x. If x_0 is the initial guess to a root, then the next approximation is given by,

$$x_1 = g(x_0)$$

Further approximation is given by,

$$x_2 = g(x_1)$$

This iteration process can be expressed in general form as,

$$x_{i+1} = g(x_i),\ i = 0, 1, 2, 3, \ldots\ldots$$

Scanned with CamScanner

Which is called the fixed point iteration formula. This method of solution is also known as the method of successive approximation or method of direct substitution.

The algorithm is simple the iteration process would be terminated when the successive approximations agree within some specified error.

Convergence of fixed point iteration method

Convergence of the iteration process depends on the nature of g(x). The process converges only when the absolute value of the slope of y = g(x) curve is less than the slope of y = x curve. Since the slope of y = x curve is I, the necessary condition for convergence is g'(x) < 1.

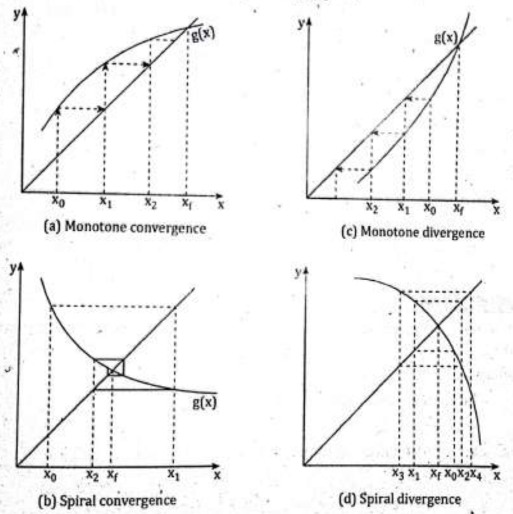


Figure 1.6: Patterns of behaviour of fixed point iteration process

We can theoretically prove this as follows;

The iteration formula is,

$$x_{i+1} = g(x_i) \quad \text{..... (3)}$$

Let, x_f be the root of the equation. Then,

$$x_f = g(x_f) \quad \text{..... (4)}$$

Subtracting equation (3) form (4) yields.

$$x_f - x_{i+1} = g(x_f) - g(x_i) \quad \text{..... (5)}$$

According to the mean value theorem, there is at least one point, say, $x = R$, in the interval x_f and x_i such that,

$$g'(R) = \frac{g(x_f) - g(x_i)}{x_f - x_i}$$

This gives,

$$g(x_f) - g(x_i) = g'(R)(x_f - x_i)$$

Replacing this in equation (5), we get,

$$x_f = x_{i+1} = g'(R)(x_f - x_i) \quad \text{...... (6)}$$

If e_i represents the error in the i^{th} iteration, then equation (6) becomes,

$$e_{i+1} = g'(R)\, e_i$$

This shows that the error will decrease with each iteration only if $g'(R) < 1$

Equation (6) implies the following,

i) Error decreases if $g'(R) < 1$

ii) Error grows if $g'(R) > 1$

iii) If $g'(R)$ is positive, the convergence is monotonic

iv) If $g'(R)$ is negative, the convergence will be oscillatory

v) The error is roughly proportional to (or less than) the error in the previous step; the fixed point method is, therefore, said to be linearly convergent.

Example 1.11

Locate root of the equation $x^2 + x - 2 = 0$ using the fixed point iteration method.

Solution:

The given equation can be expressed as,

$x = 2 - x^2$

Let us start with an initial value of $x_0 = 0$

$x_1 = 2 - 0 = 2$

$x_2 = 2 - 4 = -2$

$x_3 = 2 - 4 = -2$

Since $x_3 - x_2 = 0$, -2 is one of the roots of the equation

Let us assume that $x_0 = -1$,

Then,

$x_1 = 2 - 1 = 1$

$x_2 = 2 - 1 = 1$

Another root is 1.

Scanned with CamScanner

1.9.1 Algorithm for Fixed Point Iteration Method for a System

1.	Start.
2.	Define iteration function $F(x, y)$ and $G(x, y)$
3.	Decide starting points x_0 and y_0 and error tolerance E
4.	$x_1 = F(x_0, y_0)$ $y_1 = G(x_0, y_0)$
5.	If $\|x_1 - x_0\| < E$ and $\|y_1 - y_0\| < E$, then solution obtained; go to step 7
6.	Otherwise, set $x_0 = x_1$ $y_0 = y_1$ go to step 4
7.	Write values of x_1 and y_1
8.	Stop.

1.9.2 Advantages of Fixed Point Iteration Method

i) Ease of implementation

ii) Constraints satisfied

iii) Low cost per iteration

Scanned with CamScanner

BOARD EXAMINATION SOLVED QUESTIONS

1. **Find the positive root of the equation $f(x) = \cos x - 3x + 1$ correct upto 3 decimal places using Bisection method.** **[2013/Fall]**

Solution:

$f(x) = \cos x - 3x + 1$

Let initial guess be

$x = 0, \quad f(0) = \cos(0) - 3 \times 0 + 1 = 2 > 0$

$x = 1, \quad f(1) = \cos(1) - 3(1) + 1 = -1.4596 < 0$

So root lies between x = 0 and x = 1

$\therefore \quad x_L = 0$ and $x_U = 1$

Now, first approximated root using bisection method

$$x_N = \frac{x_L + x_U}{2} = \frac{0 + 1}{2} = 0.5$$

$f(x_N) = 0.3775 > 0$, so now root lies between 0.5 and 1

Remaining iterations are solved in tabular form

Iteration	x_L	$f(x_L) = \cos x_L - 3x_L + 1$	x_U	$f(x_U) = \cos x_U - 3x_U + 1$	x_N	$f(x_N) = \cos x_N - 3x_N + 1$
1	0	2	1	-1.4596	0.5	0.3775
2	0.5	0.3775	1	-1.4596	0.75	-0.5183
3	0.5	0.3775	0.75	-0.5183	0.625	-0.0640
4	0.5	0.3775	0.625	-0.0640	0.5625	0.1584
5	0.5625	0.1584	0.625	-0.0640	0.5937	0.0477
6	0.5937	0.0477	0.625	-0.0640	0.6093	-7.85×10^{-3}
7	0.5937	0.0477	0.6093	-7.85×10^{-3}	0.6015	0.0199
8	0.6015	0.0199	0.6093	-7.85×10^{-3}	0.6054	6.07×10^{-3}
9	0.6054	6.07×10^{-3}	0.6093	-7.85×10^{-3}	0.6073	-7.08×10^{-4}
10	0.6054	6.07×10^{-3}	0.6073	-7.08×10^{-4}	0.6063	2.86×10^{-3}
11	0.6063	2.86×10^{-3}	0.6073	-7.08×10^{-4}	0.6068	1.07×10^{-3}

Here, the value of x_N do not change upto 3 decimal places.

Hence, the positive root of the equation is 0.6068.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = x_U$, $C = x_N$, $D = f(x_L)$, $E = f(x_U)$, $F = f(x_N)$

Step 1: Set the calculator in radian mode.

Step 2: Set the following in calculator as shown;

$$A : B : C = \frac{A + B}{2} : D = \cos A - 3A + 1 : E = \cos B - 3B + 1 :$$
$$F = \cos C - 3C + 1$$

Scanned with CamScanner

Step 3: Press CALC then,
Enter the value of A? then press =
Enter the value of B? then press =
Step 4: Now press = only, again and again to get the values for the respective row for each column.
Step 5: Update the values when A? and B? is asked again.
Step 6: Go to step 4.

2. Calculate the root of non-linear equation 3x = cos x + 1 using secant method. **[2013/Fall]**

Solution:

Let, $f(x) = 3x - \cos x - 1$

$x_0 = 0$ and $x_1 = 1$ be two initial guesses

$f(x_0) = -2$ and $f(x_1) = 1.4596$

Then, next approximated root by secant method is given by

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = 1 - \frac{1.4596(1-0)}{1.4596 - (-2)} = 0.5781$$

$f(x) = -0.1032$ and now root lies between 1 and 0.5781.

Now, solving other iterations in tabular form as follows.

Iteration	x_{n-1}	$f(x_{n-1}) = 3x_{n-1} - \cos x_{n-1} - 1$	x_n	$f(x_n) = 3x_n - \cos x_n - 1$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = 3x_{n+1} - \cos x_{n+1} - 1$
1	0	-2	1	1.4596	0.5781	-0.1032
2	1	1.4596	0.5781	-0.1032	0.6059	-4.28×10^{-3}
3	0.5781	-0.1032	0.6059	-4.28×10^{-3}	0.6071	-5.88×10^{-6}
4	0.6059	-4.28×10^{-3}	0.6071	-5.88×10^{-6}	0.6071	5.73×10^{-9}

Here, the value of x_{n+1} do not change up to 4 decimal places. Hence, the root of given non-linear equation is 0.6071

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_{n-1}$, $B = x_n$, $C = x_{n+1}$, $D = f(x_{n-1})$, $E = f(x_n)$, $F = f(x_{n+1})$
Step 1: Set the calculator in radian mode.
Step 2: Set the following in calculator as shown;

$$A : B : D = 3A - \cos A - 1 : E = 3B - \cos B - 1 : C = B - \frac{E(B-A)}{E-D} :$$

$$F = 3C - \cos C - 1$$

Step 3: Press CALC then,
Enter the value of A? then press =
Enter the value of B? then press =
Step 4: Now press = only, again and again to get the values for the respective row for each column.
Step 5: Update the values when A? and B? is asked again.
Step 6: Go to step 4.

Scanned with CamScanner

3. Find a real root of the equation $x \log_{10} x = 1.2$ by using Newton Raphson (NR) method such that the root must have error less than 0.0001%. **[2013/Fall, 2018/Fall]**

Solution:

Let, $f(x) = x \log_{10} x - 1.2$ ..... (1)

Differentiating equation (1) with respect to x.

$f'(x) = 1 + \log_{10} x$ ..... (2)

From equation (1).

Let the initial guess be

$x_0 = 1, f(x_0) = -1.2, f'(x_0) = 1$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(-1.2)}{1} = 2.2$$

$f(x_1) = -0.4466$

Now, continuing process in tabular form

Iteration	x_n	$f(x_n) = x_n \log_{10} x_n - 1.2$	$f'(x_n) = 1 + \log_{10} x_n$	$f(x_{n+1}) = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} \log_{10} x_{n+1} - 1.2$
1	1	−1.2	1	2.2	−0.4466
2	2.2	−0.4466	1.3424	2.5326	−0.1779
3	2.5326	−0.1779	1.4035	2.6593	−0.0704
4	2.6593	−0.0704	1.4247	2.7087	−0.0277
5	2.7087	−0.0277	1.4327	2.7280	-0.0110
6	2.7280	−0.0110	1.4358	2.7356	-4.39×10^{-3}
7	2.756	-4.39×10^{-3}	1.4370	2.7386	-1.78×10^{-3}
8	2.7386	-1.78×10^{-3}	1.4375	2.7398	-7.37×10^{-4}
9	2.7398	-7.37×10^{-4}	1.4377	2.7403	-3.01×10^{-4}
10	2.7403	-3.01×10^{-4}	1.4377	2.7405	-1.27×10^{-4}
11	2.7405	-1.27×10^{-4}	1.4378	2.7405	-5.03×10^{-5}

Here, the value of x_{n+1} do not change up to 4 decimal places and have error less than 0.0001%. Hence, required root is 2.7405.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_n$, $B = f(x_n)$, $C = f'(x_n)$, $D = x_{n-1}$, $E = f(x_{n+1})$

Step 1: Set the calculator in radian mode.

Step 2: Set the following in calculator as shown;

$$A : B : A \log_{10} A - 1.2 : C = 1 + \log_{10} A : D = A - \frac{B}{C} : E = D \log_{10} D - 1.2$$

Step 3: Press CALC then,

Enter the value of A? then press =

Scanned with CamScanner

Step 4: Now press = only, again and again to get the values for the respective row for each column.
Step 5: Update the values when A? is asked again.
Step 6: Go to step 4.

4. Solve $f(x) = 3x + \sin x - e^x$ by secant method up to 5th iteration.
[2013/Spring, 2017/Fall]

Solution:

$f(x) = 3x + \sin x - e^x$

Let, $x_0 = 0$ and $x_1 = 1$ be two initial guesses.

$f(x_0) = -1$ and $f(x_1) = 1.1231$

Then, next approximated root by secant method is given by

$$x_2 = x_1 - \frac{f(x_1)\,(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 1 - \frac{1.1231\,(1 - 0)}{1.1231 - (-1)} = 0.4710$$

$f(x_2) = 0.2651$

Now, solving up to 5th iteration in tabular form as follows

Iteration	x_{n-1}	$f(x_{n-1}) = 3x_{n-1} + \sin x_{n-1} - e^{x_{n-1}}$	x_n	$f(x_n) = 3x_n + \sin x_n - e^{x_n}$	$x_{n+1} = x_n - \frac{f(x_n)\,(x_n - x_{n-1})}{f(x_n - f(x_{n-1}))}$	$f(x_{n+1}) = 3x_{n+1} + \sin x_{n+1} - e^{x_{n+1}}$
1	0	−1	1	1.1231	0.4710	0.2651
2	1	1.1231	0.4710	0.2651	0.3075	−0.1348
3	0.4710	0.2651	0.3075	−0.1348	0.3626	5.44×10^{-3}
4	0.3075	−0.1348	0.3626	5.44×10^{-3}	0.3604	-5.42×10^{-5}
5	0.3626	5.44×10^{-3}	0.3604	-5.42×10^{-5}	0.3604	-1.84×10^{-10}

Here, the value of x_{n+1} do not change up to 4 decimal places. Hence, the root of the given equation is 0.3604.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = (x_{n+1})$, $F = f(x_{n+1})$
Step 1: Set the calculator in radian mode.
Step 2: Set the following in calculator:

$$A : C : B = 3A + \sin A - e^A : D = 3C + \sin C - e^C : E = C - \frac{D(C - A)}{D - B} :$$

$$F = 3E + \sin E - e^E$$

Step 3: Press CALC then,
Enter the value of A? then press =
Enter the value of C? then press =
Step 4: Now press = only, again and again to get the values for the respective row for each column.
Step 5: Update the values when A? and C? is asked again.
Step 6: Go to step 4.

Scanned with CamScanner

5. The equation $\alpha \tan\alpha = 1$ occurs in theory of vibrations. Find one of the positive real roots by using any close-end method, correct to at least three decimal places. **[2014/Spring]**

Solution:

Let, $f(\alpha) = \alpha \tan \alpha - 1$

Initial guess value be

$\alpha = 0,\quad f(0) = -1 < 0$

$\alpha = 1,\quad f(1) = 0.5574 > 0$

so, root between $\alpha = 0$ and $\alpha = 1$

$\therefore\ X_L = 0$ and $x_U = 1$

Now, first approximated root using bisection method as closed end method

$$x_N = \frac{x_L + x_U}{2} = \frac{0+1}{2} = 0.5$$

$f(x_N) = -0.7268 < 0$

So root now lies between 0.5 and 1.

Remaining iterations are solved in tabular form.

Iteration	x_L	$f(x_L) = x_L \tan x_L - 1$	x_U	$f(x_U) = x_U \tan x_U - 1$	$x_N = \frac{x_L + x_U}{2}$	$f(x_N) = x_N \tan x_N - 1$
1	0	−1	1	0.5574	0.5	−0.7268
2	0.5	−0.7268	1	0.5574	0.75	−0.3013
3	0.75	−0.3013	1	0.5574	0.875	0.0477
4	0.75	−0.3013	0.875	0.0477	0.8125	−0.1422
5	0.8125	−0.1422	0.875	0.0477	0.8437	−0.0517
6	0.8437	−0.0517	0.875	0.0477	0.8593	-3.28×10^{-3}
7	0.8593	-3.28×10^{-3}	0.875	0.0477	0.8671	0.0217
8	0.8593	-3.28×10^{-3}	0.8671	0.0217	0.8632	9.16×10^{-3}
9	0.8593	-3.28×10^{-3}	0.8632	9.16×10^{-3}	0.8612	2.76×10^{-3}
10	0.8593	-3.28×10^{-3}	0.8612	2.76×10^{-3}	0.8602	-4.25×10^{-4}
11	0.8602	-4.25×10^{-4}	0.8612	2.76×10^{-3}	0.8607	1.16×10^{-3}

Here, the value of x_N do not change up to 3 decimal places.

Hence, the positive real root of the equation is 0.8607.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Set the following in calculator:

$$A : C : B = A\tan A - 1 : D = C \tan C - 1 : E = \frac{A + C}{2} : F = E \tan E - 1$$

CALC

Scanned with CamScanner

6. Find the root of the equation $f(x) = x^2 - 3x + 2$ in the vicinity of $x = 0$, using Newton Raphson method. **[2014/Spring]**

Solution:

$f(x) = x^2 - 3x + 2$ ..... (1)

Differentiating equation (1) with respect to x

$f'(x) = 2x - 3$ ..... (2)

Let the initial guess be

$x_0 = 0$, $f(x_0) = 0^2 - 3 \times (0) + 2 = 2$, $f'(0) = -3$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{2}{(-3)} = 0.6667$$

$f(x_1) = 0.4443$

Now, continuing process in tabular form.

Iteration	x_n	$f(x_n) = x_n^2 - 3x_n + 2$	$f(x_{n+1}) = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1}^2 - 3x_{n+1} + 2$
1	0	2	0.6667	0.4443
2	0.6667	0.443	0.9332	0.0712
3	0.9332	0.0712	0.9960	4.01×10^{-3}
4	0.9960	4.01×10^{-3}	0.9999	1.00×10^{-4}
5	0.9999	1.00×10^{-4}	0.9999	1.99×10^{-8}

Here, the value of x_{n+1} do not change up to 4 decimal places.

Hence, the root of the equation is 0.9999.

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
> Set the following in calculator:
> $A : B := A^2 - 3A + 2 : C = A - \frac{B}{2A - 3} : D = C^2 - 3C + 2$
> CALC

7. Find the square root of 7 using Newton Raphson method and fixed point iteration method correct up to 4 decimal digit. **[2014/Spring]**

Solution:

For Newton Raphson method

Let, $x = \sqrt{N}$ or $x^2 - N = 0$

Taking $f(x) = x^2 - N$

We have,

$f'(x) = 2x$

Then Newton's formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

Scanned with CamScanner

Now, taking N = 7, the above formula becomes

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{7}{x_n}\right)$$

For initial guess, taking approximate value of $\sqrt{7}$

i.e., $\sqrt{7} \approx \sqrt{9} = \sqrt{3^2} = 3$

i.e., we take $x_0 = 3$

Then, $x_1 = \frac{1}{2}\left(x_0 + \frac{7}{x_0}\right) = \frac{1}{2}\left(3 + \frac{7}{3}\right) = 2.6667$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{7}{x_1}\right) = 2.6458$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{7}{x_2}\right) = 2.6457$$

$$x_4 = \frac{1}{2}\left(x_3 + \frac{7}{x_3}\right) = 2.6457$$

Here, $x_3 = x_4$ upto 4 decimal places

Hence, the value of $\sqrt{7}$ is 2.6457

Now, for fixed point iteration method

$$x^2 = 7$$

$$f(x) = x^2 - 7$$

Differentiating with respect to x,

$$f'(x) = 2x$$

Let initial guess be $x_1 = 3$

$$f(x_1) = 3^2 - 7 = 2$$

Now, $x^2 - 7 = 0$

or, $2x^2 - x^2 = 7$

or, $x = \frac{7 + x^2}{2x}$

$\therefore$ $x_1 = \frac{\frac{7}{x} + x}{2}$ .....(

First iteration

$$x_1 = \frac{\frac{7}{3} + 3}{2} = 2.6666$$

Error = |2.6666 − 3| = 0.3333

Second iteration

$$x_2 = \frac{\frac{7}{2.6666} + 2.6666}{2} = 2.6458$$

Error = |2.6458 − 2.6666| = 0.0208

Third iteration

$$x_3 = \frac{\frac{7}{2.6458} + 2.6458}{2} = 2.6457$$

Error = $|2.6457 - 2.6458| = 0.0001$

Fourth iteration

$$x_4 = \frac{\frac{7}{2.6457} + 2.6457}{2} = 2.64575$$

Error = $|2.64575 - 2.6457| = 0.00005$

Here, $x_3 = x_4$ up to 4 decimal places.

Hence, the value of $\sqrt{7}$ is 2.64575

8. **The flux equation of an iron core electric circuit is given by $f(\phi) = 10 - 2.1\,\phi - 0.01\,\phi^3$. The steady state value of flux is obtained by solving the equation $f(\phi) = 0$. By using any close end method, estimate the steady state value of "ϕ" correct to 3 decimal places. [2014/Fall]**

Solution:

$f(\phi) = 10 - 2.1\,\phi - 0.01\,\phi^3$

Let initial guess be

$x = \phi = 4$, $f(4) = 10 - 2.1 \times 4 - 0.01\,(4)^3 = 0.96 > 0$

$x = \phi = 5$, $f(5) = 10 - 2.1 \times 5 - 0.01 \times 5^3 = -1.75 < 0$

So root lies between x = 4 and x = 5

∴ $x_L = 4$ and $x_U = 5$

Now, first approximated root using bisection method as close end method,

$$x_N = \frac{x_L + x_U}{2} = \frac{4+5}{2} = 4.5$$

$f(x_N) = -0.3612 < 0$ so now root lies between 4 and 4.5

Remaining iterations are solved in tabular form.

Iteration	x_L	$f(x_L) = 10 - 2.1x_L - 0.01x_L^3$	x_U	$f(x_U) = 10 - 2.1x_U - 0.01x_U^3$	$x_N = \frac{x_L + x_U}{2}$	$f(x_N) = 10 - 2.1x_N - 0.01x_N^3$
1	4	0.96	5	−1.75	4.5	−0.3612
2	4	0.96	4.5	−0.3612	4.25	0.3073
3	4.25	0.3073	4.5	−0.3612	4.375	−0.0249
4	4.25	0.3073	4.375	−0.0249	4.3125	0.1417
5	4.3125	0.1417	4.375	−0.0249	4.3437	0.0586
6	4.3437	0.0586	4.375	−0.0249	4.3593	0.0170
7	4.3593	0.0170	4.375	−0.0249	4.3671	-3.78×10^{-3}
8	4.3593	0.1070	4.3671	-3.78×10^{-3}	4.3632	6.63×10^{-3}

Scanned with CamScanner

9	4.3632	6.63×10^{-3}	4.3671	-3.78×10^{-3}	4.3651	1.55×10^{-3}
10	4.3651	1.55×10^{-3}	4.3671	-3.78×10^{-3}	4.3661	-1.11×10^{-3}
11	4.3651	1.55×10^{-3}	4.3661	-1.11×10^{-3}	4.3656	2.23×10^{-4}
12	4.3656	2.23×10^{-4}	4.3661	-1.11×10^{-3}	4.3658	-3.10×10^{-4}

Here, the value of x_N do not change up to 3 decimal places.

Hence, the steady state value of ϕ is 4.3658

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$
Set the following in calculator:

$A : C : B = 10 - 2.1\,A - 0.01\,A^3 : D = 10 - 2.1\,C - 0.01\,C^3 :$

$E = \frac{A + C}{2} : F = 10 - 2.1\,E - 0.01\,E^3$

CALC

9. Evaluate one of the real roots of the given equation $xe^x - \cos x = 0$ by NR method accurate to at least 4 decimal places. [2014/Fall]

Solution:

Let $f(x) = xe^x - \cos x$ ..... (1)

Differentiating equation (1) with respect to x.

$f'(x) = x\,e^x + e^x + \sin x$ ..... (2)

From equation (1)

Let the initial guess be

$x_0 = 0$

$f(x_0) = 0e^0 - \cos(0) = -1$

$f'(x_0) = 0e^0 + e^0 + \sin(0) = 1$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-1)}{1} = 1$$

$f(x_1) = 2.1779$

Now, continuing process in tabular form.

Iteration	x_n	$f(x_n) = x_ne^{x_n} - \cos x_n$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1}e^{x_{n+1}} - \cos x_{n+1}$
1	0	-1	1	2.1779
2	1	2.1779	0.6530	0.4603
3	0.6530	0.4603	0.5313	0.0416
4	0.5313	0.0416	0.5179	4.33×10^{-4}
5	0.5179	4.33×10^{-4}	0.5177	-1.74×10^{-4}
6	0.5177	-1.74×10^{-4}	0.5177	-4.90×10^{-7}

Here, the value of x_{n+1} do not change up to 4 decimal places.

Scanned with CamScanner

Hence, the desired root is 0.5177 of the equation.

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = x_N$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$.
> Set the following in calculator:
> $$A : B = Ae^A - \cos A : C = A - \frac{B}{Ae^A + e^A + \sin A} : D = Ce^C - \cos C$$
> CALC

10. Determine the root of $e^x = x^3 + \cos 25x$ using secant method correct to four decimal place. **[2015/Fall]**

Solution:

Let $f(x) = e^x - x^3 - \cos 25x$

Let $x_0 = 4$ and $x_1 = 5$ be two initial guesses

$f(x_0) = -10.2641$ and $f(x_1) = 22.6254$

Then, next approximated root by secant method is given by

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 5 - \frac{22.6254(5-4)}{22.2654 - (-10.2641)} = 4.3210$$

$f(x_2) = -6.1371$

Now, solving other iterations in tabular form as follows

Itn.	x_{n-1}	$f(x_{n-1}) = e^{x_{n-1}} - x_{n-1}^3 - \cos 25x_{n-1}$	x_n	$f(x_n) = e^{x_n} - x_n^3 - \cos 25x_n$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = e^{x_{n+1}} - x_{n+1}^3 - \cos 25x_{n+1}$
1	4	−10.2641	5	22.6254	4.3120	−6.1371
2	5	22.6254	4.3120	−6.1371	4.4587	−2.2048
3	4.3120	−6.1371	4.4587	−2.2048	4.5409	−0.7681
4	4.4587	−2.2048	4.5409	−0.7681	4.5848	1.5611
5	4.5409	−0.7681	4.5848	1.5611	4.5553	−0.0979
6	4.5848	1.5611	4.5553	−0.0979	4.5570	−0.0112
7	4.5553	−0.0979	4.5570	−0.0112	4.5572	-9.43×10^{-4}
8	4.5570	−0.0112	4.5572	-9.43×10^{-4}	4.5572	3.39×10^{-6}

Here, the value of x_{n+1} do not change up to four decimal place.

Hence, the root of the equation is 4.5572.

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = x_n$, $E = x_{n+1}$, $F = f(x_{n+1})$
> Set the following in calculator:
> $$A : C : B = e^A - A^3 - \cos 25A : D = e^C - C^3 - \cos 25C : E = C - \frac{D(C-A)}{D-B} :$$
> $$F = e^E - E^3 - \cos 25E$$
> CALC

Scanned with CamScanner

11. The current I in an electric circuit is given by $I = 10\,e^{-x}\sin 2\pi x$ where x is in seconds. Using NR method, find the value of x correct up to 3 decimal places for I = 2 ampere. **[2015/Fall]**

Solution:

Given that;

$$i = 10e^{-x}\sin 2\pi x$$

At i = 2 Ampere

$$2 = 10e^{-x}\sin 2\pi x$$

Let, $f(x) = 10e^{-x}\sin 2\pi x - i$ ..... (1)

or, $f(x) = (10e^{-x}\sin 2\pi x) - 2$ for i = 2 amp

Differentiating equation (1) with respect to x,

$$f'(x) = 10\,(e^{-x}\,2\pi\cos 2\pi x - \sin 2\pi x \cdot e^{-x})$$

$$= 10\,e^{-x}(2\pi\cos 2\pi x - \sin 2\pi x) \quad \text{.....(2)}$$

From equation (1),

Let the initial guess be,

$x_0 = 0$, $f(x_0) = 10e^0\,\sin 0 - 2 = -2$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-2)}{62.8318} = 0.0318$$

$$f(x_1) = -0.0773$$

Now, continuing process in tabular form

Iteration	x_n	$f(x_n) = 10e^{-x_n}\sin 2\pi x_n$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = 10e^{-x_{n+1}}\sin 2\pi x_{n+1}$
1	0	−2	0.0318	−0.0773
2	0.0318	−0.0773	0.0331	-2.45×10^{-3}
3	0.0331	-2.45×10^{-3}	0.0331	-1.20×10^{-6}

Here, the value of x_{n+1} do not change up to 3 decimal places.

Hence, the value of x is 0.0331 seconds.

12. Solve the equation log x − cos x = 0 correct to three significant digit after decimal using bracketing method. **[2015/Fall]**

Solution:

Let $f(x) = \log x - \cos x$

Let initial guess be

$x = 1$, $f(1) = \log(1) - \cos(1) = -0.5403 < 0$

$x = 2$, $f(2) = \log(2) - \cos(2) = 0.7171 > 0$

so, root lies between x = 1 and x = 2

$\therefore$ $x_L = 1$ and $x_U = 2$

Scanned with CamScanner

Now, first approximated root using bisection method.

$$x_N = \frac{x_L + x_U}{2} = \frac{1+2}{2} = 1.5$$

$f(x_N) = 0.1053 > 0$ so now root lies between 1 and 1.5.

Remaining iterations are carried out in tabular form

Itn.	x_L	$f(x_L) = \log x_L - \cos x_L$	x_U	$f(x_U) = \log x_U - \cos x_U$	$x_N = \frac{x_L + x_U}{2}$	$f(x_N) = \log x_N - \cos x_N$
1	1	−0.5403	2	0.7171	1.5	0.1053
2	1	−0.5403	1.5	0.1053	1.25	−0.2184
3	1.25	−0.2184	1.5	0.1053	1.375	−0.0562
4	1.375	−0.0562	1.5	0.1053	1.4375	0.0247
5	1.375	−0.0562	1.4375	0.0247	1.4062	−0.0157
6	1.4062	−0.0157	1.4375	0.0247	1.4218	4.39×10^{-3}
7	1.4062	−0.0157	1.4218	4.39×10^{-3}	1.4140	-5.70×10^{-3}
8	1.4140	5.70×10^{-3}	1.4218	4.39×10^{-3}	1.4179	-6.55×10^{-4}
9	1.4179	-6.55×10^{-4}	1.4218	4.39×10^{-3}	1.4198	1.80×10^{-3}
10	1.4179	-6.55×10^{-4}	1.4198	1.8×10^{-3}	1.4188	5.09×10^{-4}
11	1.4179	-6.55×10^{-4}	1.4188	5.09×10^{-4}	1.4183	-1.37×10^{-4}

Here, the value of x_N do not change up to three significant digits after decimal. Hence, the root of the equation is 1.4183.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Set the following in calculator:

$$A : C : B = \log A - \cos A : D = \log C - \cos C : E = \frac{A+C}{2} :$$

$$F \log E - \cos E$$

CALC

13. Find the root of the equation $x - 1.5 \sin x - 2.5 = 0$ using Newton Raphson method so that relative error is less than 0.01%. [2015/Spring]

Solution:

Let $f(x) = x - 1.5 \sin x - 2.5$ ..... (1)

Differentiating equation (1) with respect to x,

$f'(x) = 1 - 1.5 \cos x$ ..... (2)

From equation (1),

Let the initial guess be

$x_0 = 3$

$f(x_0) = 3 - 1.5 \sin(3) - 2.5 = 0.2883$

$f'(x_0) = 1 - 1.5 \cos(3) = 2.4849$

Scanned with CamScanner

Now, using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{0.2883}{2.4849} = 2.8839$$

$$f(x_1) = 1.624 \times 10^{-3}$$

Now continuing process in tabular form.

Iteration	x_n	$f(x_n)$ $x_n - 1.5 \sin x_n - 2.5$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} - 1.5 \sin x_{n+1} - 2.5$
1	3	0.2883	2.8839	1.624×10^{-3}
2	2.8839	1.624×10^{-3}	2.8832	-9.034×10^{-5}
3	2.8832	-9.034×10^{-5}	2.8832	-5.250×10^{-9}

Here, the value of x_{n+1} do not change up to 4 decimal places and relative error is also less than 0.01%. Hence, the root of the equation is 2.8832.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
Set the following in calculator:

$$A : B = A - 1.5 \sin A - 2.5 : C = A - \frac{B}{1 - 1.5 \cos A} :$$

$$D = C - 1.5 \sin C - 2.5$$

CALC

14. Find the root of the equation $xe^x = \cos x$ using secant method correct to four decimal place. [2015/Spring]

Solution:

Let, $f(x) = xe^x - \cos x$

$x_0 = 0$ and $x_1 = 1$ be the initial guesses

$f(x_0) = 0e^0 - \cos(0) = -1$

$f(x_1) = 1 \times e^1 - \cos(1) = 2.1779$

Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = 1 - \frac{2.1779(1-0)}{2.1779 - (-1)} = 0.3146$$

$$f(x_2) = -0.5200$$

Now, solving other iterations in tabular form as follows

Itn.	x_{n-1}	$f(x_{n-1}) = x_{n-1} e^{x_{n-1}} - \cos x_{n-1}$	x_n	$f(x_n) = x_n e^{x_n} - \cos x_n$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = x_{n+1} e^{x_{n+1}} - \cos x_{n+1}$
1	0	−1	1	2.1779	0.3146	−0.5200
2	1	2.1779	0.3146	−0.5200	0.4467	−0.2036
3	0.3146	−0.5200	0.4467	−0.2036	0.5317	0.0429
4	0.4467	−0.2036	0.5317	0.0429	0.5169	-2.60×10^{-3}
5	0.5317	0.0429	0.5169	-2.60×10^{-3}	0.5177	-1.74×10^{-4}
6	0.5169	-2.60×10^{-3}	0.5177	-1.74×10^{-4}	0.5177	4.47×10^{-8}

Scanned with CamScanner

Here, the value of x_{n+1} do not change up to four decimal places.
Hence, the root of the equation is 0.5177.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = x_{n+1}$, $F = f(x_{n+1})$
Set the following in calculator:

$$A : C : B = Ae^A - \cos A : D = Ce^C - \cos C : E = C - \frac{D(C-A)}{D-B} :$$
$$F = Ee^E - \cos E$$

CALC

15. Using the bisection method, find the approximate root of the equation $\sin x = \frac{1}{x}$ that lies between x = 1 and x = 1.5 (in radian's). Carry out up to 7th stage. [2013/Spring, 2015/Spring, 2017/Fall]

Solution:

Let $f(x) = \sin x - \frac{1}{x}$

The initial guess be,

$x = 1$, $\quad f(1) = \sin 1 - \frac{1}{1} = -0.1585 < 0$

$x = 1.5$, $\quad f(1.5) = \sin(1.5) - \frac{1}{1.5} = 0.3308 > 0$

As root lies between x = 1 and x = 1.5,

$\therefore \quad x_L = 1$ and $x_U = 1.5$

Now, first approximated root using bisection method,

$$x_N = \frac{x_L + x_U}{2} = \frac{1 + 1.5}{2} = 1.25$$

$f(x_N) = 0.1489 > 0$ so now root lies between 1 and 1.25.

Performing the iterations up to 7th stage in tabular form.

Itn.	x_L	$f(x_L) = \sin x_L - \frac{1}{x_L}$	x_U	$f(x_U) = \sin x_U - \frac{1}{x_U}$	$x_N = \frac{x_L + x_U}{2}$	$f(x_N) = \sin x_N - \frac{1}{x_N}$
1	1	−0.1585	1.5	0.3308	1.25	0.1489
2	1	−0.1585	1.25	0.1489	1.125	0.0133
3	1	−0.1585	1.125	0.0133	1.0625	−0.0676
4	1.0625	−0.0676	1.125	0.0133	1.09375	−0.0259
5	1.09375	−0.0259	1.125	0.0133	1.109375	-5.98×10^{-3}
6	1.109375	-5.98×10^{-3}	1.125	0.0133	1.1171875	3.76×10^{-3}
7	1.109375	-5.98×10^{-3}	1.1171875	3.76×10^{-3}	1.11328125	-1.09×10^{-3}

Thus, the desired approximation to the root carried out up to 7th stage is 1.11328125.

Scanned with CamScanner

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Set the following in calculator:

$$A : C : B = \sin A - \frac{1}{A} : D = \sin C - \frac{1}{C} : E = \frac{A + C}{2} : F = \sin E - \frac{1}{E}$$

CALC

16. Find a real root of the equation $xe^x = 3$ by using any bracketing method correct to three decimal places (Take $x_1 = 1$ and $x_2 = 1.5$). **[2016/Fall]**

Solution:

Let $f(x) = x e^x - 3$

And, initial guess be the provided value

i.e., $x = 1$, $f(1) = 1e^1 - 3 = -0.2817 < 0$

$x = 1.5$, $f(1.5) = 1.5e^{1.5} - 3 = 3.7225 > 0$

Root lies between $x = 1$ and $x = 1.5$,

$\therefore$ $x_L = 1$ and $x_U = 1.5$

Now, first approximated root using bisection method as bracketing method

$$x_N = \frac{x_L + x_U}{2} = \frac{1 + 1.5}{2} = 1.25$$

$f(x_N) = 1.3629 > 0$ so now root lies between 1 and 1.25.

Remaining iterations are carried out in tabular form.

Itn.	x_L	$f(x_L) = x_L e^{x_L} - 3$	x_U	$f(x_U) = x_U e^{x_U} - 3$	$x_N = \frac{x_L + x_U}{2}$	$f(x_N) = x_N e^{x_N} - 3$
1	1	−0.2817	1.5	3.7225	1.25	1.3629
2	1	−0.2817	1.25	1.3629	1.125	0.4652
3	1	−0.2817	1.125	0.4652	1.0625	0.0744
4	1	−0.2817	1.0625	0.0744	1.0312	−0.1077
5	1.0312	−0.1077	1.0625	0.0744	1.0468	−0.0181
6	1.0468	−0.0181	1.0625	0.0744	1.0546	0.0275
7	1.0468	−0.0181	1.0546	0.0275	1.0507	4.63×10^{-3}
8	1.0468	−0.0181	1.0507	4.63×10^{-3}	1.0487	-7.07×10^{-3}
9	1.0487	-7.07×10^{-3}	1.0507	4.63×10^{-3}	1.0497	-1.22×10^{-3}
10	1.0497	-1.22×10^{-3}	1.0507	4.63×10^{-3}	1.0502	1.70×10^{-3}
11	1.0497	-1.22×10^{-3}	1.0502	1.70×10^{-3}	1.0499	-5.21×10^{-5}
12	1.0499	-5.21×10^{-5}	1.0502	1.70×10^{-3}	1.0500	5.33×10^{-4}
13	1.0499	-5.21×10^{-5}	1.0500	5.33×10^{-4}	1.0499	-5.21×10^{-5}
14	1.0499	-5.21×10^{-5}	1.0500	5.33×10^{-4}	1.0499	-5.21×10^{-5}

Here, the value of x_N do not change up to 3 decimal places.

Scanned with CamScanner

Hence, the real root of the equation is 1.0499.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Set the following in calculator:

$$A : C : B = Ae^A - 3 : D = Ce^C - 3 : E = \frac{A + C}{2} : F = Ee^E - 3$$

CALC

17. **Obtain a real root of the equation sin x + 1 = 2x by using secant method such that the real root must have relative error less than 0.0001.** **[2016/Fall]**

Solution:

Let $f(x) = \sin x + 1 - 2x$

Let $x_0 = 0$ and $x_1 = 1$ be two initial guesses.

$f(x_0) = 1$ and $f(x_1) = -0.1585$

Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)\,(x_1 - x_0)}{f(x_1) - f(x_0)} = 1 - \frac{-0.1585(1 - 0)}{-0.1585 - 1} = 0.8631$$

$f(x_2) = 0.0336$

Now, solving other iterations in tabular form as follows

Itn.	x_{n-1}	$f(x_{n-1}) = \sin x_{n-1} + 1 - 2x_{n-1}$	x_n	$f(x_n) = \sin x_n + 1 - 2x_n$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = \sin x_{n+1} + 1 - 2x_{n+1}$
1	0	1	1	-0.1585	0.8631	0.0336
2	1	-0.1585	0.8631	0.0336	0.8870	1.18×10^{-3}
3	0.8631	0.0336	0.8870	1.18×10^{-3}	0.8878	8.51×10^{-5}
4	0.8870	1.18×10^{-3}	0.8878	8.51×10^{-5}	0.8878	4.43×10^{-6}

Here, the value of x_{n+1} do not change up to 4 decimal places and have relative error less than 0.0001.

Hence, the real root of the equation is 0.8878

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = x_{n+1}$, $F = f(x_{n+1})$

Set the following in calculator:

$$A : C : B = \sin A + 1 - 2A : D = \sin C + 1 - 2C : E = C - \frac{D(C - A)}{D - B} :$$

$$F = \sin E + 1 - 2E$$

CALC

18. **Find the root of the equation x sin x + cos x = 0 using Newton Raphson's method so that relative error is less than 0.1.** **[2016/Fall]**

Scanned with CamScanner

Solution:

Let, $f(x) = x \sin x + \cos x$ ..... (1)

Differentiating equation (1) with respect to x,

$f'(x) = x \cos x$ ..... (2)

From equation (1)

Let the initial guess be

$x_0 = 2$, $f(x_0) = 1.4024$, $f'(x_0) = -0.8322$

Using NR method, next approximated root is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{1.4024}{-0.8322} = 3.6851$$

$f(x_1) = -2.7616$

Now, continuing the process in tabular form.

Itn.	x_n	$f(x_n) = x_n \sin x_n + \cos x_n$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} \sin x_{n+1} + \cos x_{n+1}$
1	2	1.4024	3.6851	-2.7616
2	3.6851	-2.7616	2.8095	-0.0294
3	2.8095	-0.0294	2.7984	-0.03×10^{-3}
4	2.7984	-0.03×10^{-3}	2.7983	2.26×10^{-4}
5	2.7983	2.26×10^{-4}	2.7983	7.32×10^{-7}

Here, the value of x_{n+1} do not change up to 4 decimal places. And, relative error is also less than 0.1.

$$\text{Relative error} = \left(\frac{|x_{n+1} - x_n|}{x_{n+1}}\right) = \left(\frac{|2.7983 - 2.7984|}{2.7983}\right) = 0.003574$$

Hence, the desired root of the equation is 2.7983.

NOTE:

Procedure to iterate in programmable calculator:

Let, A = x_n, B = $f(x_n)$, C = x_{n+1}, D = $f(x_{n+1})$

Set the following in calculator:

$A : B : A \sin A + \cos A : C = A - \frac{B}{A \cos A} : D = C \sin C + \cos C$

CALC

19. Using Newton-Raphson method find a root of the equation $xe^x = 2$. **[2016/Spring]**

Solution:

Let, $f(x) = xe^x - 2$ ..... (1)

Differentiating equation (1) with respect to x,

$f'(x) = e^x + x e^x$ ..... (2)

Scanned with CamScanner

From equation (1),

Let the initial guess be,

$x_0 = 0, \quad f(x_0) = 0e^0 - 2 = -2, \quad f'(x_0) = e^0 + 0e^0 = 1$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-2}{1} = 2$$

$$f(x_1) = 12.7781$$

Now, continuing process in tabular form.

Iteration	x_n	$f(x_n) = x_n e^{x_n} - 2$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} e^{x_{n+1}} - 2$
1	0	-2	2	12.7781
2	2	12.7781	1.4235	3.9098
3	1.4235	3.9098	1.0349	0.9130
4	1.0349	0.9130	0.8755	0.1012
5	0.8755	0.1012	0.8530	1.71×10^{-3}
6	0.8530	1.71×10^{-3}	0.8526	-2.39×10^{-5}
7	0.8526	-2.39×10^{-5}	0.8526	-1.01×10^{-8}

Here, the value of x_{n+1} do not change up to 4 decimal places.

Hence, a root of the equation is 0.8526.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$

Set the following in calculator:

$$A : B = Ae^A - 2 : C = A - \frac{B}{e^A + Ae^A} : D = Ce^C - e$$

CALC

20. Find a real root of the cos x = 3x − 1, correct to three decimal places, using fixed point method. **[2016/Spring]**

Solution:

Let, $f(x) = \cos x - 3x + 1 = 0$ ..... (1)

or, $\cos x + 1 = 3x$

or, $x = \frac{1 + \cos x}{3}$

i.e., $g(x) = \frac{1 + \cos x}{3}$ ..... (2)

Let initial guess be $x_0 = 1$ then,

$$|g'(x_0)| = \left|\frac{1}{3}(-\sin x)\right| = \left|\frac{1}{3}(-\sin 1)\right| = 0.2804$$

Here, $|0.2804| < 1$

Scanned with CamScanner

Then next approximated root by fixed point method is given by,

$$g(x_0) = x_1 = \frac{1 + \cos(1)}{3} = 0.5134$$

Now, continuing the process in tabular form.

Itn.	x_n	$f(x_n) = \cos x_n - 3x_{n+1}$	$x_{n+1} = g(x_n) = \frac{1 + \cos x_n}{3}$	$f(x_{n+1}) = \cos x_{n+1} - 3x_{n+1} + 1$
1	1	-1.45	0.5134	0.3308
2	0.5134	0.3308	0.6236	-0.0590
3	0.6236	-0.0590	0.6039	0.0114
4	0.6039	0.0114	0.6077	-2.13×10^{-3}
5	0.6077	-2.13×10^{-3}	0.6069	7.19×10^{-4}
6	0.6069	7.19×10^{-4}	0.6071	5.88×10^{-6}
7	0.6071	5.88×10^{-6}	0.6071	-1.11×10^{-6}

Here, the value of $g(x_n)$ do not change up to 4 decimal places.

Hence, the real real of the equation is 0.6071.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1} = g(x_n)$, $D = f(x_{n+1})$

Step 1: Set the calculator in radian mode.

Step 2: Set the following in calculator:

$$A : B = \cos A - 3A + 1 : C = \frac{1 + \cos A}{3} : D = \cos C - 3C + 1$$

Step 3: Press CALC then,

Enter the value of A? then press =

Step 4: Now press = only, again and again to get the values for the respective row for each column.

Step 5: Update the values when A? is asked again.

Step 6: Go to step 4.

21. Find a real root of $e^{\cos x} - \sin x - 1 = 0$ correct to 4 decimal places using false position method. **[2017/Spring]**

Solution:

Let, $f(x) = e^{\cos x} - \sin x - 1$

The initial guess be,

$x_L = x_0 = 0$, $f(x_0) = e^{\cos(0)} - \sin(0) - 1 = 1.71828 > 0$

$x_U = x_1 = 1$, $f(x_1) = e^{\cos(1)} - \sin(1) - 1 = -0.12494 < 0$

i.e., Root lies between 0 and 1.

Now, using false position method,

$$x_2 = x_0 - \frac{(x_1 - x_0)\, f(x_0)}{f(x_1) - f(x_0)} = 0 - \frac{(1 - 0) \times 1.71828}{(-0.12494 - 1.71828)} = 0.93221$$

$\therefore$ $f(x_2) = 0.01201$

Since the value of $f(x_2)$ is positive, now root lies between 0.9322 and 1.

Scanned with CamScanner

Solving other iterations in tabular form as follows,

Itn.	x_L	$f(x_L) = e^{\cos x_L} - \sin x_L - 1$	x_U	$f(x_U) = e^{\cos x_U} - \sin x_U - 1$	$x_N = x_L - \frac{f(x_L)(x_U - x_L)}{f(x_U) - f(x_L)}$	$(x_N) = e^{\cos x_N} - \sin x_N - 1$
1	0	1.71828	1	-0.12494	0.93221	0.01201
2	0.93221	0.01201	1	-0.12494	0.93815	-1.64×10^{-4}
3	0.93221	0.01201	0.93815	-1.64×10^{-4}	0.93806	1.95×10^{-5}
4	0.93806	1.95×10^{-5}	0.93815	-1.64×10^{-4}	0.93806	1.95×10^{-5}

Here, the value of x_N do not change up to 4 decimal places.

Hence, the root of the equation is 0.93806.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Step 1: Set the calculator in radian mode.

Step 2: Set the following in calculator:

$$A : C : B = e^{\cos A} - \sin A - 1 : D = e^{\cos C} - \sin C - 1 : E = A - \frac{(C - A)B}{D - B} :$$

$$F = e^{\cos E} - \sin E - 1$$

Step 3: Press CALC then,

Enter the value of A? then press =

Enter the value of C? then press =

Step 4: Now press = only, again and again to get the values for the respective row for each column.

Step 5: Update the values when A? and C? is asked again.

Step 6: Go to step 4.

22. **Find the root of the equation 3x = cos x + 1 using NR method with the tolerance is 10E – 5.** **[2017/Spring]**

Solution:

Let, $f(x) = 3x - \cos x - 1$ ..... (1)

Differentiating equation (1) with respect to x,

$f'(x) = 3 + \sin x$ ..... (2)

From equation (1),

Let the initial guess be,

$x_0 = 0,\quad f(x_0) = 3 \times 0 - \cos 0 - 1 = -2 < 0$

$x_1 = 1,\quad f(x_1) = 3 \times 1 - \cos(1) - 1 = 1.4596 > 0$

so, a root lies between 0 and 1.

Using Newton Raphson method, next approximated root is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-2)}{3} = 0.6667$$

$f(x_1) = 0.2142$

Now, continuing process in tabular form.

Itn.	x_n	$f(x_n) = 3x_n - \cos x_n - 1$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = 3x_{n+1} - \cos x_{n+1} - 1$
1	0	-2	0.6667	0.2142
2	0.6667	0.2142	0.6075	1.422×10^{-3}
3	0.6075	1.422×10^{-3}	0.6071	-5.88×10^{-6}
4	0.6071	-5.88×10^{-6}	0.6071	-4.53×10^{-9}

Here, the value of x_{n+1} do not change up to 4 decimal places.
Hence, the desired root is 0.6071 of the equation.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
Set the following in calculator:

$$A : B = 3A - \cos A - 1 : C = A - \frac{B}{3 + \sin A} : D = 3C - \cos C - 1$$

CALC

23. Find the root of $e^x \tan x = 1$ by creating iterative formula of Newton-Raphson method. **[2018/Spring]**

Solution:

Let $f(x) = e^x \tan x - 1$ ..... (1)

Differentiating equation (1) with respect to x,

$f'(x) = e^x (\tan x + \sec^2 x)$ ..... (2)

From equation (1),
Let the initial guess be,

$x_0 = 0$

$f(x_0) = e^0 \tan 0 - 1 = -1$

$f'(x_0) = e^0(\tan 0 + \sec^2 0) = 1$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-1)}{1} = 1$$

$f(x_1) = 3.2334$

Now, continuing process in tabular form.

Itn.	x_n	$f(x_n) = e^{x_n} \tan x_n - 1$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = e^{x_{n+1}} \tan x_{n+1} - 1$
1	0	-1	1	3.2334
2	1	3.2334	0.7612	1.0396
3	0.7612	1.0396	0.5914	0.2132
4	0.5914	0.2132	0.5357	0.0142
5	0.5357	0.0142	0.5314	3.007×10^{-5}
6	0.5314	3.007×10^{-5}	0.5313	-2.988×10^{-4}
7	0.5313	-2.988×10^{-4}	0.5313	3.311×10^{-8}

Scanned with CamScanner

Here, the value of x_{n+1} do not change up to 4 decimal places.
Hence, the desired root is 0.5313 of the equation.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
Set the following in calculator:

$$A : B = e^A \tan A - 1 : C = A - \frac{B}{e^A(\tan A + \sec^2 A)} : D = e^C \tan C - 1$$

CALC

24. Solve f(x) = xe^x – 1 by secant method for tolerance value 0.0001.

Solution: **[2018/Spring]**

$f(x) = x\,e^x - 1$

Let $x_0 = 0$ and $x_1 = 1$ be two initial guesses.

Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)\,(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 1 - \frac{(1.7182)\,(1 - 0)}{1.7182 - (-1)} = 0.3678$$

$f(x_2) = -0.4686$

Now, solving other iterations in tabular form as follows,

Itn.	x_{n-1}	$f(x_{n-1}) = x_{n-1}e^{x_{n-1}} - 1$	x_n	$f(x_n) = x_n e^{x_n} - 1$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = x_{n+1}e^{x_{n+1}} - 1$
1	0	−1	1	1.7182	0.3678	−0.4686
2	1	1.7182	0.3678	−0.4686	0.5032	−0.1677
3	0.3678	−0.4686	0.5032	−0.1677	0.5786	0.0319
4	0.5032	−0.1677	0.5786	0.0319	0.5665	−0.0017
5	0.5786	0.0319	0.5665	−0.0017	0.5671	−0.0001
6	0.5665	−0.0017	0.5671	−0.0001	0.5671	−0.0001

Here, the value of x_{n+1} do not change up to 4 decimal places with the tolerance value of 0.0001.

Hence, the root of the equation is 0.5671.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = x_{n+1}$, $F = f(x_{n+1})$
Set the following in calculator:

$$A : C : B = Ae^A - 1 : D = Ce^C - 1 : E = C - \frac{D(C - A)}{D - B} : F = Ee^E - 1$$

CALC

Scanned with CamScanner

25. Using secant method, find a root of the equation $e^x \sin x - x^2 = 0$ correct up to three decimal places. **[2018/Fall]**

Solution:

Let, $f(x) = e^x \sin x - x^2$

and, $x_0 = 2$ and $x_1 = 3$ be two initial guesses.

$f(x_0) = e^2 \sin(2) - 2^2 = 2.7188$ and $f(x_1) = -6.1655$

Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 3 - \frac{-6.1655(3-2)}{-6.1655 - 2.7188}$$

$$= 2.3060$$

$f(x_2) = 2.1246$

Now, solving other iterations in tabular form as follows,

Itn.	x_{n-1}	$f(x_{n-1}) = e^{x_{n-1}} \sin x_{n-1} - x_{n-1}^2$	x_n	$f(x_n) = e^{x_n} \sin x_n - x_n^2$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = e^{x_{n+1}} \sin x_{n+1} - x_{n+1}^2$
1	2	2.7188	3	−6.1655	2.3060	2.1246
2	3	−6.1655	2.3060	2.1246	2.4838	1.1590
3	2.3060	2.1246	2.4838	1.1590	2.6972	−0.8958
4	2.4838	1.1590	2.6972	−0.8958	2.6041	0.1401
5	2.6972	−0.8958	2.6041	0.1401	2.6166	0.0144
6	2.6041	0.1401	2.6166	0.0144	2.6180	1.43×10^{-4}
7	2.6166	0.0144	2.6180	1.43×10^{-4}	2.6180	-8.70×10^{-7}

Here, the value of x_{n+1} do not change up to three decimal places.

Hence, the root of given equation is 2.6180.

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = x_{n+1}$, $F = f(x_{n+1})$
> Set the following in calculator:
> $A : C : B = e^A \sin A - A^2 : D = e^C \sin C - C^2 : E = C - \frac{D(C-A)}{D-B} :$
> $F = e^E \sin E - E^2$
> CALC

26. Find where the graph of $y = x - 3$ and $y = ln(x)$ intersect using bisection method. Get the intersection value correct to four decimal places. **[2019/Fall]**

Solution:

$y = x - 3$ and $y = ln(x)$

$f(x_1) = x - 3$, $f(x_2) = ln(x)$

Scanned with CamScanner

In order to intersect

$$f(x_1) - f(x_2) = 0$$

i.e., $f(x) = x - 3 - ln(x) = 0$

Let initial guess be,

$x = 1,\quad f(1) = 1 - 3 - ln(1) = -2$

$x = 2,\quad f(2) = -1.6991 < 0$

$x = 3,\quad f(3) = -1.0986 < 0$

$x = 4,\quad f(4) = -0.3862 < 0$

$x = 5,\quad f(5) = 0.3905 > 0$

so, root lies between $x = 4$ and $x = 5$

$\therefore \quad x_L = 4$ and $x_U = 5$

Now, first approximated root using bisection method.

$$x_N = \frac{x_L + x_U}{2} = \frac{4 + 5}{2} = 4.5$$

$f(x_N) = -0.0040 < 0$ so now root lies between 4.5 and 5.

Remaining iterations are solved in tabular form.

Itn.	x_L	$f(x_L) = x_L - 3 - ln(x_L)$	x_U	$f(x_U) = x_U - 3 - ln(x_U)$	x_N	$f(x_N) = x_N - 3 - ln(x_N)$
1	4	-0.3862	5	0.3905	4.5	-0.0040
2	4.5	-0.0040	5	0.3905	4.75	0.1918
3	4.5	-0.0040	4.75	0.1918	4.625	0.0935
4	4.5	-0.0040	4.625	0.0935	4.5625	0.0446
5	4.5	-0.0040	4.5625	0.0446	4.53125	0.0202
6	4.5	-0.0040	4.53125	0.0202	4.515625	0.0080
7	4.5	-0.0040	4.515625	0.0080	4.5078125	0.0020
8	4.5	-0.0040	4.5078125	0.0020	4.50390625	-0.0010
9	4.50390625	-0.0010	4.5078125	0.0020	4.505859	0.0004
10	4.503906	-0.0010	4.505859	0.0004	4.504882	-0.0002
11	4.504882	-0.0002	4.505859	0.0004	4.505370	-0.0001
12	4.504882	-0.0002	4.505370	0.0001	4.505126	-8.985×10^{-5}
13	4.505126	-8.985×10^{-5}	4.505370	0.0001	4.505248	5.060×10^{-6}
14	4.505126	-8.985×10^{-5}	4.505248	5.060×10^{-6}	4.505187	-4.239×10^{-5}
15	4.505187	-4.239×10^{-5}	4.505248	5.060×10^{-6}	4.5052175	-1.866×10^{-5}
16	4.5052175	-1.866×10^{-5}	4.505248	5.060×10^{-6}	4.505232	-6.804×10^{-6}

Scanned with CamScanner

Here, the value of x_N do not change up to 4 decimal places.

Hence, the graph of $y = x - 3$ and $y = ln(x)$ intersects at $x = 4.505232$.

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$
> Set the following in calculator:
> $$A : C : B = A - 3 - \ln(A) : D = C - 3 - \ln(C) : E = \frac{A + C}{2} : F = E - 3 - \ln(E)$$
> CALC

27. Find value of $\sqrt{18}$ using Newton Raphson method. **[2019/Fall]**

Solution:

Let $x = \sqrt{N}$ or $x^2 - N = 0$

Taking $f(x) = x^2 - N$, we have $f'(x) = 2x$

Then Newton's formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

Now, taking N = 18, the above formula becomes

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{18}{x_n}\right)$$

For initial guess, taking approximate value of $\sqrt{18}$

i.e., $\sqrt{18} = \sqrt{4^2} = \sqrt{16} = 4$

i.e., we take $x_0 = 4$

Then,

$$x_1 = \frac{1}{2}\left(x_n + \frac{18}{x_0}\right) = \frac{1}{2}\left(4 + \frac{18}{4}\right) = 4.25.$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{18}{x_1}\right) = \frac{1}{2}\left(4.25 + \frac{18}{4.25}\right) = 4.2426$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{18}{x_2}\right) = \frac{1}{2}\left(4.2426 + \frac{18}{4.2426}\right) = 4.2426$$

Here, $x_2 = x_3$ up to 4 decimal places.

Hence, the value of $\sqrt{18}$ is 4.2426.

28. Using secant method, find the zero of function $f(x) = 2x - \log_{10} x - 7$ correct up to three decimal places. **[2019/Spring]**

Solution:

$f(x) = 2x - \log_{10} x - 7$

Let, $x_0 = 1$ and $x_1 = 2$ be two initial guesses.

> **NOTE:**
> 0 is not taken as initial guess because it gives the undetermined value of f(x) at x = 0.

Scanned with CamScanner

Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 2 - \frac{(-3.3010)(2-1)}{-3.3010 - (-5)}$$

$$= 3.9429$$

$$f(x_2) = 0.2899$$

Now, solving other iterations in tabular form as follows

Itn.	x_{n-1}	$f(x_{n-1}) = 2x_{n-1} - \log_{10}x_{n-1} - 7$	x_n	$f(x_n) = 2x_n - \log_{10}x_n - 7$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = 2x_{n+1} - \log_{10}x_{n+1} - 7$
1	1	-5	2	-3.3010	3.9429	0.2899
2	2	-3.3010	3.9429	0.2899	3.7860	-6.180×10^{-3}
3	3.9429	0.2899	3.7860	-6.180×10^{-3}	3.7892	-1.475×10^{-3}
4	3.7860	-6.180×10^{-3}	3.7892	-1.475×10^{-3}	3.7902	-0.1508
5	3.7892	-1.475×10^{-3}	3.7902	-0.1508	3.7891	-3.360×10^{-4}
6	3.7902	-0.1508	3.7891	-3.360×10^{-4}	3.7890	-5.246×10^{-4}
7	3.7891	-3.360×10^{-4}	3.7890	-5.246×10^{-4}	3.7892	-1.475×10^{-4}

Here, the value of x_{n+1} do not change up to 3 decimal places.

Hence, the zero of function f(x) is 3.7892.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = x_{n+1}$, $F = f(x_{n+1})$

Set the following in calculator:

$A : C : B = 2A - \log_{10} A - 7 : D = 2C - \log_{10} C - 7 : E = C - \frac{D(C-A)}{D-B} :$

$F = 2E - \log_{10} E - 7$

CALC

29. Find the root of the equation log x − cos x = 0 correct up to three decimal placed by using N-R method. [2019/Spring]

Solution:

Let, $f(x) = \log x - \cos x$ ..... (1)

Differentiating equation (1) with respect to x,

$$f'(x) = \frac{1}{x} + \sin x \quad \text{..... (2)}$$

From equation (1),

Let the initial guess be,

$x_0 = 1$, $f(x_0) = -0.5403$, $f'(x_0) = 1.8414$

Scanned with CamScanner

Using Newton Raphson method, next approx...

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-0.5403}{1.8414} = 1.2934$$

$f(x_1) = -0.1621$

Now, continuing process in tabular form.

Iteration	x_n	$f(x_n) = \log x_n - \cos x_n$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = \log x_{n-1} - \cos x_{n+1}$
1	1	−0.5403	1.2934	−0.1621
2	1.2934	−0.1621	1.3868	−0.0409
3	1.3868	−0.0409	1.4107	-9.97×10^{-3}
4	1.4107	-9.97×10^{-3}	1.4165	-2.46×10^{-3}
5	1.4165	-2.46×10^{-3}	1.4179	-6.55×10^{-4}
6	1.4179	-6.55×10^{-4}	1.4182	-2.67×10^{-4}
7	1.4182	-2.67×10^{-4}	1.4183	-1.37×10^{-4}

Here, the value of x_{n+1} do not change up to 3 decimal places.

Hence, the desired root is 1.4188 of the equation.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$

Set the following in calculator:

$$A : B = \log A - \cos A : C = A - \frac{B}{\sin A + \frac{1}{A}} : D = \log C \cos C$$

CALC

30. Find the positive real root of the equation $\cos x + e^x + x^2 = 3$. Using false position method, correct to 3 decimal places **[2020/Fall]**

Solution:

Let, $f(x) = \cos x + e^x + x^2 - 3$

The initial guess be,

$x_0 = 0$, $\quad f(x_0) = \cos 0 + e^0 + 0^2 - 3 = -1 < 0$

$x_1 = 1$, $\quad f(x_1) = \cos(1) + e^1 + 1^2 - 3 = 1.2585 > 0$

so, root lies between 0 and 1.

Now, using false position method,

$$x_2 = x_0 - \frac{(x_1 - x_0)\, f(x_0)}{f(x_1) - f(x_0)}$$

$$= 0 - \frac{(1-0)(-1)}{1.2585 - (-1)} = 0.4427$$

$\therefore \quad f(x_2) = -0.3435$

Since the value of $f(x_2)$ is negative, now root lies between 0.4427 and 1.

Solving other iterations in tabular form as follows,

Scanned with CamScanner

Itn.	x_L	$f(x_L) = \cos x_L + e^{x_L} + x_L^2 - 3$	x_U	$f(x_U) = \cos x_U + e^{x_U} + x_U^2 - 3$	$x_N = x_L - \frac{f(x_L)(x_U - x_L)}{f(x_U) - f(x_L)}$	$f(x_N) = \cos x_N + e^{x_N} + x_N^2 - 3$
1	0	−1	1	1.2585	0.4427	−0.3435
2	0.4427	−0.3435	1	1.2585	0.5621	−0.0835
3	0.5621	−0.0835	1	1.2585	0.5893	−0.0186
4	0.5893	−0.0186	1	1.2585	0.5952	-4.30×10^{-3}
5	0.5952	-4.30×10^{-3}	1	1.2585	0.5965	-1.12×10^{-3}
6	0.5965	-1.12×10^{-3}	1	1.2585	0.5968	-3.94×10^{-4}

Here, the value of x_N do not change up to three decimal places.

Hence, the positive real root of the equation is 0.5968.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Set the following in calculator:

$$A : C : B = \cos A + e^A + A^2 - 3 : D = \cos C + e^C + C^2 - 3 :$$

$$E = A - \frac{(C - A)B}{D - B} : F = \cos E + e^E + E^2 - 3$$

CALC

31. Find the real root of the equation x sin x − cos x = 0 using Newton-Raphson method, correct to 3 decimal places. [2020/Fall]

Solution:

Let, $f(x) = x \sin x - \cos x$ ..... (1)

Differentiating equation (1) with respect to x,

$$f'(x) = x \cos x + \sin x + \sin x \quad \text{..... (2)}$$

$$= x \cos x + 2 \sin x$$

From equation (1),

Let the initial guess be,

$x_0 = 1$, $f(x_0) = 0.3011$, $f'(x_0) = 2.2232$

Using NR method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{0.3011}{2.2232} = 0.8645$$

$f(x_1) = 8.66 \times 10^{-3}$

Now, continuing process in tabular form.

Itn.	x_n	$f(x_n) = x_n \sin x_n - \cos x_n$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} \sin x_{n+1} - \cos x_{n+1}$
1	1	0.3011	0.8645	8.66×10^{-3}
2	0.8645	8.66×10^{-3}	0.8603	-6.97×10^{-5}
3	0.8603	-6.97×10^{-5}	0.8603	-7.02×10^{-8}

Here, the value of x_{n+1} do not change up to 3 decimal places.

Scanned with CamScanner

Hence, the desired root of the equation is 0.8603.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
Set the following in calculator:

$$A : B = A \sin A - \cos A : C = A - \frac{B}{A \cos A + 2 \sin A} :$$

$$D = C \sin C - \cos C$$

CALC

32. Find the real root of the equation $x \log_{10} x - 1.2 = 0$ correct to four places of decimal using bracketing method. [2014/Fall]

Solution:

Let, $f(x) = x \log_{10} x - 1.2$

Let initial guess be,

$x = 2.5$, $f(2.5) = -0.2051 < 0$

$x = 3$, $f(3) = 0.2313 > 0$

so, root lies between $x = 2.5$ and $x = 3$

$\therefore$ $x_L = 2.5$ and $x_U = 3$

Now, first approximated root using bisection method,

$$x_N = \frac{x_L + x_U}{2} = \frac{2.5 + 3}{2} = 2.75$$

$f(x_N) = 8.16 \times 10^{-3} > 0$ so now root lies between 2.5 and 2.75

Remaining iterations are solved in tabular form.

Itn.	x_L	$f(x_L) = x_L \log_{10} x_L - 1.2$	x_U	$f(x_U) = x_U \log_{10} x_U - 1.2$	$x_N = \frac{x_L + x_U}{2}$	$f(x_N) = x_N \log_{10} x_N - 1.2$
1	2.5	-0.2051	3	0.2313	2.75	8.16×10^{-3}
2	2.5	-0.2051	2.75	8.160×10^{-3}	2.625	-0.0997
3	2.625	-0.0997	2.75	8.16×10^{-3}	2.6875	-0.0461
4	2.6875	-0.0461	2.75	8.16×10^{-3}	2.7187	-0.0191
5	2.7187	-0.0191	2.75	8.16×10^{-3}	2.7343	-5.53×10^{-3}
6	2.7343	-5.53×10^{-3}	2.75	8.16×10^{-3}	2.7421	1.26×10^{-3}
7	2.7343	-5.53×10^{-3}	2.7421	1.26×10^{-3}	2.7382	-2.13×10^{-3}
8	2.7382	-2.13×10^{-3}	2.7421	1.26×10^{-3}	2.7401	-4.76×10^{-4}
9	2.7401	-4.76×10^{-4}	2.7421	1.26×10^{-3}	2.7411	3.95×10^{-4}
10	2.7401	-4.76×10^{-4}	2.7411	3.95×10^{-4}	2.7406	-4.02×10^{-5}
11	2.7406	-4.02×10^{-5}	2.7411	3.95×10^{-4}	2.7408	1.34×10^{-4}
12	2.7406	-4.02×10^{-5}	2.7408	1.34×10^{-4}	2.7407	4.70×10^{-5}
13	2.7406	-4.02×10^{-5}	2.7407	4.70×10^{-5}	2.7406	-4.02×10^{-5}
14	2.7406	-4.02×10^{-5}	2.7407	4.70×10^{-5}	2.7406	-4.022×10^{-5}

Scanned with CamScanner

Here, the value of x_N do not change up to 4 decimal places.

Hence, the real of the equation is 2.7406.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$.

Set the following in calculator:

$$A : C : B = A \log_{10} A - 1.2 : D = C \log_{10} C - 1.2 : E = \frac{A + C}{2} :$$

$$F = E \log_{10} E - 1.2$$

CALC

33. Write short notes on error in numerical calculations.

[2013/Spring, 2014/Spring, 2016/Spring, 2019/Fall]

Solution: See the topic 1.3.

34. Write short notes on monotonic and oscillatory divergence in fixed point iteration method. **[2014/Fall]**

Solution:

If $g:[a, b] \longrightarrow [a, b]$ is continuous, then $g(x)$ has a fixed point, x^*, such that $g(x^*) = x^*$. Furthermore, if $|g'(x)| < 1$ for all $x \in [a, b]$, then this fixed point is unique on $[a, b]$ and the fixed point iteration $x_{n+1} = g(x_n)$ will coverage to x^* for all choices of $x_0 \in [a, b]$.

We have,

$$|g(x) - (y)| \le |g'(z)|\ |x - y| < |x - y|$$

where, $Z \in [x, y] \subset [a, b]$. Thus the mapping contracts and by the contraction mapping theorem $x_n \longrightarrow x^*$, the unique fixed point.

The plane R^2 may be divided into four regions for some $g(x)$ with respect to one of its fixed point x^*, depending on the behaviour $g(x)$ takes while in a given region. Chiefly, one is concerned with when $|g(x) - x^*| < |x - x^*|$, so that $g(x)$ is said to converge.

The region where $g(x)$ converges is bounded by points satisfying $|g(x) - x^*| < |x - x^*|$. For such a boundary, either $g(x) - x^* = x - x^*$ or $g(x) - x^* = x^* - x$. The former indicates that the boundary consists of additional fixed points, $g(x) = x$. The latter gives the boundary $g(x) = 2x^* - x$. One may summarize that if $g(x)$ lies between the lines x and $2x^* - x$, then $g(x)$ will be closer to the fixed point x^* than x.

The second division of behaviour is whether sign $[g(x - x^*)]$ = sign $(x - x^*)$. If this is true, then $g(x)$ converges or diverges monotonically towards or away from x^* in this region. If this is false then $g(x)$ oscillates around x^*. The boundary between these two regions is where sign $[g(x) - x^*) = 0]$ or where $g(x) = x^*$.

Scanned with CamScanner

We are now prepared to describe the four regions of a fixed point function is 1D, based on these behaviours (convergence/divergence, monotonic/ oscillation).

Region 1

If $g(x) < x < x^*$ or $g(x) > x > x^*$, then g(x) diverges monotonically from x^*

Region 2

If $x < g(x) < x^*$ or $x > g(x) > x^*$, then g(x) converges monotonically towards x^*

Region 3

If $x < x^* < g(x) < 2x^* - x$ or, $x > x^* > g(x) > 2x^* - x$, then g(x) converges with oscillations towards x^*.

Region 4

If $x < x^* < 2x^* - x < g(x)$ or $x > x^* > 2x^* - x > g(x)$, then g(x) diverges with oscillations from x^*.

35. Write short notes on an algorithm for NR-method. [2014/Spring]

Solution: See the topic 1.8.1.

36. Write short notes on convergence of Newton-Raphson methods. [2015/Fall]

Solution: See the topic 1.8.

37. Write short notes on importance of numerical methods in Engineering. [2018/Fall]

Solution:

Numerical simulation is a powerful tool to solve scientific and engineering problem. it plays an important role in many aspects of fundamental research and engineering applications. For example mechanism of turbulent flow with/without visco-elastic additives, optimization of processes, prediction of oil/gas production and online control of manufacturing. The soul of numerical simulation is numerical method which is driven by the above demands and in return pushes science and technology by the successful applications of advanced numerical methods. With the development of mathematical theory and computer hardware, various numerical methods are proposed. The new numerical methods or their new applications lead to important progress in the related fields. For example, parallel computing largely promote the precision of direct numerical simulations of turbulent flow to capture undiscovered flow structures. Proper orthogonal decomposition method greatly reduces the simulation time of oil pipelining transportation. Thus, numerical methods become more and more important and their modern developments are worth exploring.

A numerical method is a complete and definite set of procedures for the solution of a problem, together with computable error estimates. The study and implementation of such methods is the province of numerical analysis.

Scanned with CamScanner

Numerical methods may be regard as a new 'philosophy' in the development of the computer based scientific methods. Even the computer based approaches are deterministic or randomness based i.e., semi-numerical methods. The major advantage of numerical methods is that a numerical value can be obtained even when the problem has no analytical solution.

In many aspects of our life, a huge amount of different materials are used. Glass, wood, metals, concrete, which are directly used almost every minute in our everyday life. Thus, the modification of materials and prediction of their properties are very important objectives for the manufactures. In order to produce high quality materials, the engineers in industry, among other problems, are very much interested in the elastic behaviour or loading capacity of the material. While it is known that the bonding forces between the atoms of the material are responsible for their physical and chemical properties. So to manufacture a new product with higher quality, a detailed investigation of the material on the atomic level is not required in most cases. A mathematical model is needed for the quantitative description of the change of material properties under external influences. The concept of differential equations come to help us as an excellent tool for the development of such a model.

38. Write short notes on convergence of fixed point iteration method. **[2018/Spring]**

Solution: See the topic 1.9

39. Write short notes on: algorithm of Bisection method. **[2019/Spring]**

Solution: See the topic 1.6.1

40. Write an algorithm to find a real of a non-linear equation using secant method. **[2016/Spring]**

Solution: See the topic 1.7.1.

Scanned with CamScanner

ADDITIONAL QUESTION SOLUTION

1 Round off the number 75462 to four significant digits and then calculate the absolute error and percentage error.

Solution:

Given that;

$x = 75462$

Now, rounding off the number up to four significant digits

$x_1 = 75460$

Then,

Absolute error (E_a) $= |x - x_1|$

$= |75462 - 75460| = 2$

and, Percentage error (E_p) $= E_r \times 100$

$$= \left|\frac{x - x_1}{x}\right| \times 100$$

$$= \left|\frac{75462 - 75460}{75462}\right| \times 100$$

$= 0.0027$

2. If 0.333 is the approximate value of $\frac{1}{3}$, find the absolute and relative errors.

Solution:

We have,

Exact value $(x) = \frac{1}{3}$

Approximate value $(x_1) = 0.333$

Then,

$$\text{Absolute error, } E_a = |x - x_1| = \left|\frac{1}{3} - 0.333\right| = 0.0003$$

$$\text{Relative error, } E_r = \left|\frac{x - x_1}{x}\right| = \left|\frac{\frac{1}{3} - 0.333}{\frac{1}{3}}\right| = 0.0010$$

3. The height of an observation tower was estimated to be 47 m, whereas its actual height was 45 m. Calculate the percentage relative error in the measurement.

Solution:

We have,

Actual height of tower $(x) = 45$ m

Estimated height of tower $(x_1) = 47$ m

Scanned with CamScanner

Then,

Percentage relative error, $E_p = E_r \times 100$

$$= \left|\frac{x - x_1}{x}\right| \times 100 = \left|\frac{45 - 47}{45}\right| \times 100$$

$$= 4.4444$$

4. Find a real root of the following equation, correct to six decimals, using the fixed point iteration method.

$\sin x + 3x - 2 = 0$

Solution:

Let, $f(x) = \sin x + 3x - 2 = 0$ ..... (1)

or, $3x = 2 - \sin x$

or, $x = \frac{2 - \sin x}{3}$

i.e., $g(x) = \frac{2 - \sin x}{3}$ ..... (2)

Differentiating equation (2) with respect to x

$$g'(x) = \frac{1}{3}(0 - \cos x) = \frac{-\cos x}{3}$$

Let initial guess be $x_0 = 1$ then

$$|g'(x_0)| = \left|\frac{-\cos(1)}{3}\right| = 0.180101$$

Here; $0.180101 < 1$

Then next approximated root by fixed point method is given by,

$$g(x_0) = x_1 = \frac{2 - \sin(1)}{3} = 0.386176$$

Now continuing the process in tabular form

Iteration	x_L	$f(x_n)$	$x_{n+1} = g(x_n)$	$f(x_{n+1})$
1	1	1.841471	0.386176	−0.464823
2	0.386176	−0.464823	0.541117	0.138445
3	0.541117	0.138445	0.494969	−0.040089
4	0.494969	−0.040089	0.508332	0.011717
5	0.508332	0.011717	0.514426	−0.003417
6	0.504426	−0.003417	0.505565	0.000997
7	0.505565	0.000997	0.505233	−0.000290
8	0.505233	−0.000290	0.505330	0.000086
9	0.505330	0.000086	0.505301	−0.000026
10	0.505301	−0.000026	0.505310	0.000009
11	0.505310	0.000009	0.505307	−0.000003
12	0.505307	−0.000003	0.505308	0.000001
13	0.505308	0.000001	0.505308	0.000001

Scanned with CamScanner

Here, the value of $g(x_n)$ do not change up to 6 decimal places.

Hence, the real root of the equation is 0.505308.

5. Find a real root of the equation $\sin x = e^{-x}$ correct up to four decimal places using N.R method.

Solution:

Let, $f(x) = \sin x - e^{-x}$ ...... (1)

Differentiating equation (1) with respect to x

$f'(x) = \cos x + e^{-x}$ ...... (2)

From equation (1)

Let the initial guess be

$x_0 = 0, \quad f(x_0) = -1, \quad f'(x_0) = 2$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-1)}{2} = 0.5$$

$f(x_1) = -0.12711$

Now continuing process in tabular form

Iteration	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	0	−1	0.5	−0.12711
2	0.5	−0.12711	0.58565	−0.00400
3	0.58565	−0.00400	0.58853	-3.80×10^{-6}
4	0.58853	-3.80×10^{-6}	0.58853	-6×10^{-9}

Here, the value of x_{n+1} do not change up to 4 decimal places.

Hence, the desired real root of the equation is 0.58853.

6. Find a real root of $\cos x + e^x - 5 = 0$ accurate to 4 decimal places using the secant method.

Solution:

Let, $f(x) = \cos x + e^x - 5$

Let, $x_0 = 1$ and $x_1 = 2$ be two initial guesses.

Then,

$f(x_0) = -1.74142$ and $f(x_1) = 1.97291$

Next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 2 - \frac{1.97291(2-1)}{1.97291 + 1.74142}$$

$$= 1.46884$$

$f(x_2) = -0.55403$

Scanned with CamScanner

Now solving other iterations in tabular form as

Iteration	x_{n-1}	$f(x_{n-1})$	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	1	-1.74142	2	1.97291	1.46884	-0.55403
2	2	1.97291	1.46884	-0.55403	1.58530	-0.13375
3	1.46884	-0.55403	1.58530	-0.13375	1.62236	0.01349
4	1.58530	-0.13375	1.62236	0.01349	1.61896	-0.00031
5	1.62236	0.01349	1.61896	-0.00031	1.61904	0.00002
6	1.61896	-0.00031	1.61904	0.00002	1.61904	-2.91×10^{-6}

Here, the value of x_{n+1} do not change up to 4 decimal places.
Hence, the root of the equation is 1.61904.

7. **Using the Bisection method, find a real root of the equation $f(x) = 3x - \sqrt{1 + \sin x}$ correct up to three decimal points.**

Solution:

Given that;

$$f(x) = 3x - \sqrt{1 + \sin x}$$

Let initial guess be

$x = 0$, $f(0) = -1 < 0$

$x = 1$, $f(1) = 1.64299 > 0$

so, root lies between x = 0 and x = 1

$\therefore$ $x_L = 0$ and $x_U = 1$

Now first approximated root using bisection method

$$x_N = \frac{x_L + x_U}{2} = \frac{0 + 1}{2} = 0.5$$

$$f(x_N) = 0.2837 > 0$$

so, now root lies between 0 and 0.5

Remaining iterations are solved in tabular form

Iteration	x_L	$f(x_L)$	x_U	$f(x_U)$	x_N	$f(x_N)$
1	0	-1	1	1.64299	0.5	0.2837
2	0	-1	0.5	0.2837	0.25	-0.3669
3	0.25	-0.3669	0.5	0.2837	0.375	-0.0439
4	0.375	-0.0439	0.5	0.2837	0.4375	0.1193
5	0.375	-0.0439	0.4375	0.1193	0.4063	0.0377
6	0.375	-0.0439	0.4063	0.0377	0.3907	-0.0030
7	0.3907	-0.0030	0.4063	0.0377	0.3985	0.0174
8	0.3907	-0.0030	0.3985	0.0174	0.3946	0.0072
9	0.3907	-0.0030	0.3946	0.0072	0.3927	0.0022
10	0.3907	-0.0030	0.3927	0.0022	0.3917	-0.0004
11	0.3917	-0.0004	0.3927	0.0022	0.3922	0.0009
12	0.3917	-0.0004	0.3922	0.0009	0.3920	0.0004

Scanned with CamScanner

Here, the value of x_N do not change up to three decimal places.

Hence, the real root of the equation is 0.3920.

8. Find a root of $e^x = 3x$ using bisection method and Newton Raphson method correct up to 3 decimal places.

Solution:

Let, $f(x) = e^x - 3x$

i) Using bisection method

Let initial guess be

$x = 0.5, \quad f(0.5) = 0.1487 > 0$

$x = 1, \quad f(1) = -0.2817 < 0$

so, root lies between x = 0.5 and x = 1

$\therefore \quad x_L = 0.5$ and $x_U = 1$

Now, first approximated root using bisection method

$$x_N = \frac{x_L + x_U}{2} = \frac{0.5 + 1}{2} = 0.75$$

$f(x_N) = -0.1330 < 0$ so root lies between 0.5 and 0.75

Remaining iterations are solved in tabular form .

Iteration	x_L	$f(x_L)$	x_U	$f(x_U)$	x_N	$f(x_N)$
1	0.5	0.1487	1	-0.2817	0.75	-0.1330
2	0.5	0.1487	0.75	-0.1330	0.625	-0.0068
3	0.5	0.1487	0.625	-0.0068	0.5625	0.0676
4	0.5625	0.0676	0.625	-0.0068	0.5938	0.0295
5	0.5938	0.0295	0.625	-0.0068	0.6094	0.0111
6	0.6094	0.0111	0.625	-0.0068	0.6172	0.0021
7	0.6172	0.0021	0.625	-0.0068	0.6211	-0.0023
8	0.6172	0.0021	0.6211	-0.0023	0.6192	-0.0002
9	0.6172	0.0021	0.6192	-0.0002	0.6182	0.0010
10	0.6182	0.0010	0.6192	-0.0002	0.6187	0.0004

Here, the value of x_N do not change up to three decimal places.

Hence, the root of the equation is 0.6187.

ii) Using NR method

$f(x) = e^x - 3x$ ..... (1)

Differentiating equation (1) with respect to x

$f'(x) = e^x - 3$

From equation (1)

Let the initial guess be

$x_0 = 0, f(x_0) = 1, f'(x_0) = -2$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1}{(-2)} = 0.5$$

$f(x_1) = 0.1487$

Now, continuing the iterations in tabular form.

Iteration	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	0	1	0.5	0.1487
2	0.5	0.1487	0.6100	0.0104
3	0.61	0.0104	0.6190	0.0001
4	0.619	0.0001	0.6191	−0.00004

Here, the value of x_{n+1} do not change up to 3 decimal places.

Hence, the desired root of the equation is 0.6191.

9. **Find a real root of $x^5 - 3x^3 - 1 = 0$ correct up to four decimal places using the secant method.**

Solution:

Let, $f(x) = x^5 - 3x^3 - 1$

and, $x_0 = 1$ and $x_1 = 2$ be two initial guesses

$f(x_0) = -3$ and $f(x_1) = 7$

Then next approximated root by secant method is given by

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = 2 - \frac{7(2-1)}{7+3} = 1.3$$

$f(x_2) = -3.8781$

Now, solving other iterations in tabular form as,

Iteration	x_{n-1}	$f(x_{n-1})$	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	1	−3	2	7	1.3	−3.8781
2	2	7	1.3	−.38781	1.5496	−3.2279
3	1.3	−3.8781	1.5496	−3.2279	2.7887	102.5969
4	1.5496	−3.2279	2.7887	102.5969	1.5874	−2.9206
5	2.7887	102.5969	1.5874	−2.9206	1.6207	−2.5893
6	1.5874	−2.9206	1.6207	−2.5893	1.8810	2.5816
7	1.6207	−2.5893	1.8810	2.5816	1.7510	−0.6457
8	1.8810	2.5816	1.7510	−0.6457	1.7770	−0.1149
9	1.7510	−0.6457	1.7770	−0.1149	1.7826	0.0064
10	1.7770	−0.1149	1.7826	0.0064	1.7823	−0.0002
11	1.7826	0.0064	1.7823	−0.0002	1.7823	−0.0002

Here, the value of x_{n+1} do not change up to 4 decimal places.

Hence, a real root of the equation is 1.7823.

Scanned with CamScanner

10. Evaluate the real root of $f(x) = 4 \sin x - e^x$ using Newton Raphson method. The absolute error of root in consecutive iteration should be less than 0.01.

Solution:

Let, $f(x) = 4 \sin x - e^x$ ..... (1)

Differentiating equation (1) with respect to x

$$f'(x) = 4 \cos x - e^x$$

From equation (1)

Let the initial guess be

$x_0 = 0$, $f(x_0) = -1$, $f'(x_0) = 3$

Using NR method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-1)}{3} = 0.3333$$

$f(x_1) = -0.0869$

Now, continuing process in tabular form.

Iteration	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	0	-1	0.3333	-0.0869
2	0.3333	-0.0869	0.3697	-0.0020
3	0.3697	-0.0020	0.3706	0.0001
4	0.3706	0.0001	0.3706	0.0001

Here, the value of x_{n+1} do not change up to 4 decimal places and have error less than 0.01. Hence, required root is 0.3706.

Chapter

2

INTERPOLATION AND APPROXIMATION

♠♠♠

2.1 INTRODUCTION

The process of construction of y(x) to fit a table of data points is called curve fitting. A table of data may belong to one of the following two categories.

1. Table of values of well-defined functions

Examples of such tables are logarithmic tables, trigonometric tables, interest tables, steam tables etc.

2. Data tabulated from measurements made during an experiment

In such experiments, values of the dependent variable are recorded at various values of the independent variable. There are numerous examples of such experiments, the relationship between stress and strain on a metal strip, relationship between voltage applied and speed on a metal strip, relationship between voltage applied and speed of a fan, relationship between time and temperature raise in heating a given volume of water, relationship between drag force and velocity of a falling body etc can be tabulated by suitable experiments.

In category-1, the table values are accurate because they are obtained from well-behaved functions. This is not the case in category 2 where the relationship between the variable is not well defined. Accordingly, we have two approaches for fitting a curve to a given set of data points.

Scanned with CamScanner

In the first case, the function is constructed such that it passes through all the data points. This method of constructing a function and estimating values at non-tabular points is called interpolation. The functions are known as interpolation polynomials.

In the second case, the values are not accurate and therefore, it will be meaningless to try to pass the curve through every point. The best strategy would be to construct a single curve that would represent the general trend of the data, without necessarily passing through the individual points. Such functions are called approximating functions. One popular approach for finding an approximate function to fit a given set of experimental data is called least squares regression. The approximate functions are known as least-squares polynomials.

The various methods of interpolation are;

a) Lagrange interpolation
b) Newton's interpolation
c) Newton-Gregory forward interpolation
d) Spline interpolation

2.2 INTERPOLATION WITH UNEQUAL INTERVALS

Interpolation formula for unequally spaced values of x can be obtained from any method given below.

i) Lagrange's interpolation formula
ii) Newton's general interpolation formula with divided differences

2.2.1 Lagrange's Interpolation Formula

Let, $y = f(x)$ takes the value $y_0, y_1, \dots\dots, y_n$ corresponding to $x = x_0, x_1, \dots\dots, x_n$, then,

$$f(x) = \frac{(x - x_1)(x - x_2) \dots\dots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \dots\dots (x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2) \dots\dots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \dots\dots (x_1 - x_n)} y_1 + \dots\dots + \frac{(x - x_0)(x - x_1) \dots\dots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \dots\dots (x_n - x_{n-1})} y_n \quad \dots (1)$$

This is known as Lagrange's interpolation formula for unequal intervals.

Proof:

Let, $y = f(x)$ be a function which takes the values $(x_0, y_0), (x_1, y_1), \dots\dots, (x_n, y_n)$. Since there are $n + 1$ pairs of values of x and y, we can represent $f(x)$ by a polynomial in x of degree n. Let this polynomial be of the form,

$$y = f(x) = a_0 (x - x_1)(x - x_2) \dots\dots (x - x_n) + a_1 (x - x_0)(x - x_2) \dots\dots (x - x_n) + a_2 (x - x_0)(x - x_1)(x - x_3) \dots\dots (x - x_n) + \dots\dots + a_n (x - x_0)(x - x_1) \dots\dots (x - x_{n-1}) \quad \dots (2)$$

Replacing $x = x_0$, $y = y_0$ in (2), we have,

$$y_0 = a_0 (x_0 - x_1)(x - x_2) \dots\dots (x - x_n)$$

Scanned with CamScanner

$$a_0 = \frac{y_0}{(x - x_1)(x - x_2) \ldots\ldots (x - x_n)}$$

Similarly putting $x = x_1, y = y_1$ in (2), we have,

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \ldots\ldots (x_1 - x_n)}$$

Proceeding the same way, we find $a_2, a_3, a_4 \ldots\ldots a_n$.

Replacing the values of $a_0, a_1, \ldots\ldots, a_n$ in (2), we get (1).

> **NOTE:**
> Lagrange's interpolation formula (1) for n points is a polynomial of degree (n – 1) which is known as the Lagrangian polynomial and is very simple to implement on a computer. This formula can also be used to split the given function into partial fractions.

On dividing both sides of (1) by $(x - x_0)(x - x_1) \ldots\ldots (x - x_n)$, we get,

$$\frac{f(x)}{(x - x_0)(x - x_1) \ldots\ldots (x_0 - x_n)} = \frac{y_0}{(x_0 - x_1)(x_0 - x_2) \ldots\ldots (x_0 - x_n)} \cdot \frac{1}{(x - x_0)}$$

$$+ \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \ldots\ldots (x_1 - x_n)} \cdot \frac{1}{(x - x_1)}$$

$$+ \ldots\ldots + \frac{y_n}{(x_n - x_0)(x_0 - x_1) \ldots\ldots (x_n - x_{n-1})} \cdot \frac{1}{(x - x_n)}$$

Example 2.1

Given the values:

x	5	7	11	13	17
f(x)	150	392	1452	2366	5202

Evaluate f(9) using Lagrange's formula.

Solution:

Here;

$x_0 = 5, \quad x_1 = 7, \quad x_2 = 11, \quad x_3 = 13, \quad x_4 = 17$

and, $y_0 = 150, \quad y_1 = 392, \quad y_2 = 1452, \quad y_3 = 2366, \quad y_4 = 5202$

$$f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \times y_0$$

$$+ \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \times y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \times y_2$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \times y_3$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_0)(x_4 - x_1)(x_4 - x_1)(x_4 - x_3)} \times y_4$$

Putting x = 9 and replacing the above values in Lagrange's formula, we get,

$$f(9) = \frac{(9 - 7)(9 - 11)(9 - 13)(9 - 17)}{(5 - 7)(5 - 11)(5 - 13)(5 - 17)} \times 150$$

Scanned with CamScanner

$$+\frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}\times 392$$

$$+\frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}\times 1452$$

$$+\frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}\times 2366$$

$$+\frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}\times 5202$$

$$=\left(-\frac{50}{3}\right)+\frac{3136}{15}+\frac{3872}{3}+\frac{2366}{3}+\frac{578}{5}$$

$\therefore$ $f(9) = 810$

Example 2.2

Find the polynomial f(x) by using Lagrange's formula and hence find f(3) for

x	0	1	2	5
f(x)	2	3	12	147

Solution:

Here;

$x_0 = 0,\quad x_1 = 1,\quad x_2 = 2,\quad x_3 = 5$

and, $y_0 = 2,\quad y_1 = 3,\quad y_2 = 12,\quad y_3 = 147$

Lagrange's formula is,

$$y=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\times y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\times y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\times y_2$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\times y_3$$

$$=\frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}\times 2$$

$$+\frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}\times 3$$

$$+\frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}\times 12$$

$$+\frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}\times 147$$

Hence, $f(x) = x^3 + x^2 - x + 2$

$\therefore$ $f(3) = 3^3 + 3^2 - 3 + 2 = 27 + 9 - 3 + 2 = 35$

Scanned with CamScanner

Example 2.3

Find the missing term in the following table using interpolation.

x	0	1	2	3	4
f(x)	1	3	9	–	81

Solution:

Since the given data is unevenly spaced, we use Lagrange's interpolation formula.

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

Given that;

$x_0 = 0,\quad x_1 = 1,\quad x_2 = 2,\quad x_3 = 4$

$y_0 = 1,\quad y_1 = 3,\quad y_2 = 9,\quad y_3 = 81$

$$\therefore \quad y = \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)} \times 1 + \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)} \times 3$$

$$+ \frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)} \times 9 + \frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)} \times 81$$

When x = 3, then,

$$\therefore \quad y = \frac{(3-1)(3-2)(3-4)}{-8} + \frac{3(3-2)(3-4)}{1} + \frac{3(3-1)(3-4)}{-4} \times 9$$

$$+ \frac{3(3-1)(3-2)}{24} \times 81$$

$$= \frac{1}{4} - 3 + \frac{27}{2} + \frac{81}{24} = 31$$

Hence the missing term for x = 3 is y = 31.

Example 2.4

Using Lagrange's formula, express the function $\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ is a sum of partial functions.

Solution:

Let us evaluate: $y = 3x^2 + x + 1$ for x = 1, x = 2 and x = 3.

These values are:

x	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$
y	$y_0 = 5$	$y_1 = 15$	$y_2 = 31$

Lagrange's formula is,

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \times y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \times y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \times y_2$$

Scanned with CamScanner

Replacing the above values, we get,

$$y = \frac{(x-2)(x-3)}{(1-2)(1-3)} \times 5 + \frac{(x-1)(x-3)}{(2-1)(2-3)} \times 15 + \frac{(x-1)(x-2)}{(3-1)(3-2)} \times 31$$

$$= 2.5(x-2)(x-3) - 15(x-1)(x-3) + 15.5(x-1)$$

Thus, $\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}$

$$= \frac{2.5(x-2)(x-3) - 15(x-1)(x-3) + 15.5(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$= \frac{25}{(x-1)} - \frac{15}{(x-2)} + \frac{15.5}{(x-3)}$$

2.2.2 Divided Differences

The Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated. This labor of recomposing the interpolation formula which employs what are called 'divided differences". Before deriving this formula, we shall first define these differences.

If (x_0, y_0), (x_1, y_1), (x_2, y_2) .... be given points, then the first divided difference for the arguments x_0, x_1 is defined by the relation

$$[x_0, x_1] \text{ or } \underset{x_1}{\Delta} y_0 = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly,

$$[x_1, x_2] \text{ or } \underset{x_2}{\Delta} y_0 = \frac{y_2 - y_1}{x_2 - x_1}$$

and, $$[x_2, x_3] \text{ or } \underset{x_3}{\Delta} y_0 = \frac{y_3 - y_2}{x_3 - x_2}$$

The second divided difference for x_0, x_1, x_2 is defined as

$$[x_0, x_1, x_2] \text{ or } \underset{x_1, x_2}{\Delta^2} y_0 = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

The third divided difference for x_0, x_1, x_2, x_3 is defined as

$$[x_0, x_1, x_2, x_3] \text{ or } \underset{x_1, x_2, x_3}{\Delta^3} y_0 = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_2 - x_0}$$

Properties of divided differences

1. The divided differences are symmetrical in their arguments *i.e.*, independent of the order of the arguments.

$$[x_0, x_1] = \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0} = [x_1, x_0], [x_0, x_1, x_2]$$

$$= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

$$= [x_1, x_2, x_0] \text{ or } [x_2, x_0, x_1] \text{ and so on.}$$

Scanned with CamScanner

2. The nth divided differences of a polynomial of the nth degree are constant.

Let the arguments be equally spaced so that

$$x_1 - x_0 = x_2 - x_1 = \dots\dots = x_n - x_{n-1} = h$$

Then, $[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} + \frac{\Delta y_0}{h}$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

$$= \frac{1}{2h}\left[\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right] = \frac{1}{2!h^2}\Delta^2 y_0$$

and in general,

$$[x_0, x_1, x_2, \dots\dots, x_n] = \frac{1}{n!h^n}\Delta^n y_0$$

If the tabulated function is a nth degree polynomial, then $\Delta^n y_0$ will be constant. Hence, the nth divided differences will also be constant.

I. Newton's Divided Difference Formula

Let $y_0, y_1, y_2, \dots\dots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, x_2, \dots\dots, x_n$. Then from the definition of divided differences, we have,

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

So that,

$$y = y_0 + (x - x_0)\,[x, x_0] \quad \dots\dots (1)$$

Again,

$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

which gives,

$$[x, x_0] = [x_0, x_1] + (x - x_1)\,[x, x_0, x_1]$$

Hence the equation (1) becomes,

$$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x, x_0, x_1] \quad \dots\dots (2)$$

Also, $[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x, x_0, x_2]}{x - x_2}$

which gives $[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)\,[x, x_0, x_1, x_2]$

Replacing this value in equation (2), we get,

$$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\,[x, x_0, x_1, x_2]$$

Preceding in this manner, we get,

$$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)(x - x_1)\dots\dots(x - x_n)\,[x, x_0, x_1, \dots\dots, x_n] + (x - x_0)(x - x_1)(x - x_2)\,[x, x_0, x_1, x_2] + \dots\dots \quad \dots\dots (3)$$

Scanned with CamScanner

Which is called Newton's general interpolation formula with divided differences.

II. Relation between Divided and Forward Differences

If (x_0, y_0), (x_1, y_1), (x_2, y_2), ... be the given points, then,

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Also, $\Delta y_0 = y_1 - y_0$

If x_0, x_1, x_2, ... are equispaced,

then, $x_1 - x_0 = h$, so that,

$$[x_0, x_1] = \frac{\Delta y_0}{h}$$

Similarly,

$$[x_1, x_2] = \frac{\Delta y_1}{h}$$

Now,

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_1} = \frac{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}}{2h} \qquad [\because x_2 - x_0 = 2h]$$

$$= \frac{\Delta y_1 - \Delta y_0}{2h^2}$$

Thus, $[x_0, x_1, x_2] = \dfrac{\Delta^2 y_0}{2!h^2}$

Similarly,

$$[x_0, x_1, x_2] = \frac{\Delta^2 y_1}{2!h^2}$$

$$\therefore \quad [x_0, x_1, x_2, x_3] = \frac{\frac{\Delta^2 y_1}{2h^2} - \frac{\Delta^2 y_0}{2h^2}}{x_3 - x_0} = \frac{\Delta^2 y_1 - \Delta^2 y_0}{2h^2(3)} \qquad [\because x_3 - x_0 = 3h]$$

Thus, $[x_0, x_1, x_2, x_3] = \dfrac{\Delta^3 y_0}{3!h^3}$

In general,

$$[x_0, x_1, x_2, \ldots\ldots\ldots, x_n] = \frac{\Delta^n y_0}{n!h^n}$$

This is the relation between divided and forward differences.

Example 2.5

Given the values:

x	5	7	11	13	17
f(x)	150	392	1452	2366	5202

Scanned with CamScanner

Evaluate f(9), using Newton's divided difference formula.

Solution:

Creating difference table from Newton's dividend difference formula as

x	$f(x_n)$	$f(x_n, x_{n+1})$	$f(x_n, x_{n+1}, x_{n+2})$	$f(x_n, x_{n+1}, x_{n+2}, x_{n+3})$	$f(x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4})$
5	**150**				
		$\frac{392-150}{7-5}$ = **121**			
7	392		$\frac{265-121}{11-5}$ = **24**		
		$\frac{1452-392}{11-7}$ = 265		$\frac{32-24}{13-5}$ = **1**	
11	1452		$\frac{457-265}{13-7}$ = 32		$\frac{1-1}{17-5}$ = **0**
		$\frac{2366-1452}{13-11}$ = 457		$\frac{42-32}{17-7}$ = 1	
13	2366		$\frac{709-457}{17-11}$ = 42		
		$\frac{5202-2366}{17-13}$ = 709			
17	5202				

Here, we have,

$[x_0, x_1] = 121$

$[x_0, x_1, x_2] = 24$

$[x_0, x_1, x_2, x_3] = 1$

$[x_0, x_1, x_2, x_3, x_4] = 0$

Then using Newton's Gregory divided difference formula

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + (x - x_0)(x - x_1)(x - x_2)(x - x_3)[x_0, x_1, x_2, x_3, x_4]$$

Then at x = 9

$$y = 150 + (9-5)(121) + (9-5)(9-7)(24) + (9-5)(9-7)(9-11)(1) + 0$$

$$= 150 + 484 + 192 - 16$$

$\therefore$ $y = 810$

Hence, the value of $f(9)$ is 810.

Example 2.6

Using Newton's divided differences formula, evaluate f(8) and f(15) given:

Given the values:

x	4	5	7	10	11	13
y = f(x)	48	100	294	900	1210	2028

Solution:

Creating divided difference table

x	$f(x_n)$	$f(x_n, x_{n+1})$	$f(x_n, x_{n+1}, x_{n+2})$	$f(x_n, x_{n+1}, x_{n+2}, x_{n+3})$	$f(x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4})$	$f(x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5})$
4	48					
		$\frac{100-48}{5-4} = 52$				
5	100		$\frac{97-52}{7-4} = 15$			
		$\frac{294-100}{7-5} = 97$		$\frac{21-15}{10-4} = 1$		
7	294		$\frac{220-97}{10-5} = 21$		$\frac{1-1}{11-4} = 0$	
		$\frac{900-294}{10-7} = 202$		$\frac{27-21}{11-5} = 1$		$\frac{0-0}{13-4} = 0$
10	900		$\frac{310-202}{11-7} = 27$		$\frac{1-1}{13-5} = 0$	
		$\frac{1210-900}{11-10} = 310$		$\frac{33-27}{13-7} = 1$		
11	1210		$\frac{409-310}{13-10} = 33$			
		$\frac{2028-1210}{13-11} = 409$				
13	2028					

Here, we have,

$[x_0, x_1] = 52$

$[x_0, x_1, x_2] = 15$

$[x_0, x_1, x_2, x_3] = 1$

$[x_0, x_1, x_2, x_3, x_4] = [x_0, x_1, x_2, x_3, x_4, x_5] = 0$

Scanned with CamScanner

Using Newton's divided difference formula,

$$f(8) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + 0 + 0$$

$$= 48 + (8 - 4)(52) + (8 - 4)(8 - 5)(15) + (8 - 4)(8 - 5)(8 - 7)(1) + 0 + 0$$

$\therefore \quad y = 448$

Similarly for $x = 15$,

$$f(15) = 48 + (15 - 4)(52) + (15 - 4)(15 - 5)(15) + (15 - 4)(15 - 5)(15 - 7)(1) + 0 + 0$$

$$= 48 + 572 + 1650 + 880$$

$\therefore \quad f(15) = 3150$

Example 2.7

Using Newton's divided difference formula, find the missing value form the table:

x	1	2	4	5	6
y	14	15	5	...	9

Solution:

Creating dividend difference table

x	$y = f(x_n)$	$f(x_n, x_{n+1})$	$f(x_n, x_{n+1}, x_{n+2})$	$f(x_n, x_{n+1}, x_{n+2}, x_{n+3})$
1	14			
		$\frac{15-14}{2-1} = 1$		
2	15		$\frac{-5-1}{4-1} = -2$	
		$\frac{5-15}{4-2} = -5$		$\frac{\frac{7}{4}+2}{6-1} = \frac{3}{4}$
4	5		$\frac{2+5}{6-2} = \frac{7}{4}$	
		$\frac{9-5}{6-4} = 2$		
6	9			

Here, we have,

$[x_0, x_1] = 1$

$[x_0, x_1, x_2] = -2$

$[x_0, x_1, x_2, x_3] = \frac{3}{4}$

Now, using Newton's divided difference formula,

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3]$$

Scanned with CamScanner

Then at x = 5

$$y = 14 + (5-1)(1) + (5-1)(5-2)(-2) + (5-1)(5-2)(5-4)\left(\frac{3}{4}\right)$$
$$= 14 + 4 - 24 + 9$$
$$\therefore \quad y = 3$$

Hence, the missing value at x = 5 is 3.

2.3 NEWTON'S FORWARD INTERPOLATION FORMULA

Interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable while the process of computing the value of the function outside the given range is called extrapolation.

Let the function $y = f(x)$ takes the values $y_0, y_1, \ldots, y_n$ correspond to $x_0, x_1, \ldots, x_n$ of x. Let these values of x be equispaced such that $x_i = x_0 + ih$ $(i = 0, 1, \ldots)$.

Assuming $y(x)$ to be a polynomial of the n^{th} degree in x such that $y(x_0) = y_0$, $y(x_1) = y_1, \ldots, y(x_n) = y_n$. We can write,

$$y(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \ldots + a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1}) \quad \ldots (1)$$

Putting, $x = x_0, x_1, \ldots, x_n$ successively in (1), we get,

$$y_0 = a_0,\ y_1 = a_0 + a_1(x_1 - x_0),\ y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

and so on.

From these, we find that $a_0 = y_0$, $\Delta y_0 = y_1 - y_0 = a_1(x_1 - x_0) = a_1 h$

$$\therefore \quad a_1 = \frac{1}{h}\Delta y_0$$

Also, $\Delta y_1 = y_2 - y_1 = a_1(x_2 - x_1) + a_2(x_2 - x_0)(x_2 - x_1)$

$$= a_1 h + a_2 hh = \Delta y_0 + 2h^2 a_2$$

$$\therefore \quad a_2 = \frac{1}{2h^2}(\Delta y_1 - \Delta y_0) = \frac{1}{2!h^2}\Delta^2 y_0$$

Similarly $a_3 = \frac{1}{3!h^3}\Delta^3 y_0$ and so on.

Replacing values in (1), we get,

$$y(x) = y_0 + \frac{\Delta y_0}{h}(x-x_0) + \frac{\Delta^2 y_0}{2!h^2}(x-x_0)(x-x_1) + \frac{\Delta^3 y_0}{3!h^3}(x-x_0)(x-x_1)(x-x_2) + \ldots \quad \ldots (2)$$

Now, if it is required to evaluate y for $x = x_0 + ph$, then,

$$(x - x_0) = ph,\ x - x_1 = x - x_0 - (x - x_0) = ph - h = (p-1)h$$
$$(x - x_0) = x - x_0 - (x - x_0) = (p-1)h - h = (p-2)h \text{ etc}$$

Scanned with CamScanner

Hence, $y(x) = y(x_0 + ph) = y_p$, then, (2) becomes,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+ \dots\dots + \frac{p(p-1)\dots\dots(p-\overline{n-1})}{n!}\Delta^n y_0 \quad \text{..... (3)}$$

It is called Newton's forward interpolation formula as (3) contains y_0 and the forward differences of y_0.

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+ \dots\dots + \frac{p(p-1)\dots\dots(p-\overline{n-1})}{n!}\Delta^n y_0$$

NOTE:

1. This formula is used for interpolating the values of a set of tabulated values and extrapolating values of y a little backward. (*i.e.*, to the left) of y_0.
2. The first two terms of this formula give the linear interpolation which the first three terms give a parabolic interpolation and so on.

2.4 NEWTON'S BACKWARD INTERPOLATION FORMULA

Le the function $y = f(x)$ take the values $y_0, y_1, y_2, \dots\dots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \dots\dots$ of x. Suppose it is required to evaluate f(x) for $x = x^n + ph$, where p is any real number. Then we have,

$$y_p = f(x_n + ph) = Ep\, f(x_n) = (1 - \nabla)^{-p} y_n \qquad [\because E^{-1} = 1 - \nabla]$$

$$= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \dots\dots\right] y_n$$

[using Binomial theorem]

i.e., $$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots\dots \quad \text{..... (1)}$$

It is called the Newton's backward interpolation formula as (1) contains y_n and backward differences of y_n. This formula is used for interpolating the values of y near the end of a set of tabulted values and also for extrapolating values of y a little ahead (to the right) of y_n.

Example 2.8

Using Newton's backward difference formula, construct an interpolation polynomial of degree 3 for the data:

$f(-0.75) = -0.0718125$, $f(-0.5) = -0.02475$, $f(-0.25) = 0.3349375$,

$f(0) = 1.10100$. Hence find $f\left(-\frac{1}{3}\right)$.

Solution:

Creating the difference table from the given datas;

Scanned with CamScanner

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
-0.75	-0.0718125			
		0.0470625		
-0.5	-0.02475		0.312625	
		0.3596875		0.09375
-0.25	0.3349375		0.406375	
		0.7660625		
0	1.10100			

Now, using Newton's backward difference formula

$$y(x) = y_3 + p\nabla y_3 + \frac{p(p+1)}{2!}\nabla^2 y_3 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_3$$

Taking $x_3 = 0$, $p = \frac{x-0}{h} = \frac{x}{0.25} = 4x$

$$y(x) = 1.10100 + 4x(0.7660625) + \frac{4x(4x+1)}{2}(0.406375)$$

$$+ \frac{4x(4x+1)(4x+2)}{6}(0.09375)$$

$$= 1.101 + 3.06425x + 3.251x^2 + 0.81275x + x^3 + 0.75x^2 + 0.125x$$

$\therefore$ $y = x^3 + 4.001x^2 + 4.002x + 1.101$

is the required interpolating polynomial.

At $x = \left(-\frac{1}{3}\right)$,

$$y\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 + 4.001\left(-\frac{1}{3}\right)^2 + 4.002\left(-\frac{1}{3}\right) + 1.101$$

$$= 0.174518$$

Example 2.9

Using Newton's forward formula, find the value of f(1.6) if

x	1	1.4	1.8	2.2
f(x)	3.49	4.82	5.96	6.5

Solution:

Creating difference table

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
1	3.49			
		1.33		
1.4	4.82		-0.19	
		1.14		-0.41
1.8	5.96		-0.6	
		0.54		
2.2	6.5			

We have,

$x = 1.6$, $x_0 = 1$, $h = 1.4 - 1 = 0.4$

$x = x_0 + ph$

Scanned with CamScanner

or, $p = \frac{1.6 - 1}{0.4} = 1.5$

Now, using Newton's forward formula

$$y_{1.6} = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$= 3.49 + 1.5(1.33) + \frac{1.5(1.5-1)}{2}(-0.19)$$

$$+ \frac{1.5(1.5-1)(1.5-2)}{6}(-0.41)$$

$$= 3.49 + 1.995 - 0.07125 + 0.025625$$

$\therefore$ $y_{1.6} = 5.439375$

Hence the required value of f(1.6) is 5.439375.

Example 2.10

Given, sin 45° = 0.7071, sin 50° = 0.7660, sin 55° = 0.8192 sin 60° = 0.8660, find sin 52° using Newton's forward formula.

Solution:

Creating difference table from the given data,

$x = \theta$	$y = \sin\theta$	Δy	$\Delta^2 y$	$\Delta^3 y$
45°	0.7071			
		0.0589		
50°	0.7660		−0.0057	
		0.0532		−0.0007
55°	0.8192		−0.0064	
		0.0468		
60°	0.8660			

We have,

$x = 52,\quad x_0 = 45,\quad h = 50 - 45 = 5$

$x = x_0 + ph$

or, $p = \frac{52 - 45}{5} = \frac{7}{5}$

Now, using Newton's forward formula

$$y_{52} = y_{45} + p\Delta y_{45} + \frac{p(p-1)}{2!}\Delta^2 y_{45} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{45}$$

$$= 0.7071 + \frac{7}{5}(0.0589) + \frac{\frac{7}{5}\left(\frac{7}{5}-1\right)}{2}(-0.0057)$$

$$+ \frac{\frac{7}{5}\left(\frac{7}{5}-1\right)\left(\frac{7}{5}-2\right)(-0.0007)}{6}$$

$$= 0.7071 + 0.08246 - 0.001596 + 0.0000392$$

$\therefore$ $y_{45} = 0.7880032$

Hence the required value of sin 52° is 0.7880032.

Example 2.11

Apply Newton's backward difference formula to the data below, to obtain a polynomial of degree 4 in x:

x	1	2	3	4	5
y	1	−1	1	−1	1

Solution:

Creating difference table

x	y	∇y	∇²y	∇³y	∇⁴y
1	1				
		-2			
2	-1		4		
		2		-8	
3	1		-4		16
		-2		8	
4	-1		4		
		2			
5	1				

We have,

$x_4 = 5, \quad h = 5 - 4 = 1$

$x = x_4 + ph$

or, $p = \frac{x - x_4}{h} = \frac{x-5}{1} = x - 5$

Now, using Newton's backward difference formula,

$$y(x) = y_4 + p\nabla y_4 + \frac{p(p+1)}{2!}\nabla^2 y_4 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_4 + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_4$$

$$= 1 + (x-5)(2) + \frac{4}{2}(x-5)(x-5+1) + \frac{8}{6}(x-5)(x-5+1)(x-5+2) + \frac{16}{24}(x-5)(x-5+1)(x-5+2)(x-5+3)$$

$$= 1 + 2x - 10 + (2x - 10)(x-4) + \frac{4}{3}(x-5)(x-4)(x-3) + \frac{2}{3}(x-5)(x-4)(x-3)(x-2)$$

$$= 1 + 2x - 10 + 2x^2 - 18x + 40 + 1.33x^3 - 16x^2 + 62.667 - 80 + 0.667x^4 - 9.333x^3 + 47.33x^2 - 102.667x + 80$$

$\therefore \quad y(x) = 0.67x^4 - 8.003x^3 + 33.33x^2 - 56x + 31$

is the required polynomial.

Scanned with CamScanner

2.5 LINEAR INTERPOLATION

The simplest form of interpolation is to approximate two data points by a straight line. Suppose we are given two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. These two points can be connected linearly as shown in figure 2.1.

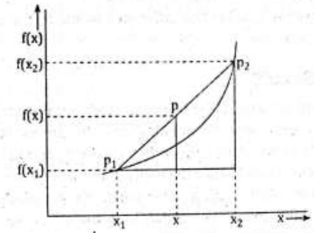


Figure 2.1: Graphical representation of linear interpolation

Using the concept of similar triangles, we can show that,

$$\frac{f(x) - f(x_1)}{x - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

On solving, we get,

$$f(x) = f(x_1) + (x - x_1)\frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{..... (1)}$$

Equation (1) is called as linear interpolation formula. Note that the term, $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ represents the slope of the line. Further, note the similarity of equation (1) with the Newton form of polynomial of first order.

$C_1 = x_1$

$a_0 = f(x_1)$

$a_1 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

The coefficient a_1 represents the first derivative of the function.

Example 2.12

The table below gives square roots for integers. Determine the square root of 2.5.

x	1	2	3	4	5
f(x)	1	4.4142	1.7321	2	2.2361

Solution:

The given value of 2.5 lies between the points. Hence,

$x_1 = 2$; $f(x_1) = 1.4142$

$x_2 = 3$; $f(x_2) = 1.7321$

Scanned with CamScanner

Then, $f(2.5) = 1.4142 + (2.5 - 2.0)\dfrac{1.7321 - 1.4142}{3.0 - 2.0}$

$$\left[\because f(x) = f(x_1) + (x_2 + x_1)\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right]$$

$$= 1.4142 + 0.5 \times 0.3179 = 1.5732$$

The correct answer is 1.5811. The difference is due to the use of a linear model to a non-linear one.

2.6 CUBIC SPLINES

The concept of splines originated from the mechanical drafting tool called "spline" used by designers for drawing smooth curves. It is a slender flexible bar made of wood or some other elastic materials. These curves resembles cubic curves and hence the name "cubic spline" has been given to the piecewise cubic interpolating polynomials. Cubic splines are popular because of their ability to interpolate data with smooth curves. It is believed that a cubic polynomial spline always appears smooth to the eyes.

In the interpolation methods so far explained, a single polynomial has been fitted to the tabulated points. If the given set of points belongs to the polynomial, then this method works well, otherwise the results are rough approximations only. If we draw lines through every two closest points, the resulting graph will not be smooth. Similarly, we may draw a quadratic curve through points A_i, A_{i+1} and another quadratic curve through A_{i+1}, A_{i+2} such that the slopes of the two quadratic curves match at A_{i+1}. The resulting curve looks better but is not quite smooth. We can ensure this by drawing a cubic curve through A_i, A_{i+1} and another cubic through A_{i+1}, A_{i+2} such that the slopes and curvatures of the two curves match at A_{i+1}. Such a curve is called a cubic spline. We may use polynomial of higher order but the resulting graph is not better. As such, cubic splines are commonly used. This technique of spline fitting is of recent origin and has important applications.

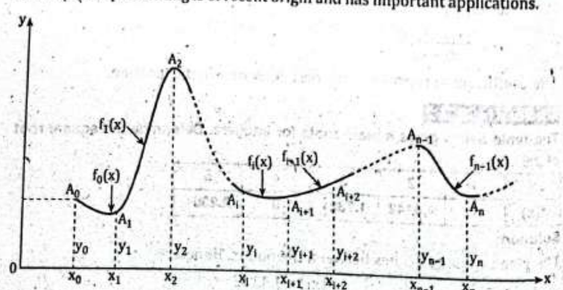


Figure 2.2

Scanned with CamScanner

Consider the problem of interpolating between the data points (x_0, y_0) (x_1, y_1), ..., (x_n, y_n) by means of spline fitting.

Then the cubic spline f(x) is such that,

i) f(x) is a linear polynomial outside the interval (x_0, x_n).

ii) f(x) is a cubic polynomial in each of the subintervals.

iii) f'(x) and f''(x) are continuous at each point.

Since f(x) is cubic in each of the subintervals f''(x) shall be linear.

$\therefore$ Taking equally spaced values of x so that $x_{i+1} - x_i = h$, we can write,

$$f''(x) = \frac{1}{h}[(x_{i+1} - x) f''(x_i) + (x - x_i) f''(x_{i+1})]$$

On integrating twice, we get,

$$f(x) = \frac{1}{h}\left[\frac{(x_{i+1} - x)}{3!} f''(x_i) + \frac{(x - x_i)}{3!} f''(x_{i+1})\right] a_i(x_{i+1} - x) + b_i(x - x_i) \quad ..... (1)$$

The constants of integration a_i, b_i are determined by substituting the values of y = f(x) at xi and x_{i+1}. Thus,

$$a_i = \frac{1}{h}\left[y_i - \frac{h^2}{3!} f''(x_i)\right]$$

$$bi = \frac{1}{h}\left[y_{i+1} - \frac{h^2}{3!} f''(x_{i+1})\right]$$

Replacing the values of a_i, b_i and writing $f''(x_i) = M_i$, (1) takes the form,

$$f(x) = \frac{(x_{i+1} - x)^3}{6h} M_i + \frac{(x - x_i)^3}{6h} M_{i+1} + \frac{x_{i+1} - x}{h}\left(y_i - \frac{h^2}{6} M_i\right)$$
$$+ \left(\frac{x - x_i}{h}\right)\left(y_{i+1} - \frac{h^2}{6} M_{i+1}\right) \quad ..... (2)$$

$$\therefore \quad f'(x) = -\frac{(x_{i+1} - x)^2}{2h} M_i + \frac{(x - x_i)^2}{6h} M_{i+1} + \frac{h}{6}(M_{i+1} - M_i) + \frac{1}{h}(y_{i+1} - y_i)$$

To impose the condition of continuity of f'(x), we get,

$$f'(x - \epsilon) = f'(x + \epsilon) \text{ as } \epsilon \to 0$$

$$\therefore \quad \frac{h}{6}(2M_i - M_{i-1}) + \frac{1}{h}(y_i - y_{i-1}) = -\frac{h}{6}(2M_i + M_{i+1}) + \frac{1}{h}(y_{i+1} - y_i)$$

or, $$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}),\ i = 1 \text{ to } n - 1 \quad ..... (3)$$

Now, since the graph is linear for $x < x_0$ and $x > x_n$, we have,

$$M_0 = 0,\ M_n = 0 \quad ..... (4)$$

Equation (3) and (4) gives (n + 1) equations in (n + 1) unknowns M_i(i = 0, 1, ..., n) which can be solved.

Replacing the value of M_i in (2) gives the concerned cubic spline.

Scanned with CamScanner

Example 2.13

Obtain the cubic spline for the following data:

x	0	1	2	3
y	2	–6	–8	2

Solution:

Since, the points are equispaced with h = 1 and n = 3, the cubic spline can be determined from

$$M_{i-1} + 4M_i + M_{i+1} = 6(y_{i-1} - 2y_i + y_{i+1}),\ i = 1, 2.$$

$$\therefore \quad M_0 + 4M_1 + M_2 = 6(y_0 - 2y_1 + y_2)$$

$$M_1 + 4M_2 + M_3 = 6(y_1 - 2y_2 + y_3)$$

i.e., $$4M_1 + M_2 = 36 \qquad [\because M_0 = 0, M_3 = 0]$$

$$M_1 + 4M_2 = 72$$

On solving, we get,

$$M_1 = 4.8 \text{ and } M_2 = 16.8$$

Now, the cubic spline in $(x_i \le x \le x_{i+1})$ is,

$$f(x) = \frac{1}{6}(x_{i+1} - x)^3 M_i + \frac{1}{6}(x - x_i)^3 M_{i+1} + (x_{i+1})\left(y_i - \frac{1}{6}M_i\right)$$

$$+ (x - x_i)\left(y_{i+1} - \frac{1}{6}M_{i+1}\right)$$

Taking i = 0,

$$f(x) = \frac{1}{6}(1 - x)^3 \times 0 + \frac{1}{6}(x - 0)^3 (4.8) + (1 - x)(x - 0) + x\left(-6 - \frac{1}{6}\right) \times 4.8$$

$$= 0.8x^3 - 8.8x + 2 \ (0 \le x \le 1)$$

Now taking i = 1, the cubic spline in $(1 \le x \le 2)$ is,

$$f(x) = \frac{1}{6}(2 - x)^3 \times (4.8) + \frac{1}{6}(x - 1)^3 (16.8) + (2 - x)$$

$$\left[-6 - \frac{1}{6} \times (4.8)\right] + (x - 1)[-8 - 1(16.8)]$$

$$= 2x^3 - 5.84x^2 - 16.8x + 0.8$$

Taking i = 2, the cubic spline in $(2 \le x \le 3)$ is,

$$f(x) = \frac{1}{6}(3 - x)^3 \times 4.8 + \frac{1}{6}(x - 2)^3 (0) + (3 - x)[-8 - 1(16.8)]$$

$$+ (x - 2)(2 - 1(2))]$$

$$= -0.8x^3 + 2.64x^2 + 9.68x - 14.8$$

2.7 CURVE FITTING: REGRESSION

In many applications, it often becomes necessary to establish a mathematical relationship between experimental values. This relationship may be used for either testing existing mathematical models or establishing

Scanned with CamScanner

new ones. The mathematical equation can also be used to predict or forecast values of the dependent variable. The process of establishing such relationships in the form of a mathematical equation is known as regression analysis or curve fitting.

Suppose the values of y for the different values of x are given. If we want to know the effect of x on y, then we may write a functional relationship $y = f(x)$.

The variable y is called the dependent variable and x the independent variable. The relationship may be either linear or non-linear as shown in figure 2.3. The type of relationship to be used should be decided by the experiment based on the nature of scatteredness of data.

It is a standard practice to prepare a scatter diagram as shown in figure 2.4 and try to determine the functional relationship needed to fit the points. The line should best fit the plotted points. This means that the average error introduced by the assumed line should be minimum. The parameters a and b of the various equations shown in figure 2.3 should be evaluated such that the equations best represent the data.

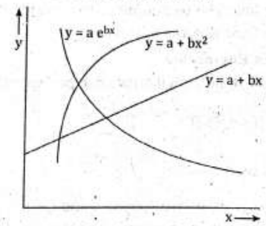


Figure 2.3: Various relationships between x and y

I. Fitting Linear Equations

Fitting a straight line is the simplest approach of regression analysis. Let us consider the mathematical equation for a straight line,

$$y = a + bx = f(x)$$

to describe the data. We know that 'a' is the intercept of the line and 'b' is its slope. Consider a point (x_i, y_i) as shown in figure 2.4. The vertical distance of this points from the line $f(x) = a + bx$ is the error q_i. Then,

$$q_i = y_i - f(x) = y_i - a - bx_i \quad \text{..... (1)}$$

There are various approaches that could be tried for fitting a 'best' line through the data. They include,

1. Minimize the sum of errors *i.e.,* minimize

$$\Sigma q_i = \Sigma y_i - a - bx_i \quad \text{..... (2)}$$

Scanned with CamScanner

2. Minimize the sum of absolute values of errors

$$\Sigma|q_i| = \Sigma|(y_i - a - bx)| \quad \text{..... (3)}$$

3. Minimize the sum of squares of errors

$$\Sigma q_i^2 = \Sigma(y_i - a - bx_i)^2 \quad \text{..... (4)}$$

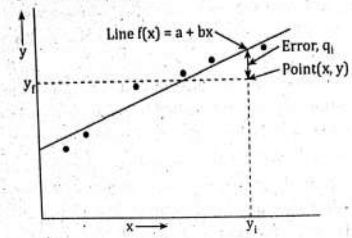


Figure 2.4: Scatter diagram

It can be easily verified that the first two strategies do not yield a unique line for a given set of data. The third strategy overcomes this problems and guarantees a unique line. The technique of minimizing the sum of squares of errors is known as least squares regression.

A. Least Squares Regression

Let the sum of squares of individual errors can be expressed as,

$$Q = \sum_{i=1}^{n} q_i^2 = \sum_{i=1}^{n} [(y_i - f(x_i)]^2$$

$$= \sum_{i=1}^{n} (y_i - a - bx_i)^2 \quad \text{..... (1)}$$

In the method of least squares, we choose a and b such that Q is minimum. Since Q depends on 'a' and 'b', a necessary condition for Q to be minimum is,

$$\frac{\partial Q}{\partial a} = 0 \text{ and } \frac{\partial Q}{\partial b} = 0$$

Then, $\frac{\partial Q}{\partial a} = -2 \sum_{i=1}^{n} (y_i - a - bx_i) = 0$

$$\frac{\partial Q}{\partial b} = -2 \sum_{i=1}^{n} x_i(y_i - a - bx_i) = 0 \quad \text{..... (2)}$$

Thus, $\Sigma y_i = na + b\Sigma x_i$

$$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 \quad \text{..... (3)}$$

These are called normal equations.

Solving for a and b, we get

$$b = \frac{n\Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2}$$

Scanned with CamScanner

$$a = \frac{\Sigma y_i}{n} - b\frac{\Sigma x_i}{n} = \bar{y} - b\bar{x}$$

when $\bar{x}$ and $\bar{y}$ are the averages of x and y values respectively.

Example 2.14

Fit a straight line to the following set of data:

x	1	2	3	4	5
y	3	4	5	6	8

Solution:

x_i	y_i	x_i^2	x_iy_i
1	3	1	3
2	4	4	8
3	5	9	15
4	6	16	24
5	8	25	40
Σ = 15	26	55	90

We know,

$$b = \frac{n\Sigma x_iy_i - \Sigma x_i\Sigma y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2}$$

Here, n = 5

so, $b = \frac{5 \times 90 - 15 \times 26}{5 \times 55 - 15^2} = 1.20$

Similarly,

$$a = \frac{\Sigma y_i}{n} - \frac{b\Sigma x_i}{n} = \frac{26}{5} - 1.20 \times \frac{15}{5} = 1.60$$

Hence, the linear equation is,

$y = a + bx = 1.60 + 1.20x$

Algorithm for Linear Regression

1. Start.
2. Read data values.
3. Compute sum of powers and products.
   $\Sigma x_i, \Sigma y_i, \Sigma x_i^2, \Sigma x_iy_i$
4. Check whether the denominator of the equation for b is zero.
5. Compute b and a.
6. Printout the equation.
7. Interpolate data, if required.
8. Stop.

B. Fitting Transcendental Equations

The relationship between the dependent and independent variables is not always linear (refer figure 2.5)

Scanned with CamScanner

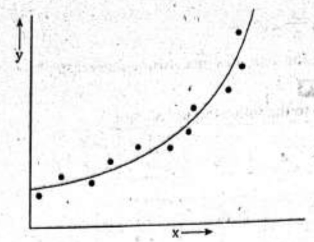


Figure 2.5: Data would fit a non-linear curve better than a linear one.

The non-linear relationship between them may exist in the form of transcendental equations (or higher order polynomials).

For example, the familiar equation for population growth is given by,

$$p = p_0 e^{kt} \quad \text{..... (1)}$$

where, p_0 is the initial population, k is the growth rate and t is the time.

Another example of non-linear model is the gas law relating to the pressure and volume as given by,

$$P = av_i^b \quad \text{..... (2)}$$

Let us consider equation (2) first. If we observe values of p for various values of v, we can then determine the parameters 'a' and 'b'. Using the method of least squares, the sum of squares of all errors can be written as,

$$Q = \sum_{i=1}^{n} [p_i - av_i^b]^2$$

To minimize Q, we have,

$$\frac{\partial Q}{\partial a} = 0 \text{ and } \frac{\partial Q}{\partial b} = 0$$

We can prove that;

$$\Sigma p_i v_i^b = a\Sigma(v_i^b)^2$$

$$\Sigma p_i v_i^b \ln v_i = a\Sigma(v_i^b)^2 \ln v_i$$

These equations can be solved for 'a' and 'b'. But since 'b' appears under the summation sign, an iterative technique must be employed to solve for 'a' and 'b'.

However, this problem can be solved by using the algorithm given in the previous section in the following ways: let us rewrite the equation using the conventional variables x and y as,

$$y = ax^b$$

If we take logarithm on both sides, we get,

$$\ln y = \ln a + b \ln x \quad \text{..... (3)}$$

Scanned with CamScanner

This equation is similar in the form to the linear equation and therefore, using the same procedure, we can evaluate the parameters a and b.

$$b = \frac{n\Sigma \ln x_i \ln y_i - \Sigma \ln x_i \Sigma \ln y_i}{n\Sigma(\ln x_i)^2 - (\Sigma \ln x_i)^2} \quad \text{..... (4)}$$

$$\ln a = R = \frac{1}{n}(\Sigma \ln y_i - b\Sigma \ln x_i)$$

$$a = e^R \quad \text{..... (5)}$$

Similarly, we can linearize the exponential model shown in equation (1) by taking logarithm on both the sides. This would yield,

$$\ln P = \ln p_0 + kt \ln e$$

Since, $\ln e = 1$

We have,

$$\ln P = \ln p_0 + kt \quad \text{..... (6)}$$

This is similar to the linear equation.

$$y = a + bx$$

where, $y = \ln P$

$a = \ln p_0$

$b = k$

$x = t$

We can now easily determine 'a' and 'b' and then p_0 and k.

There is a third form of non-linear model known as saturation growth rate equation as shown below;

$$p = \frac{k_1 t}{k_2 + t} \quad \text{..... (7)}$$

This can be linearized by taking inversion of the terms.

i.e., $$\frac{1}{p} = \left(\frac{k_2}{k_1}\right)\frac{1}{t} + \frac{1}{k_1} \quad \text{..... (8)}$$

This is again similar to the linear equation $y = a + bx$

where, $y = \frac{1}{p}$; $x = \frac{1}{t}$

$a = \frac{1}{k_1}$; $b = \frac{k_2}{k_1}$

Once we obtain 'a' and 'b', they could be transformed back into the original form for the purpose of analysis.

Example 2.15

Given the data table,

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

Fit a power-function model of the form.

$$y = ax^b$$

Scanned with CamScanner

Solution:

Given that;

$$y = ax^b$$

x_i	y_i	$ln(x_i)$	$ln(y_i)$	$(ln\, x_i)^2$	$(ln\, x_i)(ln\, y_i)$
1	0.5	0	−0.6931	0	0
2	2	0.6931	0.6931	0.4805	0.4804
3	4.5	1.0986	1.5041	1.2069	1.6524
4	8	1.3863	2.0794	1.9218	2.8827
5	12.5	1.6094	2.5257	2.5903	4.0649
Sum		4.7874	6.1092	6.1995	9.0804

We get,

$$b = \frac{n\Sigma\, ln\, x_i\, ln\, y_i - \Sigma\, ln\, x_i\, \Sigma\, ln\, y_i}{\Sigma(ln\, x_i)^2 - (\Sigma\, ln\, x_i)^2} = \frac{5 \times 9.0804 - (4.7874) \times (6.1092)}{(5)(6.1995) - (4.7874)^2}$$

$$= \frac{45.402 - 29.2472}{30.9975 - 22.9192} = 1.9998$$

$$ln\, a = \frac{1}{n}(\Sigma\, ln\, y_i - b\, \Sigma ln) = \frac{1}{5}(6.1092 - 1.9998 \times 4.7847) = -0.6929$$

$\therefore \quad a = e^{-0.6929} = 0.5001$

Thus, we obtain the power – Function as,

$$y = 0.5001\, x^{1.9998}$$

Note that the data have been derived from the equation,

$$y = \frac{x^2}{2}$$

The discrepancy in the computed coefficients is due to roundoff errors.

C. Fitting Polynomial Function

When a given set of data does not appear to satisfy a linear equations, we can try a suitable polynomial as a regression curve to fit the data. The least squares technique can be readily used to fit the data to a polynomial.

Consider a polynomial of degree m − 1,

$$y = a_1 + a_2x + a_3x^2 + \cdots\cdots + a_mx^{m-1} \quad \text{..... (1)}$$
$$= f(x)$$

If the data contains n sets of x and y values, then the sum of square of the errors is given by,

$$Q = \sum_{i=1}^{n} [(y_i - f(x_i)]^2 \quad \text{..... (2)}$$

Since f(x) is a polynomial and contains coefficients a_1, a_2, a_3, etc. We have to estimate all the m coefficients. As before, we have the following m equations that can be solved for these coefficients

$$\frac{\partial Q}{\partial a_1} = 0$$

$$\frac{\partial Q}{\partial a_2} = 0$$

....

....

....

$$\frac{\partial Q}{\partial a_m} = 0$$

Consider a general term,

$$\frac{\partial Q}{\partial a_j} = -2\sum_{i=1}^{n} [y_i - f(x_i)] \frac{\partial f(x_i)}{\partial a_j} = 0$$

$$\frac{\partial f(x_i)}{\partial a_j} = x_i^{j-1}$$

Thus, we have,

$$\sum_{i=1}^{n} [y_i - f(x_i)]\, x_i^{j-1} = 0, \qquad j = 1, 2, \ldots\ldots, m$$

$$\Sigma[y_i x_i^{j-1} - x_i^{j-1} f(x_i)] = 0$$

Replacing for $f(x_i)$,

$$\sum_{i=1}^{n} x_i^{j-1} (a_1 + a_2x_i + a_3x_i^2 + \ldots\ldots + a_m x_i^{m-1}) = \sum_{i=1}^{n} y_i\, x_i^{j-1}$$

These are m equations ($j = 1, 2, \ldots\ldots, m$) and each summation is for $i = 1$ to n.

$$a_1n + a_2\,\Sigma x_i + a_3\Sigma x_i^2 + \cdots + a_m\,\Sigma x_i^{m-1} = \Sigma y_i$$

$$a_1\Sigma x_i + a_2\Sigma x_i^2 + a_3\Sigma x_i^3 + \cdots + a_m\Sigma x_i^{m} = \Sigma y_i x_i \qquad \ldots (3)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_1\Sigma x_i^{m-1} + a_2\Sigma x_i^{m} + a_3\Sigma x_i^{m+1} + \cdots + a_m\Sigma x_i^{2m-2} = \Sigma y_i x_i^{m-1}$$

The set of m equations can be represented in matrix notation as follows:

$$CA = B$$

where,

$$C = \begin{bmatrix} n & \Sigma x_i & \Sigma x_i^2 & \cdots & \Sigma x_i^{m-1} \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 & \cdots & \Sigma x_i^{m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Sigma x_i^{m-1} & \Sigma x_i^{m} & \cdots & \cdots & \Sigma x_i^{2m-2} \end{bmatrix}$$

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_m \end{bmatrix}, \quad B = \begin{bmatrix} \Sigma y_i \\ \Sigma y_i x_i \\ \Sigma y_i x_2 \\ \vdots \\ \Sigma y_i x_i^{m-1} \end{bmatrix}$$

Scanned with CamScanner

The element of matrix C is,

$$C(j, k) = \sum_{i=1}^{n} x_i^{j+k-2} \quad ; \quad j = 1, 2, ..., m \text{ and } k = 1, 2, ..., m.$$

Similarly,

$$B(j) = \sum_{i=1}^{n} y_i x_i^{j-1} \quad ; \quad j = 1, 2, ..., m$$

Example 2.16

Fit a second order polynomial to the data in the table below;

x	1.0	2.0	3.0	4.0
y	6.0	11.0	18.0	27.0

Solution:

The order of polynomial is 2 and therefore we will have 3 simultaneous equations as shown below.

$$a_1 n + a_2 \Sigma x_i + a_3 \Sigma x_i^2 = \Sigma y_i$$

$$a_1 \Sigma x_i + a_2 \Sigma x_i^2 + a_3 \Sigma x_i^3 = \Sigma y_i x_i$$

$$a_1 \Sigma x_i^2 + a_2 \Sigma x_i^3 + a_3 \Sigma x_i^4 = \Sigma y_i x_i^2$$

The sums of power and products can be evaluated in a tabulator from as shown below;

x	y	x^2	x^3	x^4	yx	yx^2
1	6	1	1	1	6	6
2	11	4	8	16	22	44
3	18	9	27	81	54	162
4	27	16	64	256	108	432
Σ = 10	62	30	100	354	190	644

Replacing these values,

$$4a_1 + 10a_2 + 30a_3 = 62$$

op $$10a_1 + 30a_2 + 100a_3 = 190$$

$$30a_1 + 100a_2 + 354a_3 = 644$$

On solving, we get,

$$a_1 = 3$$

$$a_2 = 2$$

$$a_3 = 1$$

Hence the least squares quadratic polynomial is

$$y = 3 + 2x + x^2$$

Scanned with CamScanner

BOARD EXAMINATION SOLVED QUESTIONS

1. **Use appropriate method of interpolation to get sin θ at 45° from the given table:**

θ	10	20	30	40	50
sin θ	0.1736	0.3420	0.5000	0.6428	0.7660

Solution: **[2013/Fall]**

Here, the data of $\theta = x$ is equispaced and we have to get $\sin\theta$ at $\theta = x = 45°$ which is near the end of the provided table.

So, we use Newton's backward interpolation formula.

Now, creating difference table from given data

$x = \theta$	$y = \sin\theta$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
10	0.1736				
		0.1684			
20	0.3420		−0.0104		
		0.1580		−0.0048	
30	0.5000		−0.0152		0.0004
		0.1428		−0.0044	
40	0.6428		−0.0196		
		0.1232			
50	0.7660				

We have,

$x = 45, \quad h = 50 - 40 = 10, \quad x_n = 50$

Then,

$x = x_n + ph$

or, $45 = 50 + p10$

∴ $p = -0.5$

Now, using Newton's backward interpolation formula

$$y_p = y_4 + p\nabla y_4 + \frac{p(p+1)}{2!}\nabla^2 y_4 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_4 + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_4$$

$$= 0.7660 + (-0.5)(0.1232) + \frac{(-0.5)(-0.5+1)(-0.0196)}{2!} + \frac{(-0.5)(-0.5+1)(0.5+2)(-0.0044)}{3!} + \frac{(-0.5)(-0.5+1)(0.5+2)(-0.5+3)(0.0004)}{4!}$$

Scanned with CamScanner

$= 0.76660 - 0.0616 + 0.00245 + 0.000275 - 0.00001$

$\therefore \quad y_p = 0.7069$

Hence the value of $\sin\theta$ at $\theta = 45°$ is 0.7069.

2. From the following data:

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

Fit a power function model of the from $y = ax^b$. **[2013/Fall]**

Solution:

We have the function

$y = ax^b$

Taking $\log_{10}$ on both sides

$\log_{10} y = \log_{10} (ax^b)$

or, $\log_{10} y = \log_{10} a + b \log_{10} x$

Comparing with the equation,

$Y = A + bX$

where, $Y = \log_{10} y$

$A = \log_{10} a$

$X = \log_{10} x$

Forming normal equations as

$\Sigma Y = nA + b\Sigma X$ ...... (1)

$\Sigma XY = A\Sigma X + b\Sigma X^2$ ...... (2)

$n = 5$

x	y	$Y = \log_{10} y$	$X = \log_{10} x$	XY	X^2
1	0.5	−0.301	0	0	0
2	2	0.301	0.301	0.0906	0.0906
3	4.5	0.653	0.477	0.3114	0.2276
4	8	0.903	0.602	0.543	0.3624
5	12.5	1.096	0.698	0.765	0.4872
		$\Sigma Y = 2.652$	$\Sigma X = 2.078$	$\Sigma XY = 1.71$	$\Sigma X^2 = 1.168$

Now, equation (1) and (2), we get,

$2.625 = 5A + 2.078b$ ...... (a)

$1.71 = 2.078A + 1.168b$ ...... (b)

Solving equation (a) and (b), we get,

$A = -0.299$

or, $a = \text{anti} \log_{10} (-0.299) = 0.5$

and, $b = 1.996$

Hence, $y = 0.5\, x^{1.996}$ is the required function.

3. **If P is pull required to lift a load W by means of a pulley, find a linear law of the form P = mW + C using the following data:**

P	12	15	21	25
W	50	70	100	120

[2013/Spring]

Solution:

We have the function

$P = mW + C$ $\quad [\because Y = a + bX]$

Forming the normal equation

$\Sigma P = nC + m\Sigma W$ ..... (1)

$\Sigma WP = C\Sigma W + m\Sigma W^2$ ..... (2)

$n = 4$

W	P	WP	W^2
50	12	600	2500
70	15	1050	4900
100	21	2100	10000
120	25	3000	14400
$\Sigma Y = 340$	$\Sigma X = 73$	$\Sigma XY = 6750$	$\Sigma X^2 = 31800$

Substituting the obtained value from table to equation (1) and (2), we get,

$73 = 4\,C + 340\,m$ ..... (a)

$6750 = 340\,C + 31800\,m$ ..... (b)

On solving (a) and (b),

$C = 2.275$

$m = 0.187$

Hence a linear law is of the form P = 0.187 W + 2.275.

4. **Estimate the value of sin θ at θ = 25 using Newton-Gregory divided difference formula with the help of the following table:**

θ	10	20	30	40	50
sin θ	0.1736	0.3420	0.5	0.6428	0.7660

[2013/Spring]

Solution:

From Newton-Gregory divided difference formula for 5 data points, we have, Table

$x = \theta$	$y = f(x_n)$	$f(x_n, x_{n+1})$	$f(x_n, x_{n+1}, x_{n+2})$	$f(x_n, x_{n+1}, x_{n+2}, x_{n+3})$	$f(x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4})$
10	0.1736				
		$\frac{0.3420 - 0.1736}{20 - 10} = 0.0168$			
20	0.3420		$\frac{0.0158 - 0.0168}{30 - 10} = -0.05 \times 10^{-3}$		
		$\frac{0.5 - 0.3420}{30 - 20} = 0.0158$		$\frac{(-8\times10^{-5}) - (-0.05\times10^{-3})}{40 - 10} = -1 \times 10^{-6}$	
30	0.5		$\frac{0.0142 - 0.0158}{40 - 20} = -8 \times 10^{-5}$		$\frac{(-0.5\times10^{-6}) - (-1\times10^{-6})}{50 - 10} = 0.0125 \times 10^{-6}$
		$\frac{0.6428 - 0.5}{40 - 30} = 0.0142$		$\frac{(-9.5\times10^{-5}) - (-8\times10^{-5})}{50 - 20} = -0.5 \times 10^{-6}$	
40	0.6428		$\frac{0.0123 - 0.0142}{50 - 30} = -9.5 \times 10^{-5}$		
		$\frac{0.7660 - 0.6428}{50 - 40} = 0.0123$			
50	0.7660				

We have,

$[x_0, x_1] = 0.0168$

$[x_0, x_1, x_2] = -0.05 \times 10^{-3}$

$[x_0, x_1, x_2, x_3] = -1 \times 10^{-6}$

$[x_0, x_1, x_2, x_3, x_4] = 0.0125 \times 10^{-6}$

Then, using Newton's Gregory divided difference formula

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3, x_4]$$

$$= 0.1736 + (25 - 10)(0.0168) + (25 - 10)(25 - 20)(-0.05 \times 10^{-3}) + (25 - 10)(25 - 20)(25 - 30)(-1 \times 10^{-6}) + (25 - 10)(25 - 20)(25 - 30)(25 - 40)(0.0125 \times 10^{-6})$$

$$= 0.1736 + 0.252 - (3.75 \times 10^{-3}) + (3.75 \times 10^{-4}) + (7.031 \times 10^{-5})$$

$\therefore$ $y = 0.4222$

Hence, the value of $\sin \theta$ at $\theta = 25$ is 0.4222.

5. Find the missing term in the following table using suitable interpolation.

X	0	1	2	3	4
Y	1	3	9	?	81

[2014/Fall]

Solution:

To find the missing term from the given table, we use linear interpolation method at $x = 3$.

Here,

$x_1 = 2, \quad f(x_1) = 9$

$x_2 = 4, \quad f(x_2) = 81$

Now, from linear interpolation

$$y(x) = f(x_1) + (x - x_1)\frac{[f(x_2) - f(x_1)]}{x_2 - x_1}$$

$$= 9 + (3 - 2)\frac{(81 - 9)}{4 - 2} = 9 + 36 = 45$$

$\therefore \quad y = 45$

Hence, the required missing term is 45.

Next Method

Here, the provided data is unevenly spaced so, using Lagrange's interpolation formula:

$$y = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}y_3$$

We have,

$x_0 = 0 \quad y_0 = 1$

$x_1 = 1 \quad y_1 = 3$

$x_2 = 2 \quad y_2 = 9$

$x_3 = 4 \quad y_3 = 81$

Substituting the values

$$y = \frac{(x - 1)(x - 2)(x - 4)}{(0 - 1)(0 - 2)(0 - 4)}(1) + \frac{(x - 0)(x - 2)(x - 4)}{(1 - 0)(1 - 2)(1 - 4)}(3)$$

$$+ \frac{(x - 0)(x - 1)(x - 4)}{(2 - 0)(2 - 1)(2 - 4)}(9) + \frac{(x - 0)(x - 1)(x - 2)}{(4 - 0)(4 - 1)(4 - 2)}(81)$$

When $x = 3$, then

$$y = \frac{(3 - 1)(3 - 2)(3 - 4)}{-8} + 3(3 - 2)(3 - 4) + \frac{3(3 - 1)(3 - 4)}{-4}(9)$$

$$+ \frac{3(3 - 1)(3 - 2)}{24}(81)$$

$$= \frac{1}{4} - 3 + \frac{27}{2} + \frac{81}{4}$$

$$= 31$$

Hence the missing term for $x = 3$ is $y = 31$.

6. **The following table gives the heights, x(cm) and weights y(kg) of five persons.**

x	175	165	160	155	145
y	68	58	55	52	48

Assuming the linear relationship between x and y, obtain the regression line (x or y). Also obtain x value for y = 40. **[2014/Fall]**

Solution:

For linear relationship between x and y, we have,

$y = a + bx$

Forming the normal equation

$\Sigma Y = na + b\Sigma X$ ..... (1)

$\Sigma xy = a\Sigma x + b\Sigma x^2$ ..... (2)

$n = 5$

x	y	xy	x^2
175	68	11900	30625
165	58	9570	27225
160	55	8800	25600
155	52	8060	24025
145	48	6960	21025
$\Sigma x = 800$	$\Sigma Y = 281$	$\Sigma xy = 45290$	$\Sigma x^2 = 128500$

Substituting the values obtained in equation (1) and (2), we get,

$281 = 5a + 800b$ ..... (a)

$45290 = 800a + 128500b$ ..... (b)

Solving (a) and (b), we get,

$a = -49.4$

$b = 0.66$

We get,

$y = -49.4 + 0.66x$

which is the required linear regression equation.

Now, for y = 40

$40 = -49.4 + 0.66x$

$\therefore \quad x = 135.45.$

imp

7. **The following table gives the population of a town during the last six censuses. Estimate the increase in the population during the period from 1976 to 1978.**

Year	1941	1951	1961	1971	1981	1991
Population	12	15	20	27	39	52

[2014/Spring, 2015/Fall, 2015/Spring, 2016/Fall]

Solution:

Here, the data of given year is equispaced and we have to estimate data at 1976 and 1978 which is near the end of the table.

So, we use Newton's backward interpolation formula.

Now, creating difference table from given data.

Scanned with CamScanner

x = year	y = Population	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1941	12					
		3				
1951	15		2			
		5		0		
1961	20		2		3	
		7		3		-10
1971	27		5		-7	
		12		-4		
1981	39		1			
		13				
1991	52					

At $x = 1976$, $x_n = 1991$, $h = 1991 - 1981 = 10$

Then,

$$x = x_n + ph$$

or, $1976 = 1991 + 10p$

∴ $p = -1.5$

Now, using Newton's backward interpolation formula,

$$y_P = y_5 + p\nabla y_5 + \frac{p(p+1)}{2!}\nabla^2 y_5 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_5 + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_5 + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_5$$

$$= 52 + (-1.5)(13) + \frac{(-1.5)(-1.5+1)}{2!}(1) + \frac{(-1.5)(-1.5+1)(-1.5+2)}{3!}(-4) + \frac{(-1.5)(-1.5+1)(-1.5+2)(-1.5+3)}{4!}(-7) + \frac{(-1.5)(-1.5+1)(-1.5+2)(-1.5+3)(-1.5+4)}{5!}(-10)$$

$$= 52 - 19.5 + 0.375 - 0.25 - 0.1640 - 0.1171$$

∴ $y_P = 32.3439$

Again,

At $x = 1978$, $x_n = 1991$, $h = 1991 - 1981 = 10$

Then,

$$x = x_n + ph$$

or, $1978 = 1991 + 10p$

∴ $p = -1.3$

Then, using Newton's backward Interpolation formula and substituting the values, we get,

Scanned with CamScanner

$$y_p = 52 + (-1.3)(13) + \frac{(-1.3)(-1.3+1)}{2!}(1)$$
$$+ \frac{(-1.3)(-1.3+1)(-1.3+2)}{3!}(-4)$$
$$+ \frac{(-1.3)(-1.3+1)(-1.3+2)(-1.3+3)}{4!}(-7)$$
$$+ \frac{(-1.3)(-1.3+1)(-1.3+2)(-1.3+3)(-1.3+4)}{5!}(-10)$$
$$= 52 - 16.5 + 0.195 - 0.182 - 0.1353 - 0.1044$$

$\therefore \quad y_p = 34.8733$

Now, Increase in population during the period of 1976–1978 is given by

$$\Delta\text{Population} = y_{p \text{ at } 1978} - y_{p \text{ at } 1976}$$
$$= 34.8733 - 32.3439 = 2.5294$$

8. The pressure and volume of a gas are related by the equation $PV^{\gamma} = C$, where γ and C being constants. Fit this equation to the following set of observations.

P (kg/cm²)	0.5	1.0	1.5	2.0	2.5	3.0
V (literes)	1.62	1.00	0.75	0.62	0.52	0.46

[2016/Spring, 2015/Fall, 2014/Spring]

Solution:

Given equation $PV^{\gamma} = C$

Taking log on both sides,

$\log(PV^{\gamma}) = \log C$

or, $\log P + \log V^{\gamma} = \log C$

or, $\text{lop } P = \log C - \gamma \log V$

or, $Y = A - \gamma X$

where, $Y = \log P$

$A = \log C$

$X = \log V$

imp

Forming normal equations

$\Sigma Y = nA - \gamma \Sigma X$ .....(1)

$\Sigma XY = A\Sigma X - \gamma \Sigma X^2$ .....(2)

$n = 6$

P	V	Y = log P	X = log V	XY	X²
0.5	1.62	−0.301	0.209	−0.0629	0.0436
1.0	1.00	0	0	0	0
1.5	0.75	0.176	−0.124	−0.0218	0.0153
2.0	0.62	0.301	−0.207	−0.0623	0.0428
2.5	0.52	0.397	−0.283	−0.1123	0.0800
3.0	0.46	0.477	−0.337	−0.1607	0.1135
		ΣY = 1.050	ΣX = −0.742	ΣXY = −0.420	ΣX² = 0.2952

Scanned with CamScanner

Substituting obtained values in equation (1) and (2), we get,

$1.050 = 6A + 0.742\gamma$ ..... (a)

$-0.420 = -0.742A - 0.2952\gamma$ ..... (b)

Solving equation (a) and (b), we get,

$A = -0.00137$

$\gamma = 1.426$

and, $A = \log C$

or, $C = \text{antilog}_{10}(A) = 10^{-0.00137}$

$\therefore$ $C = 0.9968$

Hence, required equation is $PV^{1.426} = 0.9968$

9. For the following set of data, fit a parabolic curve using least square method and find f(2).

x	0.5	1	1.5	4.5	6.5	7.5
f(x)	2.5	2.7	3.5	6.5	8.4	9.5

[2015/Spring]

Solution:

We have, $y = a + bx + cx^2$ as a parabolic curve.

Forming normal equations as,

$\Sigma y = na + b\Sigma x + c\Sigma x^2$ ..... (1)

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$ ..... (2)

$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$ ..... (3)

$n = 6$

x	f(x) = y	x^2	x^3	x^4	xy	x^2y
0.5	2.5	0.25	0.125	0.0625	1.25	0.625
1	2.7	1	1	1	2.7	2.7
1.5	3.5	2.25	3.375	5.0625	5.25	7.875
4.5	6.5	20.25	91.125	410.06	29.25	131.625
6.5	8.4	42.25	274.62	1785	54.6	354.9
7.5	9.5	56.25	421.87	3164	71.25	534.375
Σx = 21.5	Σy = 33.1	Σx^2 = 122.25	Σx^3 = 792.11	Σx^4 = 5365.18	Σxy = 164.3	Σx^2y = 1032.1

Substituting the values obtained in equations (1) and (2) and (3), we get,

$33.1 = 6a + 21.5b + 122.25c$ ..... (a)

$164.3 = 21.5a + 122.25b + 792.11c$ ..... (b)

$1032.1 = 122.25a + 792.11b + 5365.18c$ ..... (c)

Solving equation (a), (b) and (c), we get,

$a = 1.8840$

$b = 1.0218$

$c = -0.0014$

Scanned with CamScanner

∴ Required parabolic curve is $y = 1.8840 + 1.0218x - 0.0014x^2$

Now, to find f(2),

$$f(2) = 1.8840 + 1.0218 \times 2 - 0.0014 \times (2)^2$$

∴ $f(2) = 3.922.$

10. Use Newton's divided difference formula to find f(3) form the following data:

x	0	1	2	4	5	6
f(x)	1	14	15	5	6	19

[2016/Spring]

Solution:

From Newton's divided difference formula for 6 data points.

Let, $y = f(x_n)$

$y_a = f(x_n, x_{n+1})$

$y_b = f(x_1, x_{n+1}, x_{n+2})$

$y_c = f(x_n, x_{n+1}, x_{n+2}, x_{n+3})$

$y_d = f(x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4})$

$y_e = f(x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5})$

Table:

x	y	y_a	y_b	y_c	y_d	y_e
0	1					
		$\frac{14-1}{1-0} = 13$				
1	14		$\frac{1-13}{2-0} = -6$			
		$\frac{15-14}{2-1} = 1$		$\frac{-2+6}{4-0} = 1$		
2	15		$\frac{-5-1}{4-1} = -2$		$\frac{1-1}{5-0} = 0$	
		$\frac{5-15}{4-2} = -5$		$\frac{2+2}{5-1} = 1$		$\frac{0-0}{6-0} = 0$
4	5		$\frac{1+5}{5-2} = 2$		$\frac{1-1}{6-1} = 0$	
		$\frac{6-5}{5-4} = 1$		$\frac{6-2}{6-2} = 1$		
5	6		$\frac{13-1}{6-4} = 6$			
		$\frac{19-6}{6-5} = 13$				
6	19					

We have,

$[x_0, x_1] = 13$

Scanned with CamScanner

$[x_0, x_1, x_2] = -6$

$[x_0, x_1, x_2, x_3] = 1$

$[x_0, x_1, x_2, x_3, x_4] = 0$

$[x_0, x_1, x_2, x_3, x_4, x_5] = 0$

Now, Newton's divided difference formula

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3]$$
$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)[x_0, x_1, x_2, x_3, x_4]$$
$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)[x_0, x_1, x_2, x_3, x_4, x_5]$$

At x = 3, we have,

$$y = 1 + (3 - 0)\,13 + (3 - 0)(3 - 1)(-6) + (3 - 0)(3 - 1)(3 - 2)(1) + 0 + 0$$
$$= 1 + 39 - 36 + 6$$

$\therefore$ $y = 10$

Hence, the value of $y = f(x)$ at $x = 3$ is 10.

11. By the method of least square methods, find the straight line that best fits the following data:

x	1	2	3	4	5
y	14	27	40	55	68

[2016/Spring]

Solution:

Using the function $y = a + bx$ to find straight line

Forming the normal equation,

$\Sigma y = na + b\Sigma x$ ..... (1)

$\Sigma xy = a\Sigma x + b\Sigma x^2$ ..... (2)

$n = 5$

x	y	xy	x^2
1	14	14	1
2	27	54	4
3	40	120	9
4	55	220	16
5	68	340	25
$\Sigma x = 15$	$\Sigma Y = 204$	$\Sigma xy = 748$	$\Sigma x^2 = 55$

Substituting the values obtained in equation (1) and (2), we get,

$204 = 5a + 15b$ ..... (a)

$748 = 15a + 55b$ ..... (b)

Solving (a) and (b), we get,

$a = 0$

$b = 13.6$

Hence, $y = 0 + 13.6x$ is the equation of best fit.

Scanned with CamScanner

12. The growth of bacteria (N) in a culture after t hours is given by the following table:

Time t(hr)	0	1	2	3	4
Bacteria (N)	32	47	65	92	132

If the relationship between bacteria N and time t is of the form $N = ab^t$. Using least square approximation estimate the N at t = 5 hr. **[2017/Spring]**

Solution:

Given that;

$N = ab^t$

Taking N = log on both sides

$\log_{10} N = \log_{10} (ab^t)$

or, $\log_{10} N = \log_{10} a + \log_{10} b^t$

or, $\log_{10} N = \log_{10} a + t \log_{10} b$

Comparing with the equation,

$Y = A + BX$

where, $Y = \log_{10} N$

$A = \log_{10} a$

$X = t$

$B = \log_{10} b$

Now, forming normal equations

$\Sigma Y = nA + B\Sigma X$ ..... (1)

$\Sigma XY = A\Sigma X + B\Sigma X^2$ ..... (2)

$n = 5$

t	N	$Y = \log_{10} N$	X = t	XY	X^2
0	32	1.505	0	0	0
1	47	1.672	1	1.672	1
2	65	1.812	2	3.624	4
3	92	1.963	3	5.889	9
4	132	2.120	4	8.48	16
		$\Sigma Y = 9.072$	$\Sigma X = 10$	$\Sigma XY = 19.665$	$\Sigma X^2 = 30$

Substituting the obtained values in equation (1) and (2), we get,

$9.072 = 5A + 10B$ ..... (a)

$19.665 + 10A + 30B$ ..... (b)

On solving (a) and (b), we get,

$A = 1.5102$

$B = 0.1521$

Then,

$A = \log_{10} a$

Scanned with CamScanner

or, $a = \text{anitlog}_{10}(A) = 10^{1.5102} = 32.374$

and, $B = \log_{10} b$

or, $b = 10^{0.1521} = 1.419$

So the relation between bacteria N and time t is

$N = 32.374 \times 1.419^t$

Now, at t = 5 hr,

$N = 32.374 \times 1.419^5$

∴ $N = 186.25$

13. **The following table given the percentage of criminals for different age groups. Using interpolation formula, find the percentage of criminals under the age of 35.**

Under age	25	30	40	50
% of criminals	52	67.3	84.1	94.4

[2017/Spring]

Solution:

The provided data in the table is unevenly spaced, so using Lagrange's interpolation formula.

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

We have,

$x_0 = 25 \qquad y_0 = 52$

$x_1 = 30 \qquad y_0 = 67.3$

$x_2 = 40 \qquad y_0 = 84.1$

$x_3 = 50 \qquad y_0 = 94.4$

When, x = 35, then,

$$y = \frac{(35-30)(35-40)(35-50)}{(25-30)(25-40)(25-50)}(52) + \frac{(35-25)(35-40)(35-50)}{(30-25)(30-40)(30-50)}(67.3) + \frac{(35-25)(35-30)(35-50)}{(40-25)(40-30)(40-50)}(84.1) + \frac{(35-25)(35-30)(35-40)}{(50-25)(50-30)(50-40)}(94.4)$$

$= -10.4 + 50.475 + 42.05 - 4.72$

∴ $y = 77.405$

Hence the percentage of criminal under age of 35 is 77.405%.

Scanned with CamScanner

14. Find the number of students securing marks between 50-55 using appropriate interpolation technique.

Marks obtained	20-30	30-40	40-50	50-60
No. of students	10	20	30	40

[2017/Fall]

Solution:

Total number of students = 10 + 20 + 30 + 40 = 100

Here, the data of marks obtained, x is equispaced at interval of 10 and 50-50 lies at near end of the provided table.

So, we use Newton's backward interpolation formula.

Now, creating difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
20	0				
		10			
30	10		10		
		20		0	
40	30		10		**0**
		30		**0**	
50	60		**10**		
		40			
60	**100**				

We have,

$x = 50 \quad x_n = 55, \quad h = 60 - 50 = 10$

Then,

$x = x_n + ph$

or, $50 = x_n + 10p$

$\therefore \quad p = -0.5$

Now, using Newton's backward interpolation formula

$$y_p = y_4 + p\nabla y_4 + \frac{p(p+1)}{2!}\nabla^2 y_4 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_4 + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_4$$

$$= 100 - 0.5 \times 40 + \frac{(-0.5)(-0.5+1)}{2!}(10) + 0 + 0$$

$$= 100 - 20 - 1.25$$

$\therefore \quad y_p = 78.75 \approx 79$

Hence, number of students securing marks between 50-55 are;

$= 79 - 60$

$= 19$ students

Scanned with CamScanner

15. **The voltage V across a capacitor at time t seconds is given by following table**

Time t(sec)	0	2	4	6	8	10
Voltage (V)	150	63	28	12	5.6	124

If the relationship between voltage V and time t is of the form $V = e^{kt}$. Using least-square approximation. Estimate the voltage at t = 2.6 sec.

[2017/Fall]

Solution:

Given that;

$V = e^{kt}$

Taking $\log_{10}$ on both sides,

$\log_{10} V = \log_{10} (e^{kt})$

or, $\log_{10} V = (kt) \log_{10} (e)$

or, $\log_{10} V = [k \log_{10} (e)]t$

Comparing with

$Y = A + BX$

where, $Y = \log_{10} V$

$B = k \log_{10} (e)$

$X = t$

$A = 0$

Now, forming normal equations,

$\Sigma Y = nA + B\Sigma X$ ..... (1)

$\Sigma XY = A\Sigma X + B\Sigma X^2$ ..... (2)

Since, A = 0, equations become

$\Sigma Y = B\Sigma X$ and $\Sigma XY = B\Sigma X^2$

n = 6

x = t	V	$Y = \log_{10} V$	XY	X^2
0	150	2.176	0	0
2	63	1.799	3.598	4
4	28	1.447	5.788	16
6	12	1.079	6.474	36
8	5.6	0.748	5.984	64
10	124	2.093	20.930	100
$\Sigma X = 30$		$\Sigma Y = 9.342$	$\Sigma XY = 42.774$	$\Sigma X^2 = 220$

Substituting the obtained values in equation (1) and (2), we get,

$9.342 = B30$

or, $B = \frac{9.342}{30} = 0.3114$

$B = k \log_{10} (e)$

or, $k = \frac{B}{\log_{10}(e)} = \frac{0.3114}{\log_{10}(e)} = 0.717$

Hence, $V = e^{0.717t}$ is the required relation.

And, at t = 2.63 seconds

$V = e^{0.717 \times 2.6} = 6.45$ volts

16. Determine the constants a and b by the method of least squares such that $y = ae^{bx}$.

X	2	4	6	8	10
Y	4.077	11.084	30.128	81.897	222.62

[2018/Spring]

Solution:

Given that;

$y = ae^{bx}$

Taking log on both sides,

$\log_{10} y = \log_{10}(ae^{bx})$

or, $\log_{10} y = \log_{10}(a) + \log_{10}(e^{bx})$

or, $\log_{10} y = \log_{10}(a) + bx \log_{10}(e)$

or, $\log_{10} y = \log_{10} a + (b \log_{10} e)\, x$

Comparing with

$Y = A + BX$

where, $Y = \log_{10} y$

$A = \log_{10} a$

$B = b \log_{10} e$

Now, forming normal equations,

$\Sigma Y = nA + B\Sigma X$ ..... (1)

$\Sigma XY = A\Sigma X + B\Sigma X^2$ ..... (2)

n = 5

X	y	$Y = \log_{10}(y)$	XY	X^2
2	4.077	0.610	1.22	4
4	11.084	1.044	4.176	16
6	30.128	1.478	8.868	36
8	81.897	1.913	15.304	64
10	222.62	2.347	23.47	100
$\Sigma X = 30$		$\Sigma Y = 7.392$	$\Sigma XY = 53.038$	$\Sigma X^2 = 220$

Substituting the obtained values in equation,

$7.392 = 5A + 30B$ ..... (a)

$53.038 = 30A + 220B$ ..... (b)

On solving (a) and (b), we get,

$A = 0.175$

$B = 0.217$

Scanned with CamScanner

Now,

$A = \log_{10}(a)$

$\therefore \quad a = \text{anitlog}_{10}(A) = 10^{0.175} = 1.496$

and, $\quad B = b \log_{10}(e)$

$\therefore \quad b = \frac{B}{\log_{10}(e)} = \frac{0.217}{\log_{10}(e)} = 0.499$

Hence, the required relation is $y = 1.496\, e^{0.499x}$.

17. From the following, find the number of students who obtained less than 45 marks.

Marks	30-40	40-50	50-60	60-70	70-80
No. of students	31	42	51	35	31

[2018/Spring, 2019/Spring]

Solution:

First we prepare the cumulative frequency table

Marks (x)	40	50	60	70	80
Students (y)	31	73	124	159	190

Now, creating the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		−25	
60	124		−16		37
		35		12	
70	159		−4		
		31			
80	190				

To find the number of students with marks less than 45.

Taking, $x = 45, \quad x_0 = 40, \quad h = 50 - 40 = 10$

$x = x_0 + ph$

$\therefore \quad p = 0.5$

Now using Newton's forward interpolation formula,

$$y_{45} = y_3 + p\nabla y_0 + \frac{p(p-1)}{2!}\nabla^2 y_{40} + \frac{p(p-1)(p-2)}{3!}\nabla^3 y_{40} + \frac{p(p-1)(p-2)(p-3)}{4!}\nabla^4 y_{40}$$

$$= 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2} \times 9 + \frac{0.5(-0.5)(-1.5)}{6}(-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24}(37)$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4453$$

$\therefore \quad y_{45} = 47.87$

Here, the number of students with marks less than 40 is 31 and the number of students with marks less than 45 is 47.87 = 48.

Also, students securing marks in between 40 and 45 = 48 − 31 = 17.

18. Generate a Lagrange's interpolating polynomial for the function $y = \cos \pi x$, taking the pivotal points $0, \frac{1}{4}$ and $\frac{1}{2}$. **[2018/Fall]**

Solution:

Given that;

$x_0 = 0$; $y_0 = \cos \pi x_0 = 1$

$x_1 = \frac{1}{4} = 0.25$; $y_1 = \cos \pi x_1 = 0.707$

$x_2 = \frac{1}{2} = 0.5$; $y_2 = \cos \pi x_2 = 0$

Now, using Lagrange's interpolation formula

$$y = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

Substituting the values

$$y = \frac{(x - 0.25)(x - 0.5)}{(0 - 0.25)(0 - 0.5)} (1) + \frac{(x - 0)(x - 0.5)}{(0.25 - 0)(0.25 - 0.5)} (0.707) + \frac{(x - 0)(x - 0.25)}{(0.5 - 0)(0.5 - 0.25)} (0)$$

$$= \frac{(x - 0.25)(x - 0.5)}{\left(\frac{1}{8}\right)} + \frac{(x - 0)(x - 0.5)}{-0.0625} (0.707) + (0)$$

$$= 8 (x - 0.25) (x - 0.5) - 11.312x (x - 0.5)$$

$$= (x - 0.5) (8x - 2 - 11.312x)$$

$$= x(-3.312x - 2) - 0.5 (-3.312x - 2)$$

$$= -3.312x^2 - 0.344x + 1$$

is the required Lagrange's interpolating polynomial for the given function.

19. The voltage V across a capacitor at a time T seconds is given by the following table. Use the principle of least squares to fit the curve of the form. $V = \alpha e^{\beta T}$ to the data.

T	0	2	4	6	8
V	150	63	28	12	5.6

[2013/Fall, 2019/Spring]

Solution:

Given that;

$V = a e^{\beta T}$

Taking log on both sides

$\log_{10} V = \log_{10} (a e^{\beta T})$

or, $\log_{10} V = \log_{10} a + \beta T \log_{10} e$

Scanned with CamScanner

Comparing with

$Y = A + BX$

where, $Y = \log_{10} V$

$A = \log_{10} \alpha$

$X = T$

$B = \beta \log_{10} e$

Forming normal equations

$\Sigma Y = nA + B\Sigma X$ ..... (1)

and, $\Sigma XY = A\Sigma X + B\Sigma X^2$ ..... (2)

$n = 5$

X = T	V	Y = $\log_{10}$ V	XY	X^2
0	150	2.176	0	0
2	63	1.799	3.598	4
4	28	1.447	5.788	16
6	12	1.079	6.474	36
8	5.6	0.748	5.984	64
$\Sigma X = 20$		$\Sigma Y = 7.249$	$\Sigma XY = 21.844$	$\Sigma X^2 = 120$

Substituting the obtained values,

$7.249 = 5A + 20B$ ..... (a)

$21.844 = 20A + 120B$ ..... (b)

On solving (a) and (b), we get,

$A = 2.165$

$B = -0.178$

Then,

$A = \log_{10} \alpha$

or, $\alpha = \text{anitlog}_{10}(A) = 10^{2.165} = 146.217$

and, $B = \beta \log_{10} e$

or, $\beta = \frac{B}{\log_{10} e} = \frac{-0.178}{\log_{10}(e)} = -0.409$

Hence, the required curve is $V = 146.217\ e^{-0.409T}$.

20. Fit a curve of the form: $y = \frac{1}{a + bx}$ by using the method of least square with the following data points.

x	1	2	3	4	5
f(x)	3.33	2.20	1.52	1.00	0.91

[2018/Fall]

Solution:

Given that;

$y = \frac{1}{a + bx}$

Scanned with CamScanner

or, $\frac{1}{y} = a + bx$

Comparing with $Y = A + BX$

$$Y = \frac{1}{y}, A = a, B = b, X = x$$

Forming normal equations

$$\Sigma X = nA + B\Sigma X \quad \text{..... (1)}$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2 \quad \text{..... (2)}$$

n = 5

x	f(x) = y	$Y = \frac{1}{y}$	XY	X^2
1	3.33	0.3	0.3	1
2	2.20	0.454	0.908	4
3	1.52	0.657	1.971	9
4	1.00	1	4	16
5	0.91	1.098	5.49	25
$\Sigma X = 15$		$\Sigma Y = 3.509$	$\Sigma XY = 12.669$	$\Sigma X^2 = 55$

Substituting the obtained values,

$$3.509 = 5A + 15B \quad \text{..... (a)}$$

$$12.669 = 15A + 55B \quad \text{..... (b)}$$

On solving (a) and (b), we get,

$$A = 0.0592$$

$$B = 0.2142$$

Then, $Y = 0.0592 + 0.2142x$

or, $\frac{1}{y} = 0.0592 + 0.2142x$

$$\therefore \quad y = \frac{1}{0.0592 + 0.2142x}$$

is the required curve of best fit.

21. The function y = f(x) is given at the points (7, 3), (8, 1) (9, 1) and (10, 9). Find the value of y = for x = 9.5 using Lagrange Interpolation formula. **[2019/Fall]**

Solution:

Given that;

$x_0 = 7$, $y_0 = 3$

$x_1 = 8$, $y_1 = 1$

$x_2 = 9$, $y_2 = 1$

$x_3 = 10$, $y_3 = 9$

Now, using Lagrange's interpolation formula

$$y = f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1 + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3$$

Scanned with CamScanner

At x = 9.5,

$$y = \frac{(9.5-8)(9.5-9)(9.5-10)(3)}{(7-8)(7-9)(7-10)} + \frac{(9.5-7)(9.5-9)(9.5-10)(1)}{(8-7)(8-9)(8-10)}$$

$$+ \frac{(9.5-7)(9.5-8)(9.5-10)(1)}{(9-7)(9-8)(9-10)} + \frac{(9.5-7)(9.5-8)(9.5-9)(9)}{(10-7)(10-8)(10-9)}$$

$$= 0.1875 - 0.3125 + 0.9375 + 2.8125$$

∴ y = 3.625 is the required answer.

22. The following table shows pressure and specific volume of dry saturated steam.

V	38.4	20	8.51	4.44	3.03
P	10	20	50	100	150

Fit a curve of the form: $PV^{\alpha} = \beta$ by using least square method. **[2019/Fall]**

Solution:

Given that;

$PV^{\alpha} = \beta$

Taking log on both sides

$\log_{10}(PV^{\alpha}) = \log_{10}\beta$

or, $\log_{10}P + \alpha\log_{10}V = \log_{10}\beta$

or, $\log_{10}P = \log_{10}\beta - \alpha\log_{10}V$

Comparing with

$Y = A + BX$

where, $Y = \log_{10}P$

$A = \log_{10}\beta$

$B = -\alpha$

$X = +\log_{10}V$

Now, forming normal equations

$\Sigma Y = nA + B\Sigma X$ ..... (1)

and, $\Sigma XY = A\Sigma X + B\Sigma X^2$ ..... (2)

n = 5

V	P	$Y = \log_{10}P$	$X = \log_{10}V$	XY	X^2
38.4	10	1	1.548	1.584	2.509
20	20	1.301	1.301	1.692	1.692
8.51	50	1.698	0.929	1.577	0.863
4.44	100	2	0.647	1.294	0.418
3.03	150	2.176	0.481	1.046	0.231
		$\Sigma Y = 8.175$	$\Sigma X = 4.942$	$\Sigma XY = 7.193$	$\Sigma X^2 = 5.713$

Substituting the obtained values,

$8.175 = 5A + 4.942B$ ..... (a)

$7.193 = 4.942A + 5.713B$ ..... (b)

Scanned with CamScanner

On solving (a) and (b), we get,

$A = 2.693$

$B = -1.071$

Then,

$A = \log_{10}(\beta)$

$\beta = \text{anitlog}_{10}(A) = 10^A = 10^{2.693} = 493.173$

and, $B = -\alpha$

or, $\alpha = 1.071$

Hence, $PV^{1.071} = 493.173$ is the required curve of best fit.

23. From following experimental data, it is known that the relation connects V and t as $V = at^b$. Find the possible values of a and b.

V	350	400	500	600
P	61	26	7	2.6

[2020/Fall]

Solution:

Given that;

$V = at^b$

Taking $\log_{10}$ on both sides,

$\log_{10} V = \log_{10}(at^b)$

or, $\log_{10} V = \log_{10} a + \log_{10} t^b$

or, $\log_{10} V = \log_{10} a + b \log_{10} t$

Comparing with

$Y = A + BX$

where, $Y = \log_{10} V$

$A = \log_{10} a$

$B = b$

$X = \log_{10} t$

Forming normal equations

$\Sigma Y = nA + B\Sigma X$ ..... (1)

and, $\Sigma XY = A\Sigma X + B\Sigma X^2$ ..... (2)

$n = 4$

V	T	$Y = \log_{10} V$	$X = \log_{10} t$	XY	X^2
350	61	2.544	1.785	4.541	3.186
400	26	2.602	1.414	3.679	1.999
500	7	2.698	0.845	2.279	0.714
600	2.6	2.778	0.414	1.150	0.171
		$\Sigma Y = 10.622$	$\Sigma X = 4.458$	$\Sigma XY = 11.649$	$\Sigma X^2 = 6.07$

Scanned with CamScanner

Substituting the obtained values

$10.622 = 4A + 4.458B$ ..... (a)

$11.649 = 4.458A + 6.07B$ ..... (b)

On solving (a) and (b), we get,

$A = 2.846$

$B = -0.171$

Then,

$A = \log_{10} a$

or, $a = \text{antilog}_{10}(A) = 10^{2.846} = 701.455$

and, $B = b = -0.171$

Hence, $V = 701.455t^{-0.171}$ is the required solution.

24. The following table gives the viscosity of oil as the function of temperature. Use Lagrange's Interpolation formula to find the viscosity of oil at a temperature of 140°C.

T(°C)	110	130	160	190
Viscosity	10.8	8.1	5.5	4.8

[2020/Fall]

Solution:

Given that;

$x_0 = 110$ $y_0 = 10.8$

$x_1 = 130$ $y_1 = 8.1$

$x_2 = 160$ $y_2 = 5.5$

$x_3 = 190$ $y_3 = 4.8$

Now, using Lagrange's interpolation formula

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

At $x = 140$,

$$y = \frac{(140-130)(140-160)(140-190)}{(110-130)(110-160)(110-190)}(10.8) + \frac{(140-110)(140-160)(140-190)}{(130-110)(130-160)(130-190)}(8.1) + \frac{(140-110)(140-130)(140-190)}{(160-110)(160-130)(160-190)}(5.5) + \frac{(140-110)(140-130)(140-160)}{(190-110)(190-130)(190-160)}(4.8)$$

$= -1.35 + 6.75 + 1.833 - 0.2$

$\therefore$ $y = 7.033$ is the required viscosity of oil.

Scanned with CamScanner

25. Write short notes on cubic spline.

[2013/Spring, 2017/Fall, 2018/Spring, 2019/Spring]

Solution: See the topic 2.6.

26. Write short notes on: An algorithm for Lagrange's interpolation polynomial. **[2014/Fall, 2018/Fall]**

Solution:

Algorithm for Lagrange's interpolation polynomial.

1.	Read x, n
2.	For i = 1 to (n + 1) in steps of 1 do read x_i, f_i end for
3.	Sum ← 0
4.	For i = 1 to (n + 1) in steps of 1 do
5.	Prodfunc ← 1
6.	For j = 1 to (n + 1) in steps of 1 do

7.	If (j + 1) then
	Prodfunc ← Prodfunc × $(x - x_j)/(x_i - x_j)$
	end for

8.	Sum ← sum + f_i × prodfunc Remarks: sum is the value of f at x end for
9.	Write x, sum s
10.	Stop.

27. Write short notes on: Linear interpolation. **[2015/Fall]**

Solution: See the topic 2.5.

28. Write short notes on: Numerical differentiation. **[2016/Fall]**

Solution:

It is the process of calculating the value of the derivative of a function at some assigned value of x from the given set of values (x_i, y_i). To compute $\frac{dy}{dx}$, we first replace the exact relation y = f(x) by the best interpolating polynomial $y = \phi(x)$ and then differentiate the latter as many times as we desire. The choice of the interpolation formula to be used, will depend on the assigned value of x at which $\frac{dy}{dx}$ is desired.

If the value of x are equi-spaced and is required near the beginning of the table, we employ Newton's forward formula. If it is required near the end of the table, $\frac{dy}{dx}$ is calculated by means of Stirlings or Bessel's formula. If the values of x are not equi-spaced, we use Lagrange's formula or Newton's divided difference formula to represent the function.

Scanned with CamScanner

Hence, corresponding to each of the interpolation formula, we can derive a formula for finding the derivative. While using this formula it must be observed that the table of values defines the function at these points only and does not completely define the function and the function may not be differentiable at all. As such, the process of numerical differentiation should be used only if the tabulated values are such that the differences of some order are constants. Otherwise, errors are bound to creep in which go on increasing as derivatives of higher order are found. This is due to the fact that the difference between f(x) and the approximating polynomial $\phi(x)$, may be small at the data points but $f'(x) - \phi'(x)$ may be large.

1. **Forward difference formulae**

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\cdots\right]$$

and so on.

2. Backward difference formula

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n - \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n - \frac{1}{4}\nabla^4 y_n + \cdots\cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n - \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n - \cdots\cdots\right]$$

and so on.

Scanned with CamScanner

ADDITIONAL QUESTION SOLUTION

1. **Estimate y(6.5) using natural cubic spline interpolation technique from the following data:**

x	3	5	7	9	11
y	8	10	9	12	5

Solution:

Since the points are equispaced with h = 2 and n = 4, the cubic spline can be determined from

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1})$$

Now, at i = 1

$$M_0 + 4M_1 + M_2 = \frac{6}{2^2}(y_0 - 2y_1 + y_2) = \frac{6}{4}[8 - 2(10) + 9] = -4.5$$

At i = 2

$$M_1 + 4M_2 + M_3 = \frac{6}{4}(y_1 - 2y_2 + y_3) = \frac{6}{4}[10 - 2(9) + 12] = 6$$

At i = 3

$$M_2 + 4M_3 + M_4 = \frac{6}{4}(y_2 - 2y_3 + y_4) = \frac{6}{4}[9 - 2(12) + 5] = -15$$

Since, $M_0 = 0$ and $M_4 = 0$

We have,

$4M_1 + M_2 = -4.5$

$M_1 + 4M_2 + M_3 = 6$

$M_2 + 4M_3 = -15$

Solving these equations, we get,

$M_1 = -1.9018$

$M_2 = 3.1071$

$M_3 = -4.5268$

Now the cubic spline in $(x_i \le x \le x_{i+1})$ is

$$f(x) = \frac{(x_{i+1} - x)^3}{6h} M_i + \frac{(x - x_i)^3}{6h} M_{i+1} + \frac{x_{i+1} - x}{h}\left(y_i - \frac{h^2}{6} M_i\right) + \frac{x - x_i}{h}\left(y_{i+1} - \frac{h^2}{6} M_{i+1}\right)$$

To estimate y (6.5),

Taking i = 1, then cubic spline in $(x_1 \le x \le x_2) = (5 \le x \le 7)$ is

$$y = \frac{(x_2 - x)^3}{12} M_1 + \frac{(x - x_1)^3}{12} M_2 + \frac{x_2 - x}{2}\left(y_1 - \frac{4}{6} M_1\right) + \frac{x - x_1}{2}\left(y_2 - \frac{4}{6} M_2\right)$$

Substituting the values at x = 6.5

$$y = \frac{(7 - 6.5)^3}{12}(-1.9018) + \frac{(6.5 - 5)^3}{12}(3.1071)$$

Scanned with CamScanner

$$+\frac{(7-6.5)}{2}\left(10-\frac{4}{6}(-1.9018)\right)+\frac{(6.5-5)}{2}\left(9-\frac{4}{6}(3.1071)\right)$$

$= -0.0198 + 0.8739 + 2.8170 + 5.1965$

$\therefore \quad y(6.5) = 8.8676$

2. Fit the curve $y = ax^b$ to the following data:

4	5	7	10	11	13
48	100	294	900	1210	2028

Solution:

We have the function

$y = ax^b$

Taking $\log_{10}$ on both sides

$\log_{10} y = \log_{10}(ax^b)$

or, $\log_{10} y = \log_{10} a + b \log_{10} x$

Comparing with the equation

$Y = A + BX$

where, $Y = \log_{10} y$

$A = \log_{10} a$

$X = \log_{10} x$

Forming normal equations

$\Sigma Y = nA + B\Sigma X$ ..... (1)

and, $\Sigma XY = A\Sigma X + B\Sigma X^2$ ..... (2)

$n = 6$

x	y	$Y = \log_{10} y$	$X = \log_{10} x$	XY	X^2
4	48	1.6812	0.6021	1.0123	0.3625
5	100	2.0	0.699	1.398	0.4886
7	294	2.4683	0.8451	2.0860	0.7142
10	900	2.9542	1	2.9542	1
11	1210	3.0828	1.0414	3.2104	1.0845
13	2028	3.3071	1.1139	3.6838	1.2408
		$\Sigma Y = 15.4936$	$\Sigma X = 5.3015$	$\Sigma XY = 14.3447$	$\Sigma X^2 = 4.8906$

Substituting the obtained values in (1) and (2), we get,

$15.4936 = 6A + 5.3015B$ ..... (a)

$14.3447 = 5.3015A + 4.8906b$ ..... (b)

On solving (a) and (b), we get,

$A = -0.2225$

$b = 3.1743$

and, $a = \text{antilog}_{10}(A) = 10^{-0.2225} = 0.5991$

Hence, $y = 0.5991x^{3.1743}$ is the required fit of the curve.

3. From the following data, compute: (a) y(3) using Newton's forward interpolation formula, (b) y(6.4) using Stirling's formula.

x	2	4	6	8	10	12
y	5.1	4.2	3.1	3.5	6.2	7.3

Scanned with CamScanner

Solution:

Creating difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2	5.1					
		-0.9				
4	4.2		-0.2			
		-1.1		1.7		
6	3.1		1.5		-0.9	
		0.4		0.8		-3.8
8	3.5		2.3		-4.7	
		2.7		-3.9		
10	6.2		-1.6			
		-1.1				
12	7.3					

a) y(3) using Newton's forward interpolation

We have,

$x = 3, \quad x_0 = 2, \quad h = 4 - 2 = 2$

$x = x_0 + ph$

or, $p = \frac{3-2}{2} = 0.5$

Now, using Newton's backward interpolation formula

$$y(3) = y_0 + p\nabla y_0 + \frac{p(p-1)}{2!}\nabla^2 y_0 + \frac{p(p-1)(p-2)}{3!}\nabla^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\nabla^4 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\nabla^5 y_0$$

$$= 5.1 + 0.5(-0.9) + \frac{0.5(0.5-1)}{2}(-0.2) + \frac{0.5(0.5-1)(0.5-2)}{6}(1.7) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24}(-0.9) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)(0.5-4)}{120}(-3.8)$$

$$= 5.1 - 0.45 + 0.025 + 0.1063 + 0.0352 - 0.1039$$

$\therefore \quad y(3) = 4.7126$

b) y(6.4) using Stirling's formula

$x = 6.4, x_0 = 6, h = 2$

$\therefore \quad p = 0.2$

Now, using Stirling's formula

$$y(6.4) = y_0 + \frac{p(\Delta y_{-1} + \Delta y_0)}{2} + \frac{p^2\Delta^2 y_{-1}}{2!} + \frac{p(p^2-1)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2}$$

Scanned with CamScanner

$$= 3.1 + 0.2 + \frac{(-1.1 + 0.4)}{2} + \frac{0.2^2}{}$$

$$+ \frac{0.2(0.2^2 - 1)}{6} \times \frac{(0.8 + 1.7)}{2}$$

$$= 3.1 - 0.07 + 0.03 - 0.04$$

$\therefore \quad y(6.4) = 3.02$

4. Using Stirling formula find U_{28}, given:

$U_{20} = 49225$, $U_{25} = 48316$, $U_{30} = 47236$, $U_{35} = 45926$, $U_{40} = 44306$

Solution:

Creating difference table from given data

	x	$y = U_n$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_{-1}	20	49225				
			-909			
x_0	25	**48316**		-171		
			-1080		-59	
x_1	30	47236		**-230**		-21
			-1310		**-80**	
x_2	35	45926		-310		
			-1620			
x_3	40	44306				

We have,

$x = 28, \quad x_0 = 25, \quad h = 5$

or, $\quad x = x_0 + ph$

$\therefore \quad p = 0.6$

Now, using Stirling's formula

$$U_{28} = y_0 + \frac{p(\Delta y_{-1} + \Delta y_0)}{2} + \frac{p^2 \Delta^2 y_{-1}}{2!}$$

$$= 48316 + \frac{0.6(-909 + 1080)}{2} + \frac{0.6^2(-171)}{2}$$

$$= 48316 - 596.7 - 30.78$$

$$= 47688.52$$

5. Fit the following data into $y = a + b\sqrt{x}$

x	500	1000	2000	4000	6000
y	0.2	0.33	0.38	0.45	0.51

Solution:

We have,

$y = a + b\sqrt{x}$

Comparing with the equation, $Y = a + bX$

$X = \sqrt{x}$

Scanned with CamScanner

Forming normal equations as

$\Sigma Y = na + b\Sigma X$ ..... (1)

$\Sigma XY = a\Sigma X + b\Sigma X^2$ ..... (2)

n = 5

x	Y	$X = \sqrt{x}$	XY	X^2
500	0.2	22.36	4.472	500
1000	0.33	31.62	10.434	1000
2000	0.38	44.72	16.993	2000
4000	0.45	63.24	28.458	4000
6000	0.51	77.45	39.499	6000
	$\Sigma Y = 1.87$	$\Sigma X = 239.39$	$\Sigma XY = 99.856$	$\Sigma X^2 = 13500$

Substituting the obtained values (1) and (2), we get,

$1.87 = 5a + 239.39b$ ..... (a)

$99.856 = 239.39a + 13500b$ ..... (b)

On solving (a) and (b), we get,

a = 0.1315

b = 0.0051

Hence the required fit of the curve is $y = 0.1315 + 0.0051\sqrt{x}$.

6. Find at x = 8 from the following data using natural cubic spline interpolation.

x	3	5	7	9
y	3	2	3	1

Solution:

Since the points are equispaced with h = 2 and n = 3, the cubic spline can be determined from

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1})$$

Now, at i = 1,

$$M_0 + 4M_1 + M_2 = \frac{6}{4}(y_0 - 2y_1 + y_2) = \frac{6}{4}(3 - 2(2) + 3) = 3$$

At i = 2,

$$M_1 + 4M_2 + M_3 = \frac{6}{4}(y_1 - 2y_2 + y_3) = \frac{6}{4}(2 - 2(3) + 1) = -4.5$$

Since, $M_0 = 0$ and $M_3 = 0$

We have,

$4M_1 + M_2 = 3$

$M_1 + 4M_2 = -4.5$

Solving these equations for M_1 and M_2, we get,

$M_1 = 1.1$

$M_2 = -1.4$

Scanned with CamScanner

Now, cubic spline in $(x_i \le x \le x_{i+1})$ is

$$f(x) = \frac{(x_{i+1} - x)^3}{6h} M_i + \frac{(x - x_i)^3}{6h} M_{i+1} + \frac{x_{i+1} - x}{h}\left(y_i - \frac{h^2}{6} M_i\right) + \frac{x - x_i}{h}\left(y_{i+1} - \frac{h^2}{6} M_{i+1}\right)$$

To find y at x = 8, take i = 2, the cubic spline in $(x_2 \le x \le x_3)$ *i.e.,* $(7 \le x \le 9)$

Substituting the values at x = 8,

$$y = \frac{(x_3 - x)^3}{12} M_2 + \frac{(x - x_2)^3}{12} M_3 + \frac{x_3 - x}{2}\left(y_2 - \frac{4}{6} M_2\right) + \frac{x - x_2}{2}\left(y_3 - \frac{4}{6} M_3\right)$$

$$= \frac{(9-8)^3}{12}(-1.4) + 0 + \frac{(9-8)}{12}\left(3 - \frac{4}{6}(-1.4)\right) + \frac{(8-7)}{2}\left(1 - \frac{4}{6}(0)\right)$$

∴ $y = 1.1$

7. Use Lagrange's interpolation formula to find the value of y when x = 3.0 from the following table.

x	3.2	2.7	1.0	4.8	5.6
y	22.0	17.8	14.2	38.3	51.7

Solution:

From Lagrange's interpolation for 5 data points, we have,

$$y = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} y_1 + \frac{(x - x_0)(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} y_3 + \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_0)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} y_4$$

At x = 3,

$$y = \frac{(3-2.7)(3-1)(3-4.8)(3-5.6)(22)}{(3.2-2.7)(3.2-1)(3.2-4.8)(3.2-5.6)} + \frac{(3-3.2)(3-1)(3-4.8)(3-5.6)(17.8)}{(2.7-3.2)(2.7-1)(2.7-4.8)(2.7-5.6)} + \frac{(3-3.2)(3-2.7)(3-4.8)(3-5.6)(14.2)}{(1-3.2)(1-2.7)(1-4.8)(1-5.6)} + \frac{(3-3.2)(3-2.7)(3-1)(3-5.6)(38.3)}{(4.8-3.2)(4.8-2.7)(4.8-1)(4.8-5.6)} + \frac{(3-3.2)(3-2.7)(3-1)(3-4.8)(51.7)}{(5.6-3.2)(5.6-2.7)(5.6-1)(5.6-4.8)}$$

$$= 14.625 + 6.4371 - 0.0610 - 1.1699 + 0.4360$$

∴ $y = 20.2672$

Scanned with CamScanner

8. **Find the values of y at x = 1.6 and x = 4.8 from the following points using Newton's interpolation technique.**

x	1	2	3	4	5
y	4	7.5	4	8.5	9.6

Solution:

Creating the difference table from the given data,

x	y	1st difference	2nd difference	3rd difference	4th difference
1	4				
		3.5			
2	7.5		-7		
		-3.5		15	
3	4		8		-26.4
		4.5		-11.4	
4	8.5		-3.4		
		1.1			
5	9.6				

At x = 1.6, which lies at the starting of table, so using Newton's forward interpolation

$x = 1.6, \quad x_0 = 1, \quad h = 2 - 1 = 1$

$x = x_0 + ph$

or, $p = 0.6$

Now using Newton's forward interpolation formula,

$$y_{1.6} = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$= 4 + (0.6 \times 3.5) + \frac{0.6(0.6-1)(-7)}{2} + \frac{0.6(0.6-1)(0.6-2)(15)}{6} + \frac{0.6(0.6-1)(0.6-2)(0.6-3)(-26.4)}{24}$$

$$= 4 + 2.1 + 0.84 + 0.84 + 0.8870$$

$\therefore \quad y_{1.6} = 8.6670$

Again,

At x = 4.8 which lies near the end of table, so using Newton's backward interpolation

$x = 4.8, x_n = 5, h = 1$

$x = x_n + ph$

or, $p = -0.2$

$$y_{4.8} = y_4 + p\nabla y_4 + \frac{p(p+1)}{2!}\nabla^2 y_4 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_4 + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_4$$

Scanned with CamScanner

$$= 9.6 + (-0.2 \times 1.1) + \frac{(-0.2)(1-0.2)(-3.4)}{2}$$
$$+ \frac{(-0.2)(1-0.2)(2-0.2)(-11.4)}{6}$$
$$+ \frac{(-0.2)(1-0.2)(2-0.2)(3-0.2)(-26.4)}{24}$$
$$= 9.6 - 0.22 + 0.272 + 0.5472 + 0.887$$

$\therefore \quad y_{48} = 11.0862$

9. Fit the following data to the curve $y = \log_e (ax + b)$

x	0	1	2	3	4	5	6
y	0.9	1.0	1.5	1.9	2.1	2.4	2.5

Solution:

We have the curve,

$y = \log_e (ax + b)$

or, $\text{antilog}_e (y) = ax + b$

or, $e^y = ax + b$

Comparing with the equation, $Y = a + bX$

$Y = e^y$

Forming normal equations as

$\Sigma Y = nb + a\Sigma x$ ..... (1)

$\Sigma XY = b\Sigma x + b\Sigma x^2$ ..... (2)

$n = 7$

x	y	$y = e^y$	xY	x^2
0	0.9	2.4596	0	0
1	1	2.7183	2.7183	1
2	1.5	4.4817	8.9634	4
3	1.9	6.6859	20.0577	9
4	2.1	8.1662	32.6648	16
5	2.4	11.0232	55.1160	25
6	2.5	12.1825	73.0950	36
$\Sigma x = 21$		$\Sigma Y = 47.7174$	$\Sigma xY = 192.6152$	$\Sigma x^2 = 91$

Substituting the obtained values (1) and (2), we get,

$47.7174 = 7b + 21a$ ..... (a)

$192.6152 = 21b + 91a$ ..... (b)

On solving (a) and (b), we get,

$a = 1.7665$

$b = 1.5172$

Hence the required fit of the curve is $y = \log_e (1.7665x + 1.5172)$

10. Fit the following set of data into a curve $y = \frac{ax}{b + x}$

x	1	2	3	4	5
y	0.5	0.667	0.75	0.8	0.833

Scanned with CamScanner

Solution:

Given curve,

$$y = \frac{ax}{b + x}$$

or, $\frac{1}{y} = \frac{b + x}{ax}$

or, $\frac{1}{y} = \frac{b}{a} \cdot \frac{1}{x} + \frac{1}{a}$

Let, $Y = \frac{1}{y}$ and $X = \frac{1}{x}$ then,

$$Y = \frac{b}{a}X + \frac{1}{a}$$

Comparing with $Y = A + BX$

$$A = \frac{1}{a} \text{ and } B = \frac{b}{a}$$

Forming normal equations

$\Sigma Y = nA + B\Sigma X$ ..... (1)

$\Sigma XY = A\Sigma X + B\Sigma X^2$ ..... (2)

$n = 5$

x	y	$X = \frac{1}{x}$	$Y = \frac{1}{y}$	XY	X^2
1	0.5	1	2	2	1
2	0.667	0.5	1.4993	0.7497	0.25
3	0.750	0.3333	1.3333	0.4444	0.1111
4	0.8	0.25	1.25	0.3125	0.0625
5	0.833	0.2	1.2	0.24	0.04
		$\Sigma X = 2.2833$	$\Sigma Y = 7.2826$	$\Sigma XY = 3.7466$	$\Sigma X^2 = 1.4636$

Substituting the obtained values (1) and (2), we get,

$7.2826 = 5A + 2.2833B$ ..... (a)

$3.7466 = 2.2833A + 1.4636B$ ..... (b)

On solving (a) and (b), we get,

$A = 0.9998 \approx 1$

$B = 1$

We have,

$A = \frac{1}{a} \Rightarrow a = \frac{1}{A} = 1$

$B = \frac{b}{a} \Rightarrow b = aB = 1 \times 1 = 1$

Hence, $y = \frac{x}{1 + x}$ is the required fit of the curve.

Scanned with CamScanner

Chapter

3

NUMERICAL DIFFERENTIATION AND INTEGRATION

♠♠♠

3.1 NUMERICAL DIFFERENTIATION

It is the process of calculating the value of the derivative of a function at some assigned value of x from the given set of values (x_i, y_i). To compute, $\frac{dy}{dx}$ we first replace the exact relation $y = f(x)$ by the best interpolating polynomial $y = \phi(x)$ and then differentiate the latter as many times as we desire. The choice of the interpolation formula to be used depend, will depend on the assigned value of x at which $\frac{dy}{dx}$ is desired.

If the values of x are equispaced and $\frac{dy}{dx}$ is required near the beginning of the table, we employ Newton's forward formula. If it is required near the end of the table, we use Newton's backward formula. For values near the middle of the table, $\frac{dy}{dx}$ is calculated by means of Stirling's or Bessel's formula. If the values of x are not equispaced, we use Lagrange's formula or Newton's divided difference formula to represent the function.

Hence corresponding to each the interpolation formula, we can derive a formula for finding the derivative.

Scanned with CamScanner

3.2 FORMULA FOR DERIVATIV[illegible]

Consider the function $y = f(x)$ which [illegible] for the values $x_i (= x_0 + ih)$, $i = 0, 1, 2, \ldots\ldots, n$.

A. Derivatives using Newton's F[illegible]ward Difference Formula

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

Differentiating both sides with respect to p, we have,

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \ldots\ldots$$

Since, $p = \frac{(x - x_0)}{h}$

Hence, $\frac{dp}{dx} = \frac{1}{h}$

Now,

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \ldots\ldots\right] \quad \text{.....(1)}$$

At $x = x_0$, $p = 0$. Hence Putting $p = 0$

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\ldots\right] \quad \text{.....(2)}$$

Again differentiating (1) with respect to x, we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dp}\right)\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2-36p^2-36p+22}{4!}\Delta^4 y_0 + \ldots\ldots\right]\frac{1}{h}$$

Putting $p = 0$, we obtain,

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \ldots\ldots\right] \quad \text{.....(3)}$$

Similarly,

$$\left(\frac{d^3y}{dx^3}\right) = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\ldots\right]$$

Otherwise;

We know that, $1 + \Delta = E = e^{hD}$

$\therefore \quad hD = \log(1 + \Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots\ldots$

Scanned with CamScanner

or, $D = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + ...\right]$.

and, $D^2 = \frac{1}{h^2}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + ...\right]^2$

$= \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 + ...\right]$

and, $D^2 = \frac{1}{h^2}\left[\Delta^3 - \frac{3}{2}\Delta^4 + ...\right]$

Now, applying the above identities to y_0, we get,

Dy_0 *i.e.*,

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\Delta y_0 - \frac{1}{2}\left[\Delta^2 y_0 \frac{1}{3}\Delta^2 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + ...\right] \text{..... (4)}$$

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + ...\right]$$

and, $$\left(\frac{d^3y}{dx^3}\right) = \frac{1}{h^3}\left[\Delta^2 y_0 - \frac{3}{2}\Delta^4 y_0 + ...\right]$$

which are same as (2), (3) and (4) respectively.

B. Derivatives using Newton's Backward Difference Formula

Newton's backward interpolation is,

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + ...$$

Differentiating both sides with respect to p, we get,

$$\frac{dy}{dp} = \nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + ...$$

Since, $p = \frac{x - x_n}{h}$,

Hence, $\frac{dp}{dx} = \frac{1}{h}$

Now,

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_0 + ...\right] \text{..... (5)}$$

At $x = x_n$, $p = 0$

Hence, putting $p = 0$, we get,

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + ...\right] \text{...(6)}$$

Differentiating equation (5), with respect to x, we have,

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx^3}$$

Scanned with CamScanner

$$= \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \dots\dots\right]$$

Putting p = 0, we get,

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\Delta^4 y_n + \frac{5}{6}\Delta^5 y_n + \frac{137}{180}\Delta^6 y_n + \dots\dots\right] \quad \text{..... (7)}$$

Similarly,

$$\left(\frac{d^3y}{dx^3}\right) = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\Delta^4 y_n + \dots\dots\right] \quad \text{..... (8)}$$

Otherwise:

We know,

$$1 - \nabla = E^{-1} = e^{-hD}$$

$$\therefore \quad hD = \log(1-\nabla) = -\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{3}\nabla^4 + \dots\dots\right]$$

or,

$$D = \frac{1}{h}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \dots\dots\right]$$

$$\therefore \quad D^2 = \frac{1}{h^2}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{2}\nabla^3 + \dots\dots\right]^2$$

$$= \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \dots\dots\right]$$

Similarly,

$$D^2 = \frac{1}{h^3}\left[\nabla^3 + \frac{3}{2}\nabla^4 + \dots\dots\right]$$

Applying these identities to y_n, we get,

Dy_n *i.e.*,

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{2}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \dots\dots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \dots\dots\right]$$

and,

$$\left(\frac{d^3y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^2 y_n + \frac{3}{2}\Delta^4 y_n + \dots\dots\right]$$

which are same as (6), (7) and (8).

C. Derivatives using Stirling's Central Difference formula

Stirling's formula is,

$$y_p = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \frac{p^2(p^2-1^2)}{4!}\Delta^4 y_{-2} + \dots\dots$$

Differentiating both sides with respect to p, we get,

$$\frac{dy}{dp} = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{2p}{2!}\Delta^2 y_{-1} + \frac{3p^2 - 1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$
$$+ \frac{4p^3 - 2p}{4!}\Delta^3 y_{-2} + \text{.........}$$

Since, $p = \frac{x - x_0}{h}$

$$\therefore \quad \frac{dp}{dx} = \frac{1}{h}$$

Now,

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}$$
$$= \frac{1}{h}\left[\left(\frac{y_0 + \Delta y_{-1}}{26}\right) + p\Delta^2_{y-1} + \frac{3p^2 - 1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)\right.$$
$$\left. + \frac{2p^3 - p}{12}\Delta^4 y_{-2} + \text{.........}\right]$$

At x = 0, p = 0. Hence putting p = 0, we get,

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \text{.........}\right] \quad \text{..... (9)}$$

Similarly,

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} + \text{.........}\right] \quad \text{..... (10)}$$

Example 3.1

The following data gives the velocity of a particle for twenty seconds at an interval of five seconds. Find the initial acceleration using the entire data.

Time, t(sec)	0	5	10	15	20
Velocity, v(m/s)	3	14	69	228	?

Solution:

The difference table is,

n = 5

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
0	0				
3					
5	3		8		
11		36		36	
10	14		44		24
55		60			
15	69		104		
159					
20	228				

Scanned with CamScanner

An initial acceleration *i.e.*, $\left(\frac{dv}{dt}\right)$ at t = 0 is required, we use Newton's forward formula.

$$\left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0 - \frac{1}{4}\Delta^4 v_0 + \ldots\ldots\right]$$

$$\therefore \quad \left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{5}\left[3 - \frac{1}{2}(8) + \frac{1}{3} \times 36 - \frac{1}{4} \times 24\right]$$

$$= \frac{1}{5}(3 - 4 + 12 - 6)$$

$$= 1$$

Hence the initial acceleration is 1 m/sec^2.

3.3 NUMERICAL INTEGRATION

The process of evaluating a definite integral from a set of tabulated values of the integrand f(x) is called numerical integration. This process when applied to a function of a single variable, is known as quadrature.

The problem of numerical integration, like that of numerical differentiation, is solved by representing f(x) by an interpolation formulation and then integrating it between the given limits. In this way, we can derive quadrature formula for approximate integration of a function defined by a set of numerical values

3.4 NEWTON-COTES QUADRATURE FORMULA

Let, $I = \int_a^b f(x)\, dx$

where, f(x) takes the values $y_0, y_1, y_2, \ldots\ldots, y_n$ for $x = x_0, x_1, x_2, \ldots\ldots, x_n$.

Let us divide the interval (a, b) into n sub-intervals of width h so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ..., $x_n = x_0 + nh = b$. Then,

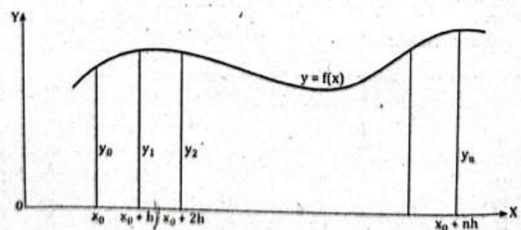


Figure 3.1

$$I = \int_{x_0}^{x_0+nh} f(x)\, dx$$

$$= h\int_0^n f(x_0 + rh)\, dr, \qquad \text{Putting } x = x_0 + rh,\ dx = hdr$$

$$= h\int_0^n \Big[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0$$

$$+ \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 y_0 \Big]$$

$$+ \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!}\Delta^6 y_0 + \ldots\ldots \Big] dr$$

[By Newton's interpolation formula]

Integrating term by term, we get,

$$\int_{x_0}^{x_0+nh} f(x)\, dx = nh\Big[y_0 + \frac{n}{2} y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0$$

$$+ \frac{n(n-2)^2}{24}\Delta^3 y_0 + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!}$$

$$+ \left(\frac{n^4}{6} - 2n^4 + \frac{34n^3}{4} - \frac{50n^2}{3} + 12n\right)\frac{\Delta^5 y_0}{5!}$$

$$+ \left(\frac{n^6}{7} - \frac{15n^5}{6} + 17n^4 - \frac{225n^3}{4} + \frac{274n^2}{3} - 60n\right)\frac{\Delta^6 y_0}{6!} + \ldots\ldots \Big] \quad \ldots (1)$$

This is known as Newton's cotes quadrature formula. From this general formula, we deduce the following important quadrature rules by taking $n = 1, 2, 3, \ldots\ldots$

I. Trapezoidalo Rule

Putting $n = 1$ in equation (1) and taking the curve through (x_0, y_0) and (x_1, y_1) as a straight line *i.e.*, a polynomial of first order so that differences of order higher than first becomes zero, we get,

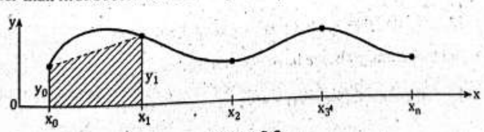


Figure 3.2

Here; $\int_{x_0}^{x_0+h} f(x)\, dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}(y_0 + y_1)$

Similarly,

$$\int_{x_0+h}^{x_0+2h} f(x)\, dx = h\left(y_1 + \frac{1}{2}\Delta y_1\right) = \frac{h}{2}(y_1 + y_2)$$

....

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\, dx = \frac{h}{2}(y_{n-1} + y_n)$$

Scanned with CamScanner

Adding these n integrals, we get,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \dots\dots + y_{n-1})\right] \quad \text{.... (2)}$$

This is known as the trapezoidal rule.

> **NOTE:**
> The area of each strips (trapezium) is found separately. Then the area under the curve and the ordinates at x_0 and x_n is approximately equal to the sum of the areas of the n trapeziums.

II. Simpson's One-third Rule

Putting n = 2 in equation (1) above and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola *i.e.*, a polynomial of the second order so that difference of order higher than the second vanish, we get,

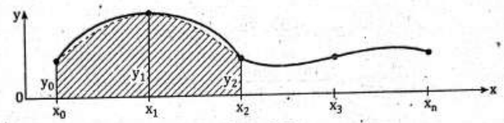


Figure 3.3

Here,

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right) = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

....

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n), \text{ n being even.}$$

Adding all these integrals, we have when n is even,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \dots\dots + y_{n-1}) + 2(y_2 + y_4 + \dots\dots + y_{n-2})\right] \quad \text{.... (3)}$$

This is known as the Simpson's one-third rule or simply Simpson's rule and is most commonly used.

> **NOTE:**
> While applying (3), the given interval must be divided into an even number of equal subintervals, since we find the area of two strips at a time.

III. Simpson's Three-eight Rule

Putting n = 3 in (1) above and taking the curve through (x_i, y_i); i = 0, 1, 2, 3 as a polynomial of the third order so that differences above the third order vanish, we get,

Scanned with CamScanner

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

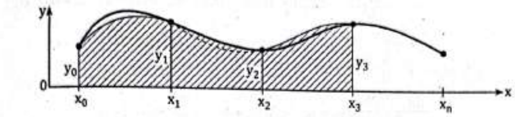


Figure 3.4

Similarly,

$$\int_{x_0+3h}^{x_0+5h} f(x)\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6) \text{ and so on.}$$

Adding all such expressions from x_0 to $x_0 + nh$, where n is a multiple of 3, we get,

$$\int_{x_0}^{x_0+h} f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{n-1})$$

$$+ 2(y_3 + y_0 + \ldots\ldots + y_{n-3})] \quad \ldots (4)$$

NOTE:
While applying equation (4), the number of sub-intervals should be taken as a multiple of 3.

Example 3.2

Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using

i) **Trapezoidal rule**

ii) **Simpson's $\frac{1}{3}$ rule**

iii) **Simpson's $\frac{3}{8}$ rule**

Solution:

Divide the interval (0, 6) into six parts, each of width h = 1. The values of $f(x) = \frac{1}{1+x^2}$ are given below;

x	0	1	2	3	4	5	6
f(x)	1	0.5	0.2	0.1	0.0588	0.0385	0.027
= y	y_0	y_1	y_2	y_3	y_4	y_5	y_6

i) By trapezoidal

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2}[(1 + 0.027) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385)]$$

$$= 1.4108$$

Scanned with CamScanner

ii) By Simpson's $\frac{1}{3}$ rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[(1 + 0.27) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)]$$

$$= 1.3662$$

iii) By Simpson's $\frac{3}{8}$ rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}[(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2 \times 0.1]$$

$$= 1.3571$$

Example 3.3

Evaluate the integral $\int_0^1 \frac{x^2}{1+x^3}dx$ by using Simpson's $\frac{1}{3}$ rule. Compare the error with the exact value.

Solution:

Let us divide the internal (0, 1) into 4 equal parts so that $h = \frac{1-0}{4} = 0.25$.

Taking $y = \frac{x^2}{(1+x^3)}$, we have,

x	0	0.25	0.50	0.75	1.00
y	0	0.06153	0.22222	0.39560	0.5
	y_0	y_1	y_2	y_3	y_4

By Simpson's $\frac{1}{3}$ rule, we have,

$$\int_0^1 \frac{x^2}{1+x^3}dx = \frac{h}{3}[(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3)]$$

$$= \frac{0.25}{3}[(0 + 0.5) + 2 \times 0.22222) + 4(0.06153 + 0.3956)]$$

$$= \frac{0.25}{3}[0.5 + 0.44444 + 1.82852]$$

$$= 0.23108$$

Also,

$$\int_0^1 \frac{x^2}{1+x^3}dx = \frac{1}{3}|\log(1+x^3)|_0^1$$

$$= \frac{1}{3}\log_e(2) = 0.23108$$

Hence the error = 0.23108 – 0.23105 = –0.00003

Scanned with CamScanner

Example 3.4

Use trapezoidal rule to evaluate $\int_0^1 x^3\,dx$ considering five sub-intervals.

Solution:

Given that,

$$I = \int_0^1 x^3\,dx$$

Also, a = 0, b = 1, sub-intervals = 5, intervals (n) = 5 − 1 = 4

Then,

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

Now table is created at the interval of 0.25 from 0 to 1.

x	0	0.25	0.5	0.75	1
y	0	0.0156	0.125	0.4219	1
	y_0	y_1	y_2	y_3	y_4

Now, using trapezoidal rule,

$$I = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.25}{2}[0 + 1 + 2\,(0.0156 + 0.125 + 0.4219)]$$

$$= 0.256$$

Also, $I_{abs} = \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1^4}{4} - 0 = 0.25$

Example 3.5

Evaluate $\int_0^1 \frac{dx}{1+x}$ applying

- **i) Trapezoidal rule**
- **ii) Simpson's $\frac{1}{3}$ rule**
- **iii) Simpson's $\frac{3}{8}$ rule**

Solution:

Given that;

$$I = \int_0^1 \frac{dx}{1+x}$$

Also, a = 0, b = 1, Taking n = 5

$$h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$$

Now, table is created at the interval of 0.2 from 0 to 1.

x	0	0.2	0.4	0.6	0.8	1
y	1	0.8333	0.7143	0.625	0.5556	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

Scanned with CamScanner

i) By trapezoidal rule,

$$I = \frac{h}{2}[y_0 + y_5 + 2(y_1 + y_2 + y_3 + y_4)]$$

$$= \frac{0.2}{2}[1 + 0.5 + 2(0.8333 + 0.7143 + 0.625 + 0.5556)]$$

$$= 0.6956$$

ii) By Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[y_0 + y_5 + 4(y_1 + y_3) + 2(y_2 + y_4)]$$

$$= \frac{0.2}{3}[1 + 0.5 + 4(0.8333 + 0.625) + 2(0.7143 + 0.5556)]$$

$$= 0.6582$$

iii) By Simpson's $\frac{3}{8}$ rule,

$$I = \frac{h}{3}[y_0 + y_5 + 3(y_1 + y_2 + y_4) + 2y_3]$$

$$= \frac{3 \times 0.2}{8}[1 + 0.5 + 3(0.8333 + 0.7143 + 0.5556) + 2(0.625)]$$

$$= 0.6795$$

Also, $I_{abs} = \int_0^1 \frac{dx}{1+x} = 0.6931$

Example 3.6

Given that;

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
log x	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Evaluate $\int_4^{5.2} \log x \, dx$ by,

- **a) Trapezoidal rule**
- **b) Simpson's $\frac{1}{3}$ rule**
- **c) Simpson's $\frac{3}{8}$ rule**

Solution:

Given that;

$$I = \int_4^{5.2} \log x \cdot dx$$

From the given table, n = 6

so, $h = \frac{b-a}{n} = \frac{5.2-4}{6} = 0.2$

Simply we can find the h from table as 4.2 − 4 = 0.2

Scanned with CamScanner

Now, from the table we have,

$y_0 = 1.3863$ $\quad$ $y_3 = 1.5261$

$y_1 = 1.4351$ $\quad$ $y_4 = 1.5686$

$y_2 = 1.4816$ $\quad$ $y_5 = 1.6094$

$y_6 = 1.6487$

Now, by Trapezoidal rule,

$$I = \frac{h}{2}[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)$$

$$= \frac{0.2}{2}[1.3863 + 1.6487 + 2(1.4351 + 1.4816 + 1.5261 + 1.5686 + 1.6094)]$$

$$= 1.8277$$

By Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{0.2}{3}[1.3863 + 1.6487 + 4(1.4351 + 1.5261 + 1.6094) + 2(1.4816 + 1.5686)]$$

$$= 1.8279$$

By Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3(0.2)}{8}[1.3863 + 1.6487 + 3(1.4351 + 1.4816 + 1.5686 + 1.6094) + 2(1.5261)]$$

$$= 1.8278$$

Also, $I_{abs} = \int_4^{5.2} \log x \, dx = 1.8278$

3.5 ERRORS IN QUADRATURE FORMULA

The error in the quadrature formula is given by,

$$E = \int_a^b y \, dx - \int_a^b p(x) \, dx$$

where, $p(x)$ is the polynomial representing the function $y = f(x)$, in the interval $[a, b]$.

I. **Error in Trapezoidal Rule**

Expanding $y = f(x)$ around $x = x_0$ by Taylor's series, we get,

$$y = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \ldots\ldots \quad \ldots (1)$$

$$\therefore \quad \int_{x_0}^{x_0+h} y\,dx = \int_{x_0}^{x_0+h}\left[y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \ldots\ldots\right]dx \quad \text{.....(2)}$$

$$= y_0 h + \frac{h^2}{2!}y_0' + \frac{h^3}{3!}y_0'' + \ldots\ldots$$

Also, A = area of the first trapezium in the interval

$$[x_0, x_1] = \frac{1}{2}h(y_0 + y_1) \quad \text{.....(3)}$$

Putting $x = x_0 + h$ and $y = y_1$ in equation (1), we get,

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \ldots\ldots$$

Replacing this value of y_1 in (3), we get,

$$A_1 = \frac{1}{2}h\left[y_0 + y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \ldots\ldots\right] \quad \text{.....(4)}$$

$$= hy_0 + \frac{h^2}{2!}y_0' + \frac{h^3}{2\times 2!}y_0'' + \ldots\ldots$$

$\therefore$ Error in the interval $[x_0, x_1] = \int_{x_1}^{x_2} y\,dx - A_1$

$$= \frac{1}{3!} - \frac{1}{2.2!}h^3 y_0'' + \ldots\ldots$$

$$= -\frac{h^3}{12}y_0'' + \ldots\ldots$$

i.e., principal part of the error in $[x_0, x_1] = -\frac{h^3}{12}y_0''$

Hence the total error, $E = -\frac{h^3}{12}[y_0'' + y_1'' + \ldots\ldots + y_{n-1}'']$

Assuming that $y''(X)$ is the largest of n quantities,

$y_0'', y_1'', \ldots\ldots, y_{n-1}''$, we get,

$$E < -\frac{nh^3}{12}y''(X) = -\frac{(b-a)h^2}{12}y''(X) \qquad [\because nh = b - a] \quad \text{.....(5)}$$

Hence the error in the trapezoidal rule is of the order h^2.

II. Error in Simpson's $\frac{1}{3}$ Rule = $-\left(\frac{b-a}{180}\right)h^4 y^{iv}(X)$

Assuming the $y^{iv}(X)$ is the largest of

$y_0^{iv}, y_2^{iv}, \ldots\ldots, y_{2n-2}^{iv}$

***i.e.,* the error in Simpson's $\frac{1}{3}$ rule is of the order h^4.**

III. Error in Simpson's $\frac{3}{8}$ Rule = $-\frac{3h^5}{80}y^{iv}$

Scanned with CamScanner

3.6 ROMBERG'S INTEGRATION

Romberg integration method is named after Werner Romberg. This method is an extrapolation formula of the trapezoidal rule for integration. It provides a better approximation of the integral by reducing the true error. We compute the value of the integral with a number of step lengths using the same method. Usually, we start with a coarse step length, then reduce the step lengths are recomputed the value of the integral. The sequence of these values converges to the exact value of the integral. Romberg method uses these values of the integral obtained with various step lengths, to refine the solution such that the new values are of higher order. That is, as if the results are obtained using a higher order method than the order of the method used. The extrapolation method is derived by studying the error of the method that is being used.

Romberg's method provides a simple modification to the quadrature formulae for finding their better approximations. As an illustrations, let us improve upon the value of the integral,

$$I = \int_a^b f(x)\,dx$$

by the trapezoidal rule.

If I_1, I_2 are the values of I with sub-intervals of width h_1, h_2 and E_1, E_2 their corresponding errors, respectively, then,

$$E_1 = -\frac{(b-a)\,h^2}{12} y''(X)$$

$$E_2 = -\frac{(b-a)^2\,h^2}{12} y''(\bar{X})$$

Since, $y''(\bar{X})$ is also the largest value of y'', we can reasonably assume that $y''(X)$ and $y''(\bar{X})$ are very nearly equal.

$$\therefore \quad \frac{E_1}{E_2} = \frac{h_1^2}{h_2^2} \text{ or } \frac{E_1}{E_2 - E_1} = \frac{h_1^2}{h_2^2 - h_1^2} \qquad \text{..... (1)}$$

Now,

$$\text{Since } I = I_1 + E_1 = I_2 + E_2$$

$$\therefore \quad E_2 - E_1 = I_1 - I_2 \qquad \text{..... (2)}$$

From (1) and (2), we get,

$$E_1 = \frac{h_1^2}{h_2^2 - h_1^2}(I_1 - I_2)$$

Hence, $I = I_1 + E_1 = I_1 + \dfrac{h_1^2}{h_2^2 - h_1^2}(I_1 - I_2)$

$$\text{i.e.,} \quad I = \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2} \qquad \text{..... (3)}$$

Scanned with CamScanner

which is a better approximation of I.

To evaluate I systematically, we take $h_1 = h$ and $h_2 = \frac{1}{2}h$.

So that (3) gives,

$$I = \frac{I_1\left(\frac{h}{2}\right)^2 - I_2 h_2^2}{\left(\frac{h}{2}\right)^2 - h^2} = \frac{4I_2 - I_1}{3}$$

i.e., $$I\left(h, \frac{h}{2}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right] \quad \text{.... (4)}$$

Now, we use trapezoidal rule several times successively halving h and apply (4) to each per pair of values as per the following scheme.

$I(h)$			
	$I\left(h, \frac{h}{2}\right)$		
$I\left(\frac{h}{2}\right)$		$I\left(h, \frac{h}{2}, \frac{h}{4}\right)$	
	$I\left(\frac{h}{2}, \frac{h}{4}\right)$		$I\left(h, \frac{h}{2}, \frac{h}{4}, \frac{h}{8}\right)$
$I\left(\frac{h}{4}\right)$		$I\left(\frac{h}{2}, \frac{h}{4}, \frac{h}{8}\right)$	
	$I\left(\frac{h}{4}, \frac{h}{8}\right)$		
$I\left(\frac{h}{8}\right)$			

The computation is continued until successive values are close to each other. This method is called Richardson's deferred approach to the limit and its systematic refinement is called Romberg's method.

Example 3.7

Evaluate $\int_0^{0.5} \left(\frac{x}{\sin x}\right) dx$ correct to three decimals places using Romberg's method.

Solution:

Taking h = 0.25, 0.125, 0.0625 respectively, let us evaluate the given integral by using Simpson's $\frac{1}{3}$ rule.

i) When h = 0.25, the values of $y = \frac{x}{\sin x}$ are,

x	0	0.25	0.5
y	1	1.0105	1.0429
	y_0	y_1	y_2

Scanned with CamScanner

∴ By Simpson's rule,

$$I = \frac{h}{3}[(y_0 + y_2) + 4y_1]$$

$$= \frac{0.25}{3}[(1 + 1.0429) + 1.0105]$$

$$= 0.5071$$

ii) When h = 0.125, the values of y are,

x	0	0.125	0.25	0.375	0.5
y	1	1.0026	1.0105	1.1003	1.0429
	y_0	y_1	y_2	y_3	y_4

∴ By Simpson's rule,

$$I = \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{0.125}{3}[(1 + 1.0429) + 4(1.0026 + 1.1003) + 2(1.0105)]$$

$$= 0.5198$$

iii) When h = 0.0625, the values of y are,

x	0	0.0625	0.125	0.1875	0.25	0.3125	0.1875	0.4375	0.5
y	1	0.0006	1.0026	1.0059	1.0157	1.0165	1.1003	1.0326	1.0429
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

∴ By Simpson's rule,

$$I = \frac{h}{3}[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{0.0625}{3}[(1 + 1.0429) + 4(1.0006 + 1.0059 + 1.0165 + 1.0326)$$

$$+ 2(1.0026 + 1.0105 + 1.1003)]$$

$$= 0.510253$$

Using Romberg's formulae, we get

$$I = \left(h, \frac{h}{2}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right] = 0.5241$$

$$I = \left(\frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{4}\right) - I\left(\frac{h}{2}\right)\right] = 0.5070$$

$$I\left(h, \frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}, \frac{h}{4}\right) - I\left(h, \frac{h}{2}\right)\right] = 0.5013$$

Hence,

$$\int_0^{0.5}\left(\frac{x}{\sin x}\right)dx = 0.501$$

Example 3.8

Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using the Romberg's method.

Solution:

Given that;

$$I = \int_0^2 \frac{dx}{x^2+4}$$

Here, a = 0, b = 2

i) Taking h = 1 and creating interval of 1 from 0 to 2

x	0	1	2
y	0.25	0.2	0.125
	y_0	y_1	y_2

Now, using Trapezoidal rule,

$$I(1) = \frac{h}{2}[y_0 + y_2 + 2y_1] = \frac{1}{2}[0.25 + 0.125 + 2(0.2)] = 0.3875$$

ii) Taking h = 0.5 and creating interval of 0.5 from 0 to 2

x	0	0.5	1	1.5	2
y	0.25	0.2353	0.2	0.16	1.125
	y_0	y_1	y_2	y_3	y_4

Now, using trapezoidal rule,

$$I(0.5) = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3]$$

$$= \frac{0.5}{2}[0.25 + 0.125 + 2(0.2353 + 0.2 + 0.16]$$

$$= 0.3914$$

iii) Taking h = 0.25 and creating interval of 0.25 from 0 to 2

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
y	0.25	0.2462	0.2353	0.2192	0.2	0.1798	0.16	0.1416	0.125

Now, using trapezoidal rule,

$$I(0.25) = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.25}{2}[0.25 + 0.125 + 2(0.2462 + 0.2353 + 0.2192$$

$$+ 0.2 + 0.1798 + 0.16 + 0.1416)]$$

$$= 0.3924$$

Now, optimizing values by Romberg integration,

$$I(1, 0.5) = \frac{1}{3}[4I(0.5) - I(1)]$$

$$= \frac{1}{3}[4(0.3914) - 0.3875]$$

$$= 0.3927$$

Scanned with CamScanner

$$I(0.5, 0.25) = \frac{1}{3}[4I(0.25) - I(0.5)]$$

$$= \frac{1}{3}[4(0.3924) - 0.3914]$$

$$= 0.3927$$

$$I(1, 0.5, 0.25) = \frac{1}{3}[4I(0.5, 0.25) - I(1, 0.5)]$$

$$= \frac{1}{3}[4(0.3927) - 0.3927]$$

$$= 0.3927$$

Hence the value of $\int_0^2 \frac{dx}{x^2+4} = 0.3927$

3.7 GAUSSIAN INTEGRATION

Gauss derived a formula which uses the same number of functional values but with different Gauss formula is expressed as,

$$\int_{-1}^{1} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2) + \ldots\ldots + w_n f(x_n)$$

$$= \sum_{i=1}^{n} w_i f(x_i) \qquad \text{.... (1)}$$

where, w_i and x_i are called the weights and abscissae, respectively. The abscissae and weights are symmetrical with respect to the middle point of the interval. There being 2n unknowns in (1), 2n relations between them are necessary so that the formula is exact for all polynomials of degree not exceeding 2n – 1. Thus, we consider,

$$f(x) = c_0 + c_1x + c_2x + \ldots\ldots + c_{2n-1}x^{2n-1} \qquad \text{.... (2)}$$

Then, (1) gives,

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} (c_0 + c_1x + c_2x + \ldots\ldots + c_{2n-1}x^{2n-1})\,dx \qquad \text{.... (3)}$$

$$= 2c_0 + \frac{2}{3}c_2 + \frac{2}{5}c_4 + \ldots\ldots$$

Putting $x = x_i$ in (2), we get,

$$f(x_i) = c_0 + c_1x_i + c_2x_i^2 + c_2x_i^3 + \ldots\ldots + c_{2n-1}x_i^{2n-1}$$

Substituting these values on the right hand side of (1), we get,

$$\int_{-1}^{1} f(x)\,dx = w_1(c_0 + c_1x_1 + c_2x_1^2 + c_3x_1^3 + \ldots\ldots + c_{2n}x_1^{2n-1})$$

$$+ w_2(c_0 + c_1x_2 + c_2x_2^2 + c_3x_2^3 + \ldots\ldots + c_{2n-1}x_2^{2n-1})$$

$$+ w_3(c_0 + c_1x_3 + c_2x_3^2 + c_3x_3^3 + \ldots\ldots + c_{2n-1}x_3^{2n-1})$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$+ w_n(c_0 + c_1x_n + c_2x_n^2 + c_3x_n^3 + \ldots\ldots + c_{2n-1}x_n^{2n-1})$$

Scanned with CamScanner

$$= c_0 (w_1 + w_2 + w_3 + \ldots\ldots + w_n) \ldots\ldots + c_1(w_1x_1 + w_2x_2 + w_3x_3 + \ldots\ldots + w_nx_n) + c_2 (w_1x_1^2 + w_2x_2^2 + w_3x_3^2 + \ldots\ldots + w_nx_n^2) + \ldots\ldots + c_{2n-1} (w_1x_1^{2n-1} + w_2x_2^{2n-1} + w_3x_3^{2n-1} + \ldots\ldots + w_nx_n^{2n-1}) \quad \ldots (4)$$

But the equation (3) and (4) are identical for all values of c_i, hence comparing coefficients of c_i, we get 2n equations in 2n unknowns in 2n unknowns w_i and x_i(i = 1, 2, 3, ..., n).

$$\left.\begin{aligned} & w_1 + w_2 + w_3 + \ldots\ldots + w_n = 2 \\ & w_1x_1 + w_2x_2 + w_3x_3 + \ldots\ldots + w_nx_n = 0 \\ & w_1x_1^2 + w_2x_2^2 + w_3x_3^2 + \ldots\ldots + w_nx_n^2 = \frac{2}{3} \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ & w_1x_1^{2n-1} + w_2x_2^{2n-1} + w_3x_n^{2n-1} + \ldots\ldots + w_nx_n^{2n-1} = 0 \end{aligned}\right\} \quad \ldots (5)$$

The solution of above equations is extremely complicated. It can however, be shown that x_i are the zeros of the $(n + 1)^{th}$ Legendre polynomial.

Gauss formula for n = 2

$$\int_{-1}^{1} f(x)\, dx = w_1 f(x_1) + w_2 f(x_2)$$

Then the equation (5) becomes,

$$w_1 + w_2 = 2$$

$$w_1x_1 + w_2x_2 = 0$$

$$w_1x_1^2 + w_2x_2^2 = \frac{2}{3}$$

$$w_1x_1^3 + w_2x_2^3 = 0$$

On solving, we get,

$$w_1 = w_2 = 1,\ x_1 = \frac{-1}{\sqrt{3}} \text{ and } x_2 = \frac{1}{\sqrt{3}}$$

Thus, gauss formula for n = 2 is,

$$\int_{-1}^{1} f(x)\, dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \quad \ldots (6)$$

which gives the correct values of the integral of f(x) in the range (–1, 1) for any function upto third order. Equation (6) is also called as Gauss-Legendre formula.

Gauss formula for n = 3 is,

$$\int_{-1}^{1} f(x)\, dx = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(-\frac{\sqrt{3}}{5}\right) + f\left(\frac{\sqrt{3}}{5}\right)\right]$$

which is exact for polynomials upto degree 5.

Scanned with CamScanner

Gauss formula imposes a restriction on the limits of integration to be form −1 to 1.

In general, the limits of the integral $\int_a^b f(x)\,dx$ are changed to −1 to 1 by means of the transformation,

$$x = \frac{1}{2}(b-a)\,u + \frac{1}{2}(b+a)$$

Example 3.9

Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ using Gauss formula for n = 2 and n = 3.

Solution:

i) Gauss formula for n = 2 is,

$$I = \int_{-1}^{1} \frac{dx}{1+x^2} = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

where, $f(x) = \frac{dx}{1+x^2}$

$$\therefore \quad I = \frac{1}{1+\left(\frac{-1}{\sqrt{3}}\right)^2} + \frac{1}{1+\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{3}{4} + \frac{3}{4} = 1.5$$

ii) Gauss formula for n = 3 is,

$$I = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(\frac{-\sqrt{3}}{5}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

where, $f(x) = \frac{1}{1+x^2}$

$$\text{Hence, } I = \frac{8}{9} + \frac{5}{9}\left(\frac{5}{8} + \frac{5}{8}\right) = \frac{8}{9} + \frac{50}{72} = 1.5833$$

Scanned with CamScanner

BOARD EXAMINATION SOLVED QUESTIONS

1. Evaluate the integral $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx$. Compare the result in both conditions for Simpson $\frac{1}{3}$ and $\frac{3}{8}$ rule. **[2013/Fall]**

Solution:

Given that;

$$I = \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx$$

$$a = 0, \quad b = \frac{\pi}{2}$$

Taking n = 6,

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

Now, table is created at the interval of $\frac{\pi}{12}$ from 0 to $\frac{\pi}{2}$.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y	0	0.508	0.707	0.840	0.930	0.982	1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Now, by Simpson's $\frac{1}{3}$ rule

$$I = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{\pi}{3 \times 12}[0 + 1 + 4(0.508 + 0.840 + 0.982) + 2(0.707 + 0.930)]$$

$$= 1.186$$

Again, by Simpson's $\frac{3}{8}$ rule

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\pi}{8 \times 12}[0 + 1 + 3(0.508 + 0.707 + 0.930 + 0.982) + 2(0.840)]$$

$$= 1.184$$

and, Absolute value of I

$$I_{abs} = \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx = 1.198$$

NOTE: Use calculator to directly obtain the absolute value in radian mode.

Scanned with CamScanner

Now,

$$\text{Error by Simpson } \frac{1}{3} \text{ rule} = |1.186 - 1.198| = 0.012$$

$$\text{Error by Simpson } \frac{3}{8} \text{ rule} = |1.184 - 1.198| = 0.014$$

Here, the error by Simpson $\frac{1}{3}$ rule is less than Simpson $\frac{3}{8}$ rule.

2. Evaluate the integral $I = \int_0^6 \frac{1}{1+x^2} dx$. Compare the absolute error in both conditions for Simpson $\frac{1}{3}$ rule and Simpson $\frac{3}{8}$ rule. [2013/Spring]

Solution:

Given that;

$$I = \int_0^6 \frac{1}{1+x^2} dx$$

$a = 0, b = 6$

Let, $n = 6$ then

$$h = \frac{b-a}{n} = \frac{6-0}{6} = 1$$

Now, Table is created at the interval of 1 from 0 to 6

Formulating the table,

x	0	1	2	3	4	5	6
y	1	0.5	0.2	0.1	0.058	0.038	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[1 + 0.027 + 4(0.5 + 0.1 + 0.038) + 2(0.2 + 0.058)]$$

$$= 1.365$$

By Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$= \frac{3}{8}[1 + 0.027 + 3(0.5 + 0.2 + 0.058 + 0.038) + 2(0.1)]$$

$$= 1.355$$

Now, Absolute value of I,

$$I = \int_0^6 \frac{1}{1+x^2} dx = \tan^{-1}(x)\Big|_0^6 = 1.405$$

Scanned with CamScanner

Now,

$$\text{Error by Simpson } \frac{1}{3} \text{ rule} = |1.405 - 1.365| = 0.04$$

$$\text{Error by Simpson } \frac{3}{8} \text{ rule} = |1.405 - 1.355| = 0.05$$

Here, the error by Simpson $\frac{1}{3}$ rule is less than Simpson $\frac{3}{8}$ rule.

3. **Find the integral value $I = \int_0^1 \frac{dx}{1+x^2}$ correct to three decimal place by using Romberg integration.** **[2013/Spring, 2018/Spring]**

Solution:

Given that;

$$I = \int_0^1 \frac{dx}{1+x^2}$$

Here, $a = 0, b = 1$

i) Taking h = 0.5 and creating interval of 0.5 from 0 to 1.

x	0	0.5	1
y = f(x)	1	0.8	0.5
	y_0	y_1	y_2

Now, using trapezoidal rule,

$$I(0.5) = \frac{h}{z}[y_0 + y_2 + 2y_1]$$

$$= \frac{0.5}{2}[1 + 0.5 + 2(0.8)]$$

$$= 0.775$$

ii) Taking h = 0.25 and creating interval of 0.25 from 0 to 1.

x	0	0.25	0.5	0.75	1
y	1	0.9411	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

Now, using trapezoidal rule,

$$I(0.25) = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.25}{2}[1 + 0.5 + 2(0.9411 + 0.8 + 0.64)]$$

$$= 0.7827$$

iii) Taking h = 0.125

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.9846	0.9411	0.8767	0.8	0.7191	0.64	0.5663	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Scanned with CamScanner

Now, using Trapezoidal rule,

$$I(0.125) = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.125}{2}[1 + 0.5 + 2(0.9846 + 0.9411 + 0.8767 + 0.8 + 0.7191 + 0.64 + 0.5663)]$$

$$= 0.7847$$

Now, optimizing values by Romberg integration,

$$I(0.5, 0.25) = \frac{1}{3}[4I(0.25) - I(0.5)]$$

$$= \frac{1}{3}[4 \times 0.7827 - 0.775]$$

$$= 0.7852$$

$$I(0.25, 0.125) = \frac{1}{3}[4I(0.125) - I(0.25)]$$

$$= \frac{1}{3}[4 \times 0.7847 - 0.7827]$$

$$= 0.7853$$

$$I(0.5, 0.25, 0.125) = \frac{1}{3}[4I(0.25, 0.125) - I(0.5, 0.25)]$$

$$= 0.7853$$

Hence the value of integral $\int_0^1 \frac{dx}{1+x^2} = 0.7853$

Also, $\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}(x)\Big|_0^1 = 0.7853$

Table of obtained values;

I(0.5)		
	I(0.5, 0.25)	
I(0.25)		I(0.5, 0.25, 0.125)
	I(0.25, 0.125)	
I(0.125)		

4. **The following table gives the displacement, x (cms) of an object at various of time, t(seconds). Find the velocity and acceleration of the object at t = 1.6 sec. Using suitable interpolation method. [2014/Fall]**

T	1.0	1.2	1.4	1.6	1.8
X	9.0	9.5	10.2	11.0	13.2

Solution:

Creating the difference table from given data

Scanned with CamScanner

x = T	y = x	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1.0	9.0				
		0.5			
1.2	9.5		0.2		
		0.7		−0.1	
1.4	10.2		0.1		1.4
		0.8		1.3	
1.6	11.0		1.4		
		2.2			
1.8	13.2				

Here the data of T is equispaced and t = 1.6 sec is near the end of the table, so using Newton's backward formula for numerical differentiation.

$$h = 1.8 - 1.6 = 0.2$$

Now, at t = 1.6 sec

From numerical differentiation, using Newton's backward formula,

$$\left(\frac{dy}{dx}\right)_{1.6} = y' = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3}\right]$$

$$= \frac{1}{0.2}\left[0.8 + \frac{0.1}{2} + \frac{-0.1}{3}\right]$$

?

= 4.083 cm/s is the required velocity of an object

Now, for acceleration

$$y'' = \frac{1}{h^2}[\nabla^2 y_n + \nabla^3 y_n] = \frac{1}{0.22}[0.1 + -0.1]$$

∴ $y'' = 0 \text{ cm/s}^2$ is the required acceleration of an object.

5. Evaluate the Integral $\int_0^{\pi}(1 + 3\cos^2 x)\,dx$ by,

i) Trapezoidal rule

ii) Simpson's $\frac{3}{8}$ rule, taking number of intervals (n) = 6

[2014/Spring]

Solution:

Given that;

$$I = \int_0^{\pi}(1 + 3\cos^2 x)\,dx$$

$$n = 6$$

Also,

$$a = 0, b = \pi$$

Then,

$$h = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

Scanned with CamScanner

Now, table is created at the interval of $\frac{\pi}{6}$ from 0 to π

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
y	4	3.25	1.75	1	1.75	3.25	4
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

i) By trapezoidal rule,

$$I = \frac{h}{2}[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{\pi}{2 \times 6}[4 + 4 + 2(3.25 + 1.75 + 1 + 1.75 + 3.25)]$$

$\therefore$ $I = 7.8539$

ii) By Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\pi}{8 \times 6}[4 + 4 + 3(3.25 + 1.75 + 1.75 + 3.25) + 2(1)]$$

$$= 7.8539$$

Also,

$$I_{abs} = \int_0^{\pi}(1 + 3\cos^2 x)\,dx = \int_0^{\pi} 1 + \frac{3}{2}(\cos 2x + 1) = 7.8539$$

6. Evaluate the integral $I = \int_0^{\frac{\pi}{2}} \sin x\,dx$ for n = 6 and compare the result in both conditions for Simpson $\frac{1}{3}$ and $\frac{3}{8}$ rule. [2015/Fall]

Solution:

Given that;

$$I = \int_0^{\frac{\pi}{2}} \sin x\,dx$$

$$a = 0,\ b = \frac{\pi}{2},\ n = 6$$

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

Now, creating table at the interval of $\frac{\pi}{12}$ from 0 to $\frac{\pi}{2}$

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y	0	0.258	0.5	0.707	0.866	0.965	1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Scanned with CamScanner

Now, By Simpson's $\frac{1}{3}$ rule

$$I = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{\pi}{3 \times 12}[0 + 1 + 4(0.258 + 0.707 + 0.965) + 2(0.5 + 0.866)]$$

$$= 0.9993$$

Again, by Simpson's $\frac{3}{8}$ rule

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\pi}{8 \times 12}[0 + 1 + 3(0.258 + 0.5 + 0.866 + 0.965) + 2(0.707)]$$

$$= 0.9995$$

and, $I_{abs} = \int_0^{\frac{\pi}{2}} \sin x \cdot dx = [-\cos x]_0^{\pi/2} = -\cos\frac{\pi}{2} + \cos 0 = 1$

Now, Error by Simpson $\frac{1}{3}$ rule = $|1 - 0.9993| = 0.0007$

Error by Simpson $\frac{3}{8}$ rule = $|1 - 0.9995| = 0.0005$

Here, the error by Simpson $\frac{1}{3}$ rule is more than Simpson $\frac{3}{8}$ rule, so Simpson $\frac{3}{8}$ rule is more accurate.

7. **Use following table of data to estimate velocity at t = 7 sec**

Time, t(s)	5	6	7	8	9
Distance Travelled, s(t) (km)	10.0	14.5	19.5	25.5	32.0

Hint: velocity is first derivative of s(t). **[2015/Spring]**

Solution:

Creating difference table

t = x	y = s(t)	1st diff	2nd diff	3rd diff	4th diff
5	10.0				
		4.5			
6	14.5		0.5		
		5		0.5	
7	19.5		1		−1
		6		−0.5	
8	25.5		0.5		
		6.5			
9	32.0				

Now, to estimate velocity at t = 7 sec which lies at the mid of table.

Scanned with CamScanner

Using Stirling's central difference formula,

We have,

$$y_p = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \ldots\ldots$$

$$= x_0 + ph$$

Differentiating with respect to p, we get,

$$\frac{dy_p}{dx} = \frac{\Delta y_0 + \Delta y_{-1}}{2} + p\Delta^2 y_{-1} + \frac{3p^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \ldots\ldots$$

and, $\frac{dx}{dp} = h$

Then,

$$\frac{dy_p}{dx} = \frac{dy_p}{dp} \times \frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} + p\Delta^2 y_{-1} + \frac{3p^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \ldots\ldots\right]$$

At $x = x_0$, $p = 0$,

so, $$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \ldots\ldots\right]$$

Now,

$$s'(t) = \frac{d(s(t))}{dt} = \left(\frac{dy}{dx}\right)_7 = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right]$$

NOTE:
Formula is placed according to the data available in difference table i.e., Δy_0 and Δy_{-1} are present but not for other $\Delta^3 y_{-1}$, $\Delta^3 y_{-2}$ etc for t = 7.

or, $s'(t) = \frac{1}{1}\left[\frac{6+5}{2}\right]$

∴ $s'(t) = 5.5$ km/s is the required velocity

8. Evaluate the integral $I = \int_0^{10} \exp\left(\frac{-1}{1+x^2}\right) dx$, using gauss quadrature formula with n = 2 and n = 3. **[2016/Fall]**

Solution:

Given that;

$$I = \int_0^{10} f(x)\, dx$$

where, $f(x) = \exp\left(\frac{-1}{1+x^2}\right)$

Using gauss quadrature formula with n = 2 and n = 3 since limit a = 0 and b = 10 is not from −1 to 1, so using,

$$x = \frac{1}{2}(b-a)\,u + \frac{1}{2}(b+a)$$

Scanned with CamScanner

or, $x = \frac{1}{2}(10 - 0)u + \frac{1}{2}(10 + 0)$

$\therefore$ $x = 5u + 5$ ..... (1)

Differentiating on both sides

$dx = 5\,du$ ..... (2)

Then, substituting the values form (1) and (2) to I,

$$I = \int_{-1}^{1} \exp\left(\frac{-1}{1 + (5u + 5)^2}\right) 5du$$

Now,

i) Gauss formula for n = 2 is

$$I = \int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= 5\exp\left[\frac{-1}{1 + \left(5\left(\frac{-1}{\sqrt{3}}\right) + 5\right)^2}\right] + 5\exp\left[\frac{-1}{1 + \left(\frac{5}{\sqrt{3}} + 5\right)^2}\right]$$

$$= 4.164 + 4.921 = 9.085$$

Then,

ii) Gauss formula for n = 3 is,

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9}\left(5\exp\left(\frac{-1}{1 + (0 + 5)^2}\right)\right)$$

$$+ \frac{5}{9}\left[5\exp\left(\frac{-1}{1 + \left(5\left(-\sqrt{\frac{3}{5}}\right) + 5\right)^2}\right)\right]$$

$$+ 5\exp\left[5\exp\left(\frac{-1}{1 + \left(5\left(\sqrt{\frac{3}{5}}\right) + 5\right)^2}\right)\right]$$

$$= 4.276 + 4.531 = 8.807$$

9. Evaluate the integral $\int_0^{0.6} e^{x^2}\,dx$, using Simpson $\frac{1}{3}$ rule and Simpson $\frac{1}{8}$ rule, dividing the interval into six parts. **[2016/Spring]**

Solution:

Given that;

$$I = \int_0^{0.6} e^{x^2}\,dx,$$

a = 0, b = 0.6 and n = 6

Then,

$$h = \frac{b - a}{n} = \frac{0.6 - 0}{6} = 0.1$$

Now, table is created at the interval of 0.1 from 0 to 0.6.

x	0	0.1	0.2	0.3	0.4	0.5	0.6
y	1	1.010	1.040	1.094	1.173	1.284	1.433
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Now, by Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{0.1}{3}[1 + 1.433 + 4(1.010 + 1.094 + 1.284) + 2(1.04 + 1.173)]$$

$$= 0.68036$$

Again, by Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3 \times 0.1}{8}[1 + 1.433 + 3(1.010 + 1.040 + 1.173 + 1.284) + 2(1.094)]$$

$$= 0.68032$$

Also, $I_{abs} = \int_0^{0.6} e^{x^2}\,dx = 0.68049$

10. Estimate the following integrals by,

i) Simpson's $\frac{3}{8}$ method

ii) Simpson's $\frac{1}{3}$ method and compare the result

$$\int_2^1 \frac{e^x\,dx}{x} \text{ (Assume n = 4)}$$

[2017/Fall]

Solution:

Given that;

$$I = \int_2^1 \frac{e^x}{x}\,dx$$

$a = 2, b = 1, n = 4$

Then,

$$h = \frac{b-a}{n} = \frac{1-2}{4} = -0.25$$

Now, creating table at the interval of (−0.25) from 2 to 1.

x	2	1.75	1.5	1.25	1
y	3.694	3.288	2.987	2.792	2.718
	y_0	y_1	y_2	y_3	y_4

Now, by Simpson's $\frac{1}{3}$ rule,

Scanned with CamScanner

$$I = \frac{h}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2(y_2)]$$

$$= \frac{-0.25}{3}[3.694 + 2.718 + 4(3.288 + 2.792) + 2(2.987)]$$

$$= -3.0588$$

And, by Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[y_0 + y_4 + 3(y_1 + y_2) + 2y_3)$$

$$= \frac{3 - 0.25}{8}[3.694 + 2.718 + 3(3.288 + 2.987) + 2(2.792)]$$

$$= -2.8894$$

Then, $I_{abs} = \int_2^1 \frac{e^x}{x} dx = -3.0591$

Now, Error by Simpson $\frac{1}{3}$ rule = $|-3.0591 + 3.0588| = 0.0003$

Error by Simpson $\frac{3}{8}$ rule = $|-3.0591 + 2.8894| = 0.1697$

So, Simpson's $\frac{1}{3}$ rule is more accurate.

11. Apply Romberg's method to evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + \sin x}} dx$$ [2017/Fall]

Solution:

Given that;

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + \sin x}} dx$$

$a = 0, b = \frac{\pi}{2}$

i) Taking $h = \frac{\pi}{4}$ and creating interval of $\frac{\pi}{4}$ from 0 to $\frac{\pi}{2}$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
y	1	0.541	0
	y_0	y_1	y_2

Now, using trapezoidal rule

$$I\left(\frac{\pi}{4}\right) = \frac{h}{2}[y_0 + y_2 + 2y_1]$$

$$= \frac{\pi}{2 \times 4}[1 + 0 + 2(0.541)] = 0.8175$$

Scanned with CamScanner

ii) Taking $h = \frac{\pi}{8}$

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
y	1	0.785	0.541	0.275	0
	y_0	y_1	y_2	y_3	y_4

$$I\left(\frac{\pi}{8}\right) = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{\pi}{2 \times 16}[1 + 0 + 2(0.785 + 0.541 + 0.275)]$$

$$= 0.8250$$

iii) Taking $h = \frac{\pi}{16}$

x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$	$\frac{5\pi}{16}$	$\frac{3\pi}{8}$	$\frac{7\pi}{16}$	$\frac{\pi}{2}$
y	1	0.897	0.785	0.667	0.541	0.410	0.275	0.138	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$$I\left(\frac{\pi}{16}\right) = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{\pi}{2 \times 16}[1 + 0 + 2(0.897 + 0.785 + 0.667 + 0.541 + 0.410 + 0.275 + 0.138)]$$

$$= 0.8272$$

Now, optimizing values by Romberg Integration

$$I\left(\frac{\pi}{4}, \frac{\pi}{8}\right) = \frac{1}{3}\left[4I\left(\frac{\pi}{8}\right) - I\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{3}[4 \times 0.8250 - 0.8175] = 0.8275$$

$$I\left(\frac{\pi}{8}, \frac{\pi}{16}\right) = \frac{1}{3}\left[4I\left(\frac{\pi}{16}\right) - I\left(\frac{\pi}{8}\right)\right]$$

$$= \frac{1}{3}[4(0.8279) - (0.8250)] = 0.8279$$

$$I\left(\frac{\pi}{4}, \frac{\pi}{8}, \frac{\pi}{16}\right) = \frac{1}{3}\left[4I\left(\frac{\pi}{8}, \frac{\pi}{16}\right) - I\left(\frac{\pi}{4}, \frac{\pi}{8}\right)\right]$$

$$= \frac{1}{3}[4 \times 0.8272 - 0.8275] = 0.8280$$

Hence the value of integral $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + \sin x}} dx = 0.8280$

Also, $I_{abs} = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + \sin x}} dx = 0.8284$

Scanned with CamScanner

12. A slider in a machine moves along a fixed straight rod 9 + s distance x (cm) along the rod is given below for various values of time t seconds. Find the velocity and the acceleration of the slider when t = 0.2. **[2017/Spring]**

t	0	0.1	0.2	0.3
x	30.13	31.62	32.87	33.95

Solution:

Creating difference table from given data

x = t	y = x	1st diff	2nd diff	3rd diff
0	30.13			
		1.49		
0.1	31.62		−0.24	
		1.25		0.07
0.2	32.87		−0.17	
		1.08		
0.3	33.95			

Here, the data of t is equispaced and t = 0.2 lies near the end of the table so using Newton's backward formula for numerical differentiation.

$h = 0.3 - 0.2 = 0.1$

Now, at t = 0.2

From, numerical differentiation using Newton's backward formula

$$y' = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2}\right] = \frac{1}{0.1}\left[1.25 + \frac{-0.24}{2}\right]$$

∴ $y' = 11.3$ cm/s is the required velocity of an object.

Now, for acceleration

$$y'' = \frac{1}{h^2}[\nabla^2 y_n] = \frac{1}{0.1^2} \times -0.24$$

∴ $y'' = -24$ cm/s²

is the required acceleration of an object

13. The velocity 'v' of a particle at a distance 's' from a point on its path is given is given by the following table.

s(m)	0	10	20	30	40	50	60
v(m/s)	47	58	64	65	61	52	38

Estimate the time taken to travel 60 metres by using Simpson's $\frac{1}{3}$ rule and Simpson's $\frac{3}{8}$ rule. **[2017/ Spring]**

Solution:

We have,

$$v = \frac{ds}{dt}$$

$$dt = \frac{1}{v} ds = y \cdot ds \quad \boxed{\Rightarrow y = \frac{1}{v}}$$

Scanned with CamScanner

On integration,

$$t = \int_0^{60} y \cdot ds$$

Here; $a = 0, b = 60, n = 6$

so, $h = \frac{60-0}{6} = 10$

Creating table at the interval of 10 from 0 to 60.

x = s	0	10	20	30	40	50	60
$y = \frac{1}{v}$	$\frac{1}{47}$	$\frac{1}{58}$	$\frac{1}{64}$	$\frac{1}{65}$	$\frac{1}{61}$	$\frac{1}{52}$	$\frac{1}{38}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Now, by Simpson's $\frac{1}{3}$ rule,

$$\therefore \quad I = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{10}{3}\left[\frac{1}{47} + \frac{1}{38} + 4\left(\frac{1}{58} + \frac{1}{65} + \frac{1}{52}\right) + 2\left(\frac{1}{64} + \frac{1}{61}\right)\right] = 1.063$$

Again, by Simpson's $\frac{3}{8}$ rule

$$\therefore \quad I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3 \times 10}{8}\left[\frac{1}{47} + \frac{1}{38} + 3\left(\frac{1}{58} + \frac{1}{64} + \frac{1}{61} + \frac{1}{52}\right) + 2\left(\frac{1}{65}\right)\right] = 1.064 \text{ s}$$

14. Evaluate the integral $I = \int_0^{\frac{\pi}{2}} (1 + 3\cos 2x)\,dx$. Compare the result in both conditions for Simpson $\frac{1}{3}$ and $\frac{3}{8}$ rule. [2018/Fall]

Solution:

Given that:

$$I = \int_0^{\frac{\pi}{2}} (1 + 3\cos 2x)\,dx$$

$$a = 0, b = \frac{\pi}{2}, n = 6$$

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

Now, table is created at the interval of $\frac{\pi}{12}$

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y	4	3.598	2.5	1	-0.5	-1.598	-2
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Now, by Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{\pi}{3 \times 12}[4 + (-2) + 4(3.598 + 1 - 1.598) + 2(2.5 - 0.5)]$$

$$= 1,57079$$

Again, by Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3)]$$

$$= \frac{3\pi}{8 \times 12}[4 + (-2) + 3(3.598 + 2.5 - 0.5 - 1.598) + 2(1)]$$

$$= 1.57079$$

Also, $I_{abs} = \int_0^{\frac{\pi}{2}} (1 + 3\cos 2x)\,dx = 1.57079$

Now, Error by Simpson $\frac{1}{3}$ rule = |1.57079 − 1.57079| = 0

Error by Simpson $\frac{3}{8}$ rule = |1.57079 − 1.57079| = 0

Hence, the Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule is accurate with zero error.

15. From the following table of values of x and y, obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for x = 1.2.

x	1.0	1.2	1.4	1.6	1.8
y	2.7183	3.3201	4.0552	4.9530	6.0496

[2018/Spring]

Solution:

Creating difference table.

x	y	Δy	Δ^2y	Δ^3y	Δ^4y
1.0	2.7183				
		0.6018			
1.2	3.3201		0.1333		
		0.7351		0.0294	
1.4	4.0552		0.1627		0.0067
		0.8978		0.0361	
1.6	4.9530		0.1988		
		1.0966			
1.8	6.0496				

Scanned with CamScanner

Here, the data of x is equispaced and x = 1.2 lies near the starting of table so using Newton's forward formula for numerical differentiation.

$h = 1.2 - 1.0 = 0.2$

Now, at x = 1.2.

From numerical differentiation, using Newton's forward formula

$$\frac{dy}{dx} = y' = \frac{1}{h}\left[\Delta y_n - \frac{\Delta^2 y_n}{2} + \frac{\Delta^3 y_n}{3}\right]$$

$$= \frac{1}{0.2}\left[0.7351 - \frac{0.1627}{2} + \frac{0.0361}{3}\right]$$

$\therefore \quad y' = 3.328$

Again, for $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = y'' = \frac{1}{h^2}[\Delta^2 y_n - \Delta^3 y_n]$$

$$= \frac{1}{0.2^2}[0.1627 - 0.0361]$$

$\therefore \quad y'' = 3.165$

16. The following data gives corresponding values of pressure 'p' and specific volume 'v' of steam.

p	105	42.7	25.3	16.7	13
v	2	4	6	8	10

Find the rate of change of volume when pressure is 105 and 13.

[2018/Fall]

Solution:

As the values of p are not equispaced, we use Newton's divided difference formula.

The divided difference table is

	x = p	y = v	1st diff	2nd diff	3rd diff	4th diff
x_0	105	2				
			−0.0321			
x_1	42.7	4		0.0010		
			−0.1149		-3.96×10^{-5}	
x_2	25.3	6		0.0045		7.06×10^{-6}
			−0.2325		-6.90×10^{-4}	
x_3	16.7	8		0.0250		
			−0.5405			
x_4	13	10				

Scanned with CamScanner

Now, Newton's divided formula for the 1st derivative.

We get,

$$f'(x) = \frac{dv}{dp} = [x_0, x_1] + (2x - x_0 - x_1)\,[x_0, x_1, x_2]$$
$$+ [3x^2 - 2x(x_0 + x_1 + x_2) + x_0x_1 + x_1x_2 + x_2x_0][x_0, x_1, x_2, x_3]$$
$$+ [4x^3 - 3x^2(x_0 + x_1 + x_2 + x_3)$$
$$+ 2x(x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 + x_1x_3 + x_0x_2)$$
$$- (x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_0x_1x_3)][x_0, x_1, x_2, x_3, x_4]$$

Now, when pressure is 105

$$\frac{dv}{dp_{at\,p=105}} = -0.0321 + (2(105) - 105 - 42.7)\,(0.0010)$$
$$+ [3(105)^2 - 2(105)(105 + 42.7 + 25.3) + (105 \times 42.7)$$
$$+ (42.7 \times 25.3) + (25.3 \times 105)]\,(-3.96 \times 10^{-5})$$
$$+ [4(105)^3 - 3(105)^2(105 + 42.7 + 25.3 + 16.7)$$
$$+ 2(105)(105 \times 42.7 + 42.7 \times 25.3 + 25.3 \times 16.7$$
$$+ 16.7 \times 105 + 42.7 \times 16.7 + 105 \times 25.3)$$
$$- (105 \times 42.7 \times 25.3 + 42.7 \times 25.3 \times 16.7$$
$$+ 25.3 \times 16.7 \times 105 + 105 \times 42.7 \times 16.7)]\,(7.06 \times 10^{-6})$$
$$= 2.9289$$

Similarly when pressure is 13, using x = 13 in the formula, we get,

$$\frac{dV}{dp_{at\,p=13}} = -0.6689$$

17. Evaluate $\int_{-2}^{2} \frac{x}{x + 2e^x}\,dx$ by using trapezoidal, Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule with n = 6. **[2019/Fall]**

Solution:

Given that;

$$I = \int_{-2}^{2} \frac{x\,dx}{x + 2e^x}$$

a = −2, b = 2, n = 6

Then,

$$h = \frac{b-a}{n} = \frac{2+2}{6} = \frac{2}{3}.$$

Now, table is created at the interval of $\frac{2}{3}$ from −2 to 2.

x	−2	$\frac{-4}{3}$	$\frac{-2}{3}$	0	$\frac{2}{3}$	$\frac{4}{3}$	2
y	1.156	1.653	−1.850	0	0.146	0.149	0.119
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Scanned with CamScanner

Now, by trapezoidal rule,

$$I = \frac{h}{2}[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{2}{2 \times 3}[1.156 + 0.119 + 2(1.653 - 1.850 + 0 + 0.146 + 0.149)] = 0.490$$

Again, by Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)$$

$$= \frac{2}{3 \times 3}[1.56 + 0.119 + 4(1.653 + 0 + 0.149) + 2(-1.850 + 0.146)]$$

$$= 1.1277$$

And, by Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3 \times 2}{8 \times 3}[1.156 + 0.119 + 3(1.653 - 1.850 + 0.146 + 0.149) + 2 \times 0]$$

$$= 0.3922$$

18. Using three-point Gaussian Quadrature formula, evaluate,

$$\int_0^1 \frac{dx}{1 + x}$$ **[2019/Fall]**

Solution:

Given that;

$$I = \int_0^1 \frac{dx}{1 + x}$$

Using gauss quadrature formula with n = 3.

Since limit a = 0 and b = 1 is not from –1 to 1 so using,

$$x = \frac{1}{2}(b - a)u + \frac{1}{2}(b + a)$$

or, $$x = \frac{1}{2}(1 - 0)u + \frac{1}{2}(1 + 0)$$

∴ $$x = \frac{u}{2} + \frac{1}{2} \quad \text{..... (1)}$$

Differentiating on both sides

$$dx = \frac{du}{2} \quad \text{..... (2)}$$

Now, substituting the values from (1) and (2) to I,

$$I = \int_{-1}^{1} \frac{\frac{du}{2}}{1 + \left(\frac{u}{2} + \frac{1}{2}\right)} = \int_{-1}^{1} \frac{du}{3 + u}$$

Scanned with CamScanner

Now, Gauss formula for n = 3 is

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{5}{3}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9} \times \frac{1}{3} + \frac{5}{9}\left[\frac{1}{3-\sqrt{\frac{3}{5}}} + \frac{1}{3+\sqrt{\frac{3}{5}}}\right]$$

$\therefore$ I = 0.69312

19. The following table gives the velocity of a velocity at various points of time.

Time, t(seconds)	1	2	4	5
Velocity, v(m/s)	0.25	1	2.2	4

Find the acceleration of the vehicle at t = 1.1 second and t = 2.5 second using any suitable differential formula. [2019/Spring]

Solution:

As the values of time are not equispaced, we use Newton's divided difference formula.

The divided difference table is

	x = t	y = v	1st diff	2nd diff	3rd diff
x_0	1	0.25			
			0.75		
x_1	2	1		−0.05	
			0.6		0.1125
x_2	4	2.2		0.4	
			1.8		
x_3	5	4			

From Newton's divided formula for the 1st derivative, we get,

$$f'(x) = [x_0, x_1] + (2x - x_0 - x_1)[x_0, x_1, x_2] + [3x^2 - 2x(x_0 + x_1 + x_2) + x_0x_1 + x_1x_2 + x_2x_0][x_0, x_1, x_2, x_3]$$

Now, when t = 1.1

$$f'(x)_{1.1} = 0.75 + [2(1.1) - 1 - 2](-0.05) + [3(1.1)^2 - 2(1.1)(1 + 2 + 4) + (1)(2) + (2)(4) + (1)(4)](0.1125)$$

$$= 0.75 + 0.04 + 0.2508$$

$\therefore$ $f'(x)_{1.1} = 1.0408$ is the required acceleration in m/s²

Again, when t = 2.5

$$f'(x)_{2.5} = 0.75 + 2(2.5) - 1 - 2)(-0.05) + [3(2.5)^2 - 2(2.5)(1 + 2 + 4) + (1)(2) + (2)(4) + (1)(4)](0.1125)$$

$$= 0.75 - 0.1 - 0.2531$$

= 0.3969 m/s² is the required acceleration.

Scanned with CamScanner

20. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin u}{u} du$ by using trapezoidal, Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule with n = 6. **[2019/Spring]**

Solution:

Given that;

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin u}{u} du$$

$$a = 0, b = \frac{\pi}{2}, n = 6$$

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

Now, table is created at the interval of $\frac{\pi}{12}$ from 0 to $\frac{\pi}{2}$

x = u	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y	1	0.988	0.954	0.9	0.826	0.737	0.636
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

NOTE:

At x = u = 0, $\frac{\sin u}{u} = \frac{0}{0}$, so we use L-Hopital's rule for 0.

Rest of the values are normally calculated.

Now, by trapezoidal rule,

$$I = \frac{h}{2} [y_0 + y_6 + 2 (y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{\pi}{24} [1 + 0.636 + 2 (0.988 + 0.954 + 0.9 + 0.826 + 0.737)]$$

$$= 1.367$$

Again, by Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3} [y_0 + y_6 + 4 (y_1 + y_3 + y_5) + 2 (y_2 + y_4)]$$

$$= \frac{\pi}{36} [1 + 0.636 + 4 (0.988 + 0.9 + 0.737) + 2 (0.954 + 0.826)]$$

$$= 1.369$$

And, by Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8} [y_0 + y_6 + 3 (y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\pi}{96} [1 + 0.636 + 3 (0.988 + 0.954 + 0.826 + 0.737) + 2 (0.9)]$$

$$= 1.369$$

Scanned with CamScanner

21. Use Gauss-Legendre 2-point and 3 point formula to evaluate;

$$\int_{0.5}^{1.5} e^{x^2}\,dx$$ **[2019/Spring]**

Solution:

Given that;

$$I = \int_{0.5}^{1.5} e^{x^2}\,dx$$

Since limit a = 0.5 and b = 1.5 is not from –1 to 1

so, $x = \frac{1}{2}(b - a)\,u + \frac{1}{2}(b + a)$

or, $x = \frac{1}{2}(1.5 - 0.5)\,u + \frac{1}{2}(1.5 + 0.5)$

or, $x = \frac{u}{2} + 1$ ..... (1)

Differentiating on both sides

$dx = \frac{du}{2}$ ..... (2)

Then, substituting the values from (1) and (2) to I,

$$I = \int_{-1}^{1} \frac{e^{\left(\frac{u}{2}+1\right)^2}}{2}\,du$$

Now,

i) Gauss formula for n = 2 is

$$I = \int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{e^{\left(-\frac{1}{2\sqrt{3}}+1\right)^2}}{2} + \frac{e^{\left(\frac{1}{2\sqrt{3}}+1\right)^2}}{2}$$

$= 0.829 + 2.631$

$= 3.46$

ii) Gauss formula for n = 3 is

$$I = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9}\left(\frac{e^{(0+1)^2}}{2}\right) + \frac{5}{9}\left[\frac{e^{\left(-\frac{1}{2}\sqrt{3/5}+1\right)^2}}{2} + \frac{e^{\left(\frac{1}{2}\sqrt{3/5}+1\right)^2}}{2}\right]$$

$= 1.208 + 2.307$

∴ $I = 3.515$

22. Obtain divided difference table for the given data set **[2019/Fall]**

x	–1	2	5	7
y	–8	3	1	12

Solution:

Creating the divided difference table

x	y	1st diff	2nd diff	3rd diff
-1	-8			
		$\frac{3+8}{2+1} = 3.667$		
			$\frac{-0.667-3.667}{5+1} = -0.722$	
2	3			
		$\frac{1-3}{5-2} = -0.667$		$\frac{1.233+0.722}{7+1} = 0.244$
			$\frac{5.5+0.667}{7-2} = 1.233$	
5	1			
		$\frac{12-1}{7-5} = 5.5$		
7	12			

23. Integrate the given integral using Romberg integration,

$$\int_1^2 \frac{1}{1+x^3}\,dx$$

[2020/Fall]

Solution:

Given that;

$$I = \int_1^2 \frac{1}{1+x^3}\,dx$$

Here, a = 1, b = 2

i) Taking h = 0.5

x	1	1.5	2
y	0.5	0.228	0.111
	y_0	y_1	y_2

Now using Trapezoidal rule

$$I(0.5) = \frac{h}{2}[y_0 + y_2 + 2y_1]$$

$$= \frac{0.5}{2}[0.5 + 0.111 + 2(0.228)] = 0.266$$

ii) Taking h = 0.25

x	1	1.25	1.5	1.75	2
y	0.5	0.338	0.228	0.157	0.111
	y_0	y_1	y_2	y_3	y_4

Now, using Trapezoidal rule

$$I(0.25) = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.25}{2}[0.5 + 0.111 + 2(0.338 + 0.228 + 0.157)] = 0.257$$

Scanned with CamScanner

iii) Taking h = 0.125

x	1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
y	0.5	0.412	0.338	0.277	0.228	0.188	0.157	0.131	0.111
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Now, using Trapezoidal rule

$$I(0.125) = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.125}{2}[0.5 + 0.111 + 2(0.412 + 0.338 + 0.277$$

$$+ 0.228 + 0.118 + 0.157 + 0.131)]$$

$$= 0.254$$

Now, optimizing values by Romberg Integration

$$I(0.5, 0.25) = \frac{1}{3}[4\,I(0.25) - I(0.5)]$$

$$= \frac{1}{3}[4(0.257) - 0.266]$$

$$= 0.254$$

$$I(0.25, 0.125) = \frac{1}{3}[4I\,(0.125) - I(0.25)]$$

$$= \frac{1}{3}[4(0.254) - 0.257]$$

$$= 0.253$$

$$I\,(0.5, 0.25, 0.125) = \frac{1}{3}[4I(0.25, 0.125) - I(0.5, 0.25)]$$

$$= \frac{1}{3}[4(0.253) - 0.254]$$

$$= 0.252$$

Hence the value of integral $\int_1^2 \frac{1}{1+x^3}dx = 0.252$

Also, $I_{abs} = \int_1^2 \frac{1}{1+x^3}dx = 0.2543$

24. Compute the integral using Gaussian 3-point formula.

$$\int_2^5 \frac{e^x + \sin x}{1+x^2}dx$$

[2020/Fall]

Solution:

Given that;

$$I = \int_2^5 \frac{e^x + \sin x}{1+x^2}dx$$

Since limit a = 2 and b = 5 is not from –1 to 1, so, $x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$

or, $x = \frac{1}{2}(5-2)u + \frac{1}{2}(5+2)$

or, $x = \frac{3}{2}u + \frac{7}{2}$ ..... (1)

Differentiating on both sides, we get,

$$dx = \frac{3}{2}du \quad \text{..... (2)}$$

Then, substituting the values from (1) and (2) to I,

$$I = \int_{-1}^{1} \frac{e^{\frac{3u+7}{2}} + \sin\left(\frac{3u+7}{2}\right)}{1+\left(\frac{3u+7}{2}\right)^2} \cdot \frac{3}{2} du$$

Now, using Gaussian 3-point formula,

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9}\left[\frac{e^{(7/2)} + \sin^{(7/2)}}{1+\left(\frac{7}{2}\right)^2} \cdot \frac{3}{2}\right] + \frac{5}{9}\left[\left(\frac{3}{2} \cdot \frac{e^{\frac{-3\sqrt{3/5}+7}{2}} + \sin^{\left(\frac{-3\sqrt{3/5}+7}{2}\right)}}{1+\left(\frac{-3\sqrt{3/5}+7}{2}\right)^2}\right)\right.$$

$$\left. + \left[\left(\frac{3}{2} \cdot \frac{e^{\frac{3\sqrt{3/5}+7}{2}} + \sin^{\left(\frac{3\sqrt{3/5}+7}{2}\right)}}{1+\left(\frac{3\sqrt{3/5}+7}{2}\right)^2}\right)\right]\right]$$

$$= 3.297 + 5.271$$

$\therefore$ $I = 8.568$

25. Write short notes on Romberg integration.

[2013/Fall, 2015/Fall, 2015/Spring]

Solution: See the topic 3.6.

Scanned with CamScanner

ADDITIONAL QUESTION SOLUTION

1. Evaluate $\int_0^{\frac{\pi}{2}} e^{\sin x}\,dx$ using Gaussian 3-point formula.

Solution:

Given that;

$$I = \int_0^{\frac{\pi}{2}} e^{\sin x}\,dx$$

Since limit a = 0 and $b = \frac{\pi}{2}$ is not from −1 to 1.

so, $x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$

or, $x = \frac{1}{2}\left(\frac{\pi}{2} - 0\right)u + \frac{1}{2}\left(\frac{\pi}{2} + 0\right)$

or, $x = \frac{\pi}{4}u + \frac{\pi}{4}$ ..... (1)

Differentiating on both sides, we get,

$dx = \frac{\pi}{4}du$ ..... (2)

Then, substituting the values from (1) and (2) to I

$$I = \int_{-1}^{1} e^{\sin\frac{\pi}{4}(u+1)} \cdot \frac{\pi}{4}du$$

Now, using Gaussian 3-point formula

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9}\left[\frac{\pi}{4} \cdot e^{\sin\frac{\pi}{4}}\right] + \frac{5}{9}\left[\left(\frac{\pi}{4} \cdot e^{\sin\frac{\pi}{4}(-\sqrt{3/5}+1)}\right) + \left(\frac{\pi}{4} \cdot e^{\sin\frac{\pi}{4}(\sqrt{3/5}+1)}\right)\right]$$

$$= 1.4159 + 1.6880$$

∴ $I = 3.1039$

2. Estimate the value of cos (1.74) from the following data:

x	1.7	1.74	1.78	1.82	1.86
sin x	0.9916	0.9857	0.9781	0.9691	0.9584

Solution:

Here the data of x are equispaced and x = 1.74 lies near the starting of table so using Newton's forward formula for numerical differentiation.

Creating the difference table,

x	y = sin x	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.7	0.9916				
		-0.0059			
1.74	0.9857		-0.0017		
		-0.0076		0.0003	
1.78	0.9781		-0.0014		-0.0006
		-0.0090		-0.0003	
1.82	0.9691		-0.0017		
		-0.0107			
1.86	0.9584				

h = 1.74 - 1.7 = 0.04

Now, at x = 1.74,

From Newton's forward formula for numerical differentiation

$$\frac{dy}{dx} = y' = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3}\right]$$

$$= \frac{1}{0.04}\left[-0.0076 + \frac{0.0014}{2} - \frac{0.0003}{3}\right]$$

$$= -0.1750$$

Hence, cos (1.74) = -0.1750

3. Find f'(3) from the following table:

x	2	4	8	12	16
f(x)	20	23	30	35	40

Solution:

Here, the data of x are not equispaced, we shall use Newton's divided difference formula.

Then, creating difference table

x	y = f(x)	1st diff	2nd diff	3rd diff	4th diff
2	20				
		1.5			
4	23		0.0417		
		1.75		-0.0104	
8	30		-0.0625		0.0011
		1.25		0.0052	
12	35		0		
		1.25			
16	40				

Now, using Newton's divided difference formula,

$$f'(x) = [x_0, x_1] + (2x - x_0 - x_1)[x_0, x_1, x_2]$$
$$+ [3x^2 - 2x(x_0 + x_1 + x_2) + x_0x_1 + x_1x_2 + x_2x_0][x_0, x_1, x_2, x_3]$$

Scanned with CamScanner

$+ [4x^3 - 3x^2 (x_0 + x_1 + x_2 + x_3)$
$+ 2x (x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 + x_1x_3 + x_0x_2)$
$- (x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_0x_1x_3)][x_0, x_1, x_2, x_3, x_4]$

At x = 3,

$= 1.5 + (6 - 2 - 4)(0.0417) + [27 - 6(2 + 4 + 8) + 8 + 32 + 16]$
$(-0.0104) + [108 - 27(2 + 4 + 8 + 12) + 6(8 + 32 + 96 + 24$
$+ 48 + 16) - (64 + 384 + 192 + 96)](0.0011)$
$= 1.5 + 0 + 0.0104 + 0.0154$

$\therefore \quad f'(3) = 1.5258$

4. Evaluate $\int_2^4 e^{-x^2} dx$ using 2-point Gauss Legendre method.

Solution:

Given that;

$$I = \int_2^4 e^{-x^2} dx$$

Since limit a = 2 and b = 4 is not from –1 to 1, so using,

$$x = \frac{1}{2}(b - a)\,u + \frac{1}{2}(b + a)$$

or, $$x = \frac{1}{2}(4 - 2)\,u + \frac{1}{2}(4 + 2)$$

or, $x = u + 3$ ..... (1)

Differentiating on both sides, we get,

$dx = du$ ..... (2)

Substituting the values from (1) and (2) to I,

$$I = \int_{-1}^{1} e^{-(u+3)^2} du$$

Now, using 2-point Gauss Legendre method

$$I = \int_{-1}^{1} e^{-(u+3)^2} du$$
$$= f\left(-\sqrt{\frac{1}{3}}\right) + f\left(\sqrt{\frac{1}{3}}\right)$$
$$= e^{-\left(\frac{-1}{\sqrt{3}}+3\right)^2} + e^{-\left(\frac{1}{\sqrt{3}}+3\right)^2}$$
$$= 0.0028$$

5. Evaluate the following using Simpson's $\frac{1}{3}$ rule. (take h = 0.2)

$$\int_0^2 \frac{4e^x}{1 + x^3} dx$$

Solution:

Given that;

$$I = \int_0^2 \frac{4e^x}{1 + x^3} dx$$

Scanned with CamScanner

$h = 0.2$

Table is created at the interval of 0.2 from 0 to 2.

x	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
y	4	4.8468	5.6084	5.9938	5.8877	5.4366	4.8682	4.3325	3.8878	3.5419	3.2840
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

Now, using Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[y_0 + y_{10} + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)]$$

$$= \frac{0.2}{3}[(4 + 3.2840 + 4(4.8468 + 5.9938 + 5.4366 + 4.3325 + 3.5419)$$

$$+ 2(5.6084 + 5.8877 + 4.8682 + 3.8878)]$$

$\therefore \quad I = 9.6263$

Also,

$$I_{abs} = \int_0^2 \frac{4e^x}{1 + x^3}\,dx = 9.62615$$

6. Evaluate $\int_0^2 f(x)\,dx$, for the function $f(x) = e^x + \sin 2x$ using composite Simpson's $\frac{3}{8}$ formula taking step h = 0.4.

Solution:

Given that;

$$I = \int_0^2 f(x)\,dx = \int_0^2 e^x + \sin 2x\,dx$$

$h = 0.4$

Table is created at the interval of 0.4 from o to 2,

x	0	0.4	0.8	1.2	1.6	2
y	1	2.2092	3.2251	3.9956	4.8747	6.6323
	y_0	y_1	y_2	y_3	y_4	y_5

Now, using Simpson's $\frac{3}{8}$ formula,

$$I = \frac{3h}{8}[y_0 + y_5 + 3(y_1 + y_2 + y_4) + 2y_3]$$

$$= \frac{3(0.4)}{8}[1 + 6.6323 + 3(2.2092 + 3.2251 + 4.8947) + 2(3.9956)]$$

$\therefore \quad I = 6.9916$

Also, $I_{abs} = \int_0^2 e^x + \sin 2x\,dx = \left[e^x - \frac{\cos 2x}{2}\right]_0^2 = 7.2159$

7. Evaluate the following integral using Romberg method corrected to two decimal places.

$$\int_0^2 \frac{e^x + \sin x}{1 + x^2}\,dx$$

Scanned with CamScanner

Solution:

Given that;

$$I = \int_0^2 \frac{e^x + \sin x}{1 + x^2} dx$$

Here, a = 0 and b = 2

i) Taking h = 1 and creating interval of 1 from 0 to 2.

x	0	1	2
y	1	1.7799	1.6597
	y_0	y_1	y_2

Now, using Trapezoidal rule,

$$I(1) = \frac{h}{2}[y_0 + y_2 + 2y_1]$$

$$= \frac{1}{2}[1 + 1.6597 + 2(1.7799)]$$

$$= 3.1098$$

ii) Taking h = 0.5 and creating interval of 0.5 from 0 to 2

x	0	0.5	1	1.5	2
y	1	1.7025	1.7799	1.6859	1.6597
	y_0	y_1	y_2	y_3	y_4

Now, using Trapezoidal rule,

$$I(0.5) = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.5}{2}[1 + 1.6597 + 2(1.7025 + 1.7799 + 1.6859)]$$

$$= 3.2491$$

iii) Taking h = 0.25 and creating interval of 0.25 from 0 to 2

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
y	1	1.4413	1.7025	1.7911	1.7799	1.7324	1.6859	1.6587	1.6597
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Now, using Trapezoidal rule,

$$I(0.25) = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.25}{2}[1 + 1.6597 + 2(1.4413 + 1.7025 + 1.7911 + 1.7799 + 1.7324 + 1.6859 + 1.6587)]$$

$$= 3.2804$$

Now, optimizing values by Romberg Integration,

$$I(1, 0.5) = \frac{1}{3}[4I(0.5) - I(1)]$$

Scanned with CamScanner

$$= \frac{1}{3}[4(3.2491) - 3.1098]$$

$$= 3.2955$$

$$I(0.5, 0.25) = \frac{1}{3}[4I(0.25) - I(0.5)]$$

$$= \frac{1}{3}[4(3.2804) - 3.2491]$$

$$= 3.2908$$

$$I(1, 0.5, 0.25) = \frac{1}{3}[4I(0.5, 0.25) - I(1, 0.5)]$$

$$= \frac{1}{3}[4(3.2908) - 3.2955]$$

$$= 3.2892 \approx 3.290$$

Hence, the value of integral is 3.290.

8. **The distance travelled by a vehicle at intervals of 2 minutes are given as follows:**

Time (min)	2	4	6	8	10	12
Distance (km)	0.25	1	2.2	4	6.5	8.5

Evaluate the velocity and acceleration of the vehicle at t = 3 minutes.

Solution:

Here, the data of time is equispaced and t = 3 min lies near the starting of table. So, we use Newton's forward formula for numerical differentiation.

Creating difference table

x = time	y = distance	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2	0.25					
		0.75				
4	1		0.45			
		1.2		0.15		
6	2.2		0.6		−0.05	
		1.8		0.1		−1.25
8	4		0.7		−1.3	
		2.5		−1.2		
10	6.5		−0.5			
		2				
12	8.5					

NOTE:
We cannot use the Newton's forward differentiation formula directly because t = 3 is not available in the table.

We have,

$x = x_0 + ph$ at $x = 3$, $x_0 = 2$, $h = 4 - 2 = 2$

Scanned with CamScanner

or, $3 = 2 + 2p$

$\therefore$ $p = 0.5$

Let, $x = x_0 + ph$ ..... (1)

And, using Newton's forward interpolation formula,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 y_0 \quad \text{..... (2)}$$

Now, differentiating (1) and (2) with respect to p, we get,

$$\frac{dx}{dp} = h$$

$$\frac{dy_p}{dp} = 0 + \Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{6}\Delta^3 y_0 + \frac{(4p^3-18p^2-22p-6)}{24}\Delta^4 y_0 + \frac{(5p^4-40p^3+105p^2-100p+24)}{120}\Delta^5 y_0$$

Then,

$$y'_p = \frac{dy_p}{dp} \times \frac{dp}{dx} = \frac{dy_p}{dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{6}\Delta^3 y_0 + \frac{(4p^3-18p^2-22p-6)}{24}\Delta^4 y_0 + \frac{(5p^4-40p^3+105p^2-100p+24)}{120}\Delta^5 y_0\right]$$

Substituting the values, we obtain,

$$y'_p = \frac{1}{2}[0.75 + 0 - 0.0063 - 0.0021 + 0.0462]$$

$\therefore$ $y'_p = 0.3939$ is the required velocity at $t = 3$ minutes.

Now, for acceleration differentiating y'_p with respect to p, we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right) \cdot \frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2}\Delta^2 y_0 + \frac{(6p-6)}{6}\Delta^3 y_0 + \frac{(12p^2-36p+22)}{24}\Delta^4 y_0 + \frac{(20p^3-120p^2+210p-100)}{120}\Delta^5 y_0\right]$$

$$= \frac{1}{2^2}[0.45 - 0.075 - 0.0146 + 0.2344]$$

$\therefore$ $y''_p = 0.1487$ is the required acceleration at $t = 3$ minutes

Scanned with CamScanner

9. A rod is rotating in a plain. The following table gives the angle in radians (q) through which the rod has turned for various values of time in seconds (t). Find the angular velocity and angular acceleration t = 0.2.

t	0	0.2	0.4	0.6	0.8
θ	0	0.122	0.493	0.123	2.022

Solution:

Here, the data of time is equispaced and t = 0.2 lies near the starting of table so we use Newton's forward differentiation formula.

Creating difference table:

x = t	y = θ	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0				
		0.122			
0.2	0.122		0.249		
		0.3710		−0.99	
0.4	0.493		−0.741		4
		−0.37		3.01	
0.6	0.123		2.269		
		1.899			
0.8	2.022				

h = 0.2 − 0 = 0.2

At t = 0.2,

From numerical differentiation, using Newton's forward formula.

$$\frac{dy}{dx} = y' = \frac{1}{h}\left[\Delta y_n - \frac{\Delta^2 y_n}{2} + \frac{\Delta^3 y_n}{3}\right]$$

$$= \frac{1}{0.2}\left[0.3710 - \frac{0.741}{2} + \frac{3.01}{3}\right]$$

∴ y' = 8.7242 is the required angular velocity.

Again, for $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = y'' = \frac{1}{h^2}[\Delta^2 y_n - \Delta^3 y_n]$$

$$= \frac{1}{0.2^2}[-0.741 - 3.01]$$

∴ y" = −98.775 is the required angular acceleration.

10. Evaluate $\int_0^{1.4} (\sin x^3 + \cos x^2)\,dx$ using Gaussian 3-point formula.

Solution:

Given that;

$$I = \int_0^{1.4} (\sin x^3 + \cos x^2)\,dx$$

Since limit a = 0 and b = 1.4 is not from −1 to 1,

so, $x = \frac{1}{2}(b - a)\,u + \frac{1}{2}(b + a)$

or, $x = \frac{1}{2}(1.4 - 0)\,u + \frac{1}{2}(1.4 + 0)$

or, $x = 0.7u + 0.7$ ..... (1)

Differentiating on both sides, we get,

$dx = 0.7\,du$ ..... (2)

Substituting the values from (1) and (2) to I,

$$I = \int_{-1}^{1} (\sin(0.7u + 0.7)^3 + \cos(0.7u + 0.7)^2\,(0.7)\,du$$

Now, using Gaussian 3-point formula

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{5}{3}}\right)\right]$$

$$= \frac{8}{9}[(\sin 0.7^3 + \cos 0.7^2)(0.7)] + \frac{5}{9}\left[\left(0.7\left(\sin\left(0.7\left(-\sqrt{\frac{3}{5}}\right) + 0.7\right)^3 + \cos\left(0.7\left(-\sqrt{\frac{3}{5}}\right) + 0.7\right)^2\right) + \left(0.7\left(\sin\left(0.7\left(\sqrt{\frac{3}{5}}\right) + 0.7\right)^3 + \cos\left(0.7\left(\sqrt{\frac{3}{5}}\right) + 0.7\right)^2\right)\right]$$

$$= 0.5303 + \frac{5}{9}(0.6854 + 0.6665)$$

$\therefore$ $I = 1.2813$

Scanned with CamScanner

Chapter

4

SOLUTION OF LINEAR EQUATIONS

♠♠♠

4.1 MATRICES AND THEIR PROPERTIES

In mathematics, a matrix is a rectangular array or table of numbers, symbols or expression arranged in rows and columns. For example, the dimension of the matrix below is 2 × 3 (read "two by three") because there are two rows and three columns.

$$\begin{bmatrix} 1 & 9 & -13 \\ 20 & 5 & -6 \end{bmatrix}$$

Provided that they have the same dimensions (each matrix has the same number of rows and the same number of columns as the other), two matrices can be added or subtracted element by element. The rule for matrix multiplication, however, is that two matrices can be multiplied only when the number of columns in the first equals the number of rows in the second (*i.e.*, the inner dimensions are the same, n for (m × n) – matrix times an (n × p) – matrix resulting in an (m × p) – matrix).

Definition

A system of mn numbers arranged in a rectangular array of m rows and n columns is called an m × n matrix. Such a matrix is denoted by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

Scanned with CamScanner

Special Matrices

a) **Row and column matrices**

A matrix having a single row is called a row matrix while a matrix having a single column is called a column matrix.

b) **Square matrix**

A matrix having n rows and n columns is called a square.

c) **Non-singular matrix**

A square matrix is said to be singular if its determinant is zero otherwise it is called non-singular matrix. The elements a_{ii} in a square matrix from the leading diagonal and their sum Σa_{ii} is called the trace of the matrix.

d) **Unit matrix**

A diagonal matrix of order n which has unity for all its diagonal elements is called a unit matrix of order n and is denoted by I_m.

e) **Null matrix or zero matrix**

If all the elements of a matrix are zero, it is called a null matrix.

f) **Triangular matrix**

A square matrix all of whose elements below the leading diagonal are zero is called an upper triangular matrix. A square matrix all of whose elements above the leading diagonal are zero is called a lower triangular matrix.

g) **Symmetric and skew-symmetric matrices**

A square matrix $[a_{ij}]$ is said to be symmetric when $a_{ij} = a_{ji}$ for all i and j. If $a_{ij} = -a_{ji}$ for all i and j so that all the leading diagonal elements are zero, then the matrix is called skew-symmetric.

Examples of symmetric and skew-symmetric matrices are respectively.

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ and } \begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$$

h) **Horizontal matrix**

A matrix of order m × n is a horizontal matrix if n > m. Example,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$$

i) **Vertical matrix**

A matrix of order m × n is a vertical matrix if m > n. Example,

$$\begin{bmatrix} 2 & 5 \\ 1 & 7 \\ 4 & 6 \\ 6 & 4 \end{bmatrix}$$

Scanned with CamScanner

j) **Diagonal matrix**

If all the elements except the principal diagonal, in a square matrix are zero, it is called a diagonal matrix. Thus a square $A = [a_{ij}]$ is a diagonal matrix, if $a_{ij} = 0$ when $i \neq j$. Example,

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

is a diagonal matrix of order 3 × 3 which can also be denoted by diagonal [2 3 4].

k) **Scalar matrix**

If all the elements in the diagonal of a diagonal matrix are equal, it is called a scalar matrix.

Thus, a square matrix $A = [a_{ij}]_{m \times n}$ is a scalar matrix if

$$a_{ij} = \begin{Bmatrix} 0 \ ; & i \neq j \\ k \ ; & i = j \end{Bmatrix}$$

where, k is a constant.

Example,

$$\begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

is a scalar matrix.

l) **Idempotent matrix**

A square matrix is idempotent, provided $A^2 = A$. For an idempotent matrix A, $A^n = A \forall n > 2, n \in N \Rightarrow A^n = A, n \geq 2$.

m) **Nilpotent matrix**

A nilpotent matrix is said to be nilpotent of index p, $(p \in N)$, if $A^p = 0$, $A^{p-1} \neq 0$, *i.e.*, p is the least positive integer for which $A^p = 0$, then A is said to be nilpotent of index p.

n) **Periodic matrix**

A square matrix which satisfies the relation $A^{k+1} = A$ for some positive integer k, then A is periodic with period k *i.e.*, if k is the least positive integer for which $A^{k+1} = A$ and A is said to be periodic with period k. If k = 1, then A is called idempotent. Example,

$$\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$$

has the period 1.

o) **Involuntary matrix**

If $A^2 = I$, the matrix is said to be an involuntary matrix. Example,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Scanned with CamScanner

4.1.1 Determinants

The expression $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a determinant of the second order and stands for '$a_1b_2 - a_2b_1$'. It contains four numbers a_1, b_1, a_2, b_2 (called elements) which are arranged along two horizontal lines (called rows) and two vertical lines (called columns).

Similarly,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{..... (1)}$$

is called a determinant of the third order. It consists of nine elements which are arranged in three rows and three columns.

In general, a determinant of the n^{th} order is of the form,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

which is a block of n^2 elements in the form of a square along n rows and n rows and n columns. The diagonal through the left-hand top corner which contains the elements a_{11}, a_{22}, a_{33}, ..., a_{nn} is called the leading diagonal.

Expansion of a Determinant

The cofactor of an element in a determinant is the determinant obtained by deleting the row and column which intersect at that element, with the proper sign. The sign of an element in the i^{th} row and j^{th} column is $(-1)^{i+j}$. The cofactor of an element is usually denoted by the corresponding capital letter.

For example, the cofactor of b_3 in (1) is

$$B_3 = (-1)^{3+2} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

A determinant can be expanded in terms of any row for column as follows: Multiply each element of the row (or column) in terms of which we intend expanding the determinant, by its cofactor and then add up all these products.

∴ Expanding (1) by R_1 (*i.e.,* 1^{st} row)

$$\Delta = a_1A_1 + b_1B_1 + c_1C_1$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Scanned with CamScanner

Similarly, expanding by C_2(*i.e.*, 2^{nd} column),

$$\Delta = b_1B_1 + b_2B_2 + b_3B_3$$

$$= -b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} - b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$= b_1(a_2c_3 - a_3c_2) - b_2(a_1c_3 - a_3c_1) + b_3(a_1c_2 - a \cdot)$$

Basic Properties

i) A determinant remains unaltered by changing its rows into columns and columns into rows.

ii) If two parallel lines of a determinant are interchanged, the determinant retains its numerical value but changes in sign.

iii) A determinant vanishes if two of its parallel lines are identical.

iv) If each element of a line is multiplied by the same factor, the whole determinant is multiplied by that factor.

v) If each element of a line consists of m terms, the determinant can be expressed as the sum of m determinants.

vi) If to each element of a line, there can be added equi-multiplies of the corresponding elements of one or more parallel lines, the determinant remains unaltered.

For instance,

$$\begin{vmatrix} a_1 + pb_1 - qc_1 & b_1 & c_1 \\ a_2 + pb_2 - qc_2 & b_2 & c_2 \\ a_3 + pb_3 - qc_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + p\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} - q\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}$$

$$= \Delta + 0 + 0 \qquad [\because \text{From (iii) property}]$$

$$= \Delta$$

Example 4.1

Solve the equation:

$$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$$

Solution:

Operating $R_3 - (R_1 + R_2)$, we get,

$$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$$

Operate $R_2 - R_1$ and $(R_1 + R_3)$,

$$\begin{vmatrix} x+2 & 2x+4 & 6x+12 \\ x+1 & x+1 & x+1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$$

Scanned with CamScanner

or, $(x+1)(x+2)\begin{vmatrix} 0 & 2 & 6 \\ 1 & 1 & 1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$

Operate $R_1 - R_2$,

$\therefore \quad (x+1)(x+2)\begin{vmatrix} 0 & 1 & 5 \\ 1 & 1 & 1 \\ 1 & 1 & 3x+8 \end{vmatrix} = 0$

Expanding by C_1,

$\therefore \quad -(x+1)(x+2)(3x+8-5) = 0$

or, $-3(x+1)(x+2)(x+1) = 0$

Hence, $x = -1, -1, -2$.

NOTE:

1. In general, $AB \neq BA$ even if both exist.
2. If A be a square matrix, then the product AA is defined as A^2. Similarly, $A \cdot A^2 = A^3$ etc.

Related Matrices

A. Transpose of a matrix

The matrix obtained from a given matrix A, by interchanging rows and columns is called the transpose of A and is denoted by A¢.

NOTE:

i) For a symmetric matrix, A¢ = A and for skew-symmetric matrix, A¢ = –A.

ii) The transpose of the product of two matrices is the product of their transposes taken in the reverse order,

i.e., $(AB)' = B'A'$

iii) Any square matrix A can be written as,

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = B + C \text{ (say)}$$

Such that;

$$B' = \frac{1}{2}(A + A')' = \frac{1}{2}(A' + A) = B$$

i.e., B is a symmetric matrix.

and, $C' = \frac{1}{2}(A - A')' = \frac{1}{2}(A' + A) = -C$

i.e., C is a skew-symmetric matrix.

Thus, every square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix

B. Adjoint of a square matrix A

Adjoint of a square matrix A is the transposed matrix of cofactors of A and is written as adj A. Thus the adjoint of the matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ is } \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

Scanned with CamScanner

C. Inverse of a matrix

If A is a non-singular matrix of order n, then a squa matrix B of the same order such that AB = BA = I, is then called the inve f A being the unit matrix.

The inverse of A is written as A^{-1} so that $AA^{-1} = A^{-1}A$

Also,

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

NOTE:

i) Inverse of a matrix, when it exists is unique.

ii) $(A^{-1})^{-1} = A$

iii) $(AB)^{-1} = B^{-1}A^{-1}$

Rank of a Matrix

If we select any r rows and r columns from any matrix A, deleting all other rows and columns, then the determinant formed by these r × r elements is called the minor of A of order r. Clearly, there will be a number of different minors of the same order, got by deleting different rows and columns from the same matrix.

A matrix is said to be of rank r when,

i). it has at least one non-zero minor of order r, and,

ii) every minor of order higher than r vanishes.

4.2 DIRECT METHODS OF SOLUTION OF LINEAR SIMULTANEOUS EQUATIONS

A. Gauss Elimination Method

In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution. The method is quite general and is well adapted for computer operations.

Consider the equations,

$$\left.\begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{array}\right\} \quad \text{..... (1)}$$

Step I: To eliminate x from the second and third equations

Assuming $a_1 \neq 0$, we eliminate x from the second equation by subtracting $\left(\frac{a_2}{a_1}\right)$ times the first equation from the second equation.

Similarly, we eliminate x from the third equation by eliminating $\left(\frac{a_3}{a_1}\right)$ times the first equation from the third equation. We thus get new system.

Assunting $a_1 \neq 0$, we eliminate x from the second equation by subtracting $\left(\frac{a_2}{a_1}\right)$ times the first equation from the second equation. Similarly, we eliminmate x from the third equation by eliminating $\left(\frac{a_3}{a_1}\right)$ times the first equation from the third equation. Thus,

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ b_3'y + c_3'z &= d_3' \end{aligned}\right\} \quad \text{..... (2)}$$

Here, the first equation is called the pivotal equation and a_1 is called the first pivot.

Step II: To eliminate y from third equation in (2)

Assuming $b_2' \neq 0$, we eliminate y from the third equation of (2) by subtracting $\left(\frac{b_3'}{b_2'}\right)$ multiplied by times the second equation from the third equation. We thus, get the new system,

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ c_3''z &= d_3'' \end{aligned}\right\} \quad \text{..... (3)}$$

Here, the second equation is the pivotal equation and b_2' is the new pivot.

Step III: To evaluate the unknowns

The values of x, y, z are found from the reduced system (3) by back substitution.

NOTE:

1. On writing the given equation as,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

i.e., $AX = D$

This method consists in transforming the coefficient matrix A to the upper triangular matrix by elementary row transformations only.

2. Clearly, this method will fail if any one of the pivots a_1, b_2' or c_3'' becomes zero. In such cases, we rewrite the equations in a different order so that the pivots are non-zero.

3. Partial and Complete Pivoting

In the first step, the numerically largest coefficient of x is chosen from all the equations and brought as the first pivot by interchanging the first equation with the equation having the largest coefficient of x. In the second step, the numerically largest coefficient of y is chosen from the remaining equations (leaving the first equation) and brought as the second pivot by interchanging the second equation with the equation having the largest coefficient of y. This process is continued until we arrive at the equation with the single variable. This modified procedure is called partial pivoting.

Scanned with CamScanner

Example 4.2

Apply Gauss elimination method to solve the equations:

$$x + 4y - z = -5;\ x + y - 6z = -12;\ 3x - y - z = 4$$

Solution:

We have,

$$x + 4y - z = -5 \quad \text{..... (1)}$$
$$x + y - 6z = -12 \quad \text{..... (2)}$$
$$3x - y - z = 4 \quad \text{..... (3)}$$

Operate (2) – (1) and (3) – 3(1) to eliminate x,

$$-3y - 5z = -7 \quad \text{..... (4)}$$
$$-13y + 2z = 19 \quad \text{..... (5)}$$

Operating (5) $-\frac{13}{3}$ (4) to eliminate y,

$$\frac{71}{3}z = \frac{148}{3} \quad \text{..... (6)}$$

By backward substitution, we get,

$$z = \frac{148}{71} = 2.0845$$

From (4),

$$y = \frac{7}{3} - \frac{5}{3}\left(\frac{148}{71}\right) = \frac{-81}{71} = -1.1408$$

From (1),

$$x = -5 - 4\left(\frac{-81}{71}\right) + \left(\frac{148}{71}\right) = \frac{117}{71} = 1.6479$$

Hence, x = 1.6479, y = –1.1408 and z = 2.0845

Otherwise:

We have,

$$\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$$

Operating $R_2 - R_1$ and $R_3 - 3R_1$,

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & 5 \\ 0 & -13 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 19 \end{bmatrix}$$

Operating $R_3 - \frac{13}{3}R_2$,

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71/3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 148/3 \end{bmatrix}$$

Thus, we have,

$$z = \frac{148}{71} = 2.0845$$

Scanned with CamScanner

or, $3y = 7 - 5z = 7 - 10.4225 = -3.4225$

$\therefore$ $y = -1.1408$

and, $x = -5 - 4y + z = -5 + 4(1.1408) + 2.0845 = 1.6479$

$\therefore$ $x = 1.6479, y = -1.1408$ and $z = 2.0845$

Example 4.3

Using the Gauss elimination method, solve the equations:

$$x + 2y + 3z - u = 10,$$
$$2x + 3y - 3z - u = 1,$$
$$2x - y + 2z + 3u = 7,$$
$$3x + 2y - 4z + 3u = 2.$$

Solution:

We have,

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & -3 & -1 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 7 \\ 2 \end{bmatrix}$$

Operate $R_2 - 2R_1, R_3 - 2R_1, R_4 - 3R_1$,

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -9 & 1 \\ 0 & -5 & -4 & 5 \\ 0 & -4 & -13 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} -10 \\ -19 \\ -13 \\ -28 \end{bmatrix}$$

Operate $R_3 - 5R_2, R_4 - 4R_2$,

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -9 & 1 \\ 0 & 0 & 41 & 0 \\ 0 & 0 & 23 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 10 \\ -19 \\ 82 \\ 48 \end{bmatrix}$$

Thus, we have,

$41z = 82$

$\therefore$ $z = 2$

or, $23z + 2u = 48$ *i.e.*, $46 + 2u = 48$ $\therefore u = 1$

or, $-y - 9z + u = -19$ *i.e.*, $-y - 18 + 1 = -19$ $\therefore y = 2$

or, $x + 2y + 3z - u = 10$ *i.e.*, $x + 4 + 6 - 1 = 10$ $\therefore x = 1$

Hence, $x = 1, y = 2, z = 2$ and $u = 1$.

B. Gauss-Jordan Method

This is the modification of the Gauss elimination method. In this method, elimination of unknowns is performed not in the equation below but in the equations above also, ultimately reducing the system to a diagonal matrix form i.e., each equation involving only one unknown. From these equations, the unknowns x, y, z can be obtained readily. Thus, in this method, the labor of back-substitution for finding the unknowns is saved at the cost of additional calculations.

Scanned with CamScanner

Example 4.4

Apply the Gauss-Jordan method to solve the equations:

$x + y + z = 9;$

$2x - 3y + 4z = 13;$

$3x + 4y + 5z = 40$

Solution:

Writing the equations as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

Operate $R_2 - 2R_1, R_3 - 3R_1$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 13 \end{bmatrix}$$

Operate $R_3 + \frac{1}{5}R_2$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 12/5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 12 \end{bmatrix}$$

Operate $- R_2 + 5R_3$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 60 \end{bmatrix}$$

Operate $R_3 + \frac{1}{6}R_3, \frac{1}{12}R_3$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \\ 5 \end{bmatrix}$$

Operate $\frac{1}{5}R_2$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 5 \end{bmatrix}$$

Operate $R_1 - R_2 - R_3$,

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

Hence, $x = 1, y = 3$ and $z = 5$.

NOTE:

Here the process of elimination of variables amounts to reducing the given coefficient matrix to a diagonal matrix by elementary row transformation only.

Scanned with CamScanner

4.3 METHOD OF FACTORIZATION

I. Triangular Factorization Method or Dolittle Method

The coefficient matrix A of a system of linear equations can be factorized (or decomposed) into two triangular matrices L and U such that,

$$A = LU \quad \text{..... (1)}$$

where, $L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}$

and, $U = \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ 0 & U_{22} & \cdots & U_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & U_{nn} \end{bmatrix}$

L is known as lower triangular matrix and U is known as upper triangular matrix.

Once A is factorized into L and U, the system of equation

$$Ax = b$$

can be expressed as follows,

$$(LU)x = b$$

or, $$L(Ux) = b \quad \text{..... (2)}$$

Let us assume that,

$$Ux = z \quad \text{..... (3)}$$

where, z is an unknown vector-replacing equation (2) in equation (1), we get,

$$Lz = b \quad \text{..... (4)}$$

Now, we can solve the system,

$$Ax = b$$

in two stages:

1. Solve the equation $Lz = b$
   For z by forward substitution.
2. Solve the equation $Ux = z$
   For x using z (found in stage 1) by back substitution.

The elements of L and U can be determined by comparing the elements of the product of L and U with those of A. The process produces a system of n^2 equations with $n^2 + n$ unknowns (l_{ij} and m_{ij}) and, therefore, L and U are not unique. In order to produce unique factors, we should reduce the number of unknowns by n.

This is done by assuming the diagonal elements of L or U to be unity. The decomposition with L having unit diagonal values is called the Dolittle LU decomposition while the other one with U having unit diagonal elements is called the Crout LU decomposition.

Scanned with CamScanner

Dolittle Algorithm

We can solve for the components of L and U, given A as follows:

A = LU

Implies that,

$a_{ij} = l_{i1}u_{1j} + l_{i2}u_{2j} + \dots + l_{ii}u_{ij}$ for $i < j$ ..... (5)

$a_{ij} = l_{i1}u_{1j} + l_{i2}u_{2j} + \dots + l_{ii}u_{jj}$ for $i = j$ ..... (6)

$a_{ij} = l_{i1}u_{1j} + l_{i2}u_{2j} + \dots + l_{ij}u_{jj}$ for $i > j$ ..... (7)

where, $u_{ij} = 0$ for $i > j$ and, $l_{ij} = 0$ for $i < j$

The Dolittle algorithm assumes that all the diagonal elements of L are unity. That is,

$l_{ii} = 1, \quad i = 1, 2, 3, \dots, n$

Using equations (5), (6) and (7), we can successively determine the elements of U and L as follows:

If $i \le j$,

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} U_{kj} \qquad j = 1, 2, 3, \dots, n$$

where, $u_{11} = a_{11}$, $u_{12} = a_{12}$, $u_{13} = a_{13}$

Similarly,

If $i > j$,

$$l_{ij} = \frac{1}{u_{jj}} \times \left[a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj} \right] \qquad j = 1, 2, 3, \dots, i-1$$

where, $l_{11} = l_{22} = l_{33} = 1$

and $\quad l_{i1} = \frac{a_{i1}}{u_{11}} \qquad$ for i = 2 to n

Note that, for computing any element, we need the values of elements in the previous columns as well as the values of elements in the column above that element. This suggest that we should compute the elements, column by column from left to right within each column from top to bottom.

Example 4.5

Solve the system,

$3x_1 + 2x_2 + x_3 = 10$

$2x_1 + 3x_2 + 2x_3 = 14$

$x_1 + 2x_2 + 3x_3 = 14$

by using Dolittle LU decomposition method.

Solution:

Factorization:

For $i = 1$, $l_{11} = 1$ and

$u_{11} = a_{11} = 3$

$u_{12} = a_{12} = 2$

$u_{13} = a_{13} = 1$

Scanned with CamScanner

For i = 2,

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{2}{3} \quad \text{and} \quad l_{22} = 1$$

$$u_{22} = a_{22} - l_{21}u_{12} = 3 - \frac{2}{3} \times 2 = \frac{5}{3}$$

$$u_{23} = a_{23} - l_{21}u_{13} = 2 - \frac{2}{3} \times 1 = \frac{4}{3}$$

For i = 3,

$$l_{31} = \frac{a_{31}}{u_{11}} = \frac{1}{3}$$

$$l_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}} = \frac{2 - \left(\frac{1}{3}\right) \times 2}{\left(\frac{5}{3}\right)} = \frac{4}{5}$$

$$l_{33} = 1$$

$$u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

$$= 3 \times \frac{1}{3} \times 1 - \frac{4}{5} \times \frac{4}{3} = \frac{24}{15}$$

Thus, we have,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 24/15 \end{bmatrix}$$

Forward substitution:

Solving $lz = b$ by forward substitution, we get,

$$z_1 = b_1 = 10$$

$$z_2 = b_2 - l_{21}z_1 = 14 - \frac{2}{3} \times 10 = \frac{22}{3}$$

$$z_3 = b_3 - l_{31}z_1 - l_{32}z_2 = 14 - \frac{1}{3} \times 10 - \frac{4}{5} \times \frac{22}{3} = \frac{72}{15}$$

Back substitution:

Solving $Ux = z$ by back substitution, we get,

$$x_3 = \frac{\left(\frac{27}{15}\right)}{\left(\frac{24}{15}\right)} = 3$$

$$x_2 = \frac{z_2 - u_{23}x_3}{u_{22}} = \frac{\left(\frac{22}{3}\right) - \left(\frac{4}{3}\right) \times 3}{\left(\frac{5}{3}\right)} = 2$$

Scanned with CamScanner

$\therefore \quad x_1 = \frac{z_1 - u_{12}x_2 - u_{13}x_3}{u_{11}} = \frac{10 - (2 \times 2) - 1 \times 3}{3} = 1$

II. Crout Algorithm

Another approach to LU decomposition is Crout algorithm. Crout algorithm assumes unit diagonal values for U matrix and the diagonal elements of L matrix may assume any values as shown below.

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & \cdots & u_{1n} \\ 0 & 1 & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

We can use an approach that is similar to the one used in Dolittle decomopsition to evaluate the elements of L and U.

III. Cholesky Method

In case A is symmetric, the LU decomposition can be modified so that the upper factor is the transpose of the lower one or vice-versa. That is, we can factorize as,

$$A = LL^T$$

$$\text{or,} \quad A = U^TU \qquad \text{..... (1)}$$

Just as for Dolittle decomposition, by multiplying the terms of equation (1) and setting them equal to each other, the following recurrence relations can be obtained.

$$\left.\begin{aligned} u_{ii} &= \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2} && (i = 1 \text{ to } n) \\ u_{ij} &= \frac{1}{u_{ii}}\left[a_{ij} - \sum_{k=1}^{i-1} u_{ki}\,u_{kj}\right] && (j > 1) \end{aligned}\right\} \qquad \text{..... (2)}$$

This decomposite is called the Cholesky's factorization or the method of square roots.

Algorithm for Cholesky's factorization

1. Given n, A
2. Set $u_{11} = \sqrt{a_{11}}$
3. Set $u_{1j} = \frac{a_{1j}}{u_{11}}$ for j = 2 to n
4. For j = 2 to n,
   For i = 2 to j
   Sum = aij
   For k = 1 to i − 1
   Sum = sum − $u_{ki}\,u_{kj}$

Repeat k

Set $u_{ij} = \frac{sum}{u_k}$ if $i < j$

Set $u_{ij} = \sqrt{sum}$ if $i = j$

Repeat i

Repeat j

5. End of factorization.

Example 4.6

Factorize the matrix using Cholesky's method

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix}$$

Solution:

We have,

$$u_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2} \qquad (i = 1 \text{ to } n)$$

$$u_{ij} = \frac{1}{u_{ii}}\left[a_{ij} - \sum_{k=1}^{i-1} u_{ki}\, u_{kj}\right] \qquad (j > 1)$$

For i = 1,

$$u_{11} = \sqrt{1} = 1$$

$$u_{12} = \frac{a_{12}}{u_{11}} = \frac{2}{1} = 2$$

$$u_{13} = \frac{a_{13}}{u_{11}} = \frac{3}{1} = 3$$

For i = 2,

$$u_{22} = \sqrt{a_{22} - u_{12}^2} = \sqrt{8 - 4} = 2$$

$$u_{23} = \frac{a_{23} - u_{12}u_{13}}{u_{22}} = \frac{22 - 2 \times 3}{2} = \frac{16}{2} = 8$$

For i = 3,

$$u_{33} = \sqrt{a_{33} - u_{13}^2 - u_{23}^2} = \sqrt{82 - 9 - 64} = 3$$

Hence, $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix}$

4.4 THE INVERSE OF A MATRIX

The inverse of a matrix A is written as A^{-1} so that $AA^{-1} = A^{-1}A = I$. Thus the inverse of a matrix exists if and only if it is a non-singular square matrix. Also inverse of a matrix, when it exists is unique.

Scanned with CamScanner

A. Gauss Elimination method

In this method, we take a unit matrix of the same order as the given matrix A and write it as AI. Now making the simultaneous row operations on AI, we try to convert A into an upper triangular matrix and then to a unit matrix. Ultimately, when A is transformed into a unit matrix, the adjacent matrix (emerged out from the transformation of I) gives the inverse of A. To increase the accuracy, the largest element in A is taken as the pivot element for performing the row operations.

B. Gauss-Jordan Method

This is similar to the guess elimination method except that instead of first converting A into upper triangular form, it is directly converted into the unit matrix.

In practice, the two matrices A and I are written side by side and the same row transformations are performed on both. As soon as A is reduced to I, the other matrix represents A^{-1}.

Example 4.7

Find the inverse of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Solution:

Here;

$$|A| = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

and, $$\text{adj } A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

Hence, $$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

Example 4.8

Using Gauss-Jordan method, find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Solution:

Writing the given matrix side by side with the unit matrix of order 3. We have,

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

(Operate $R_2 - R_1$ and $R_3 + 2R_1$)

$$\sim\begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 0 & 2 & -6 & : & -1 & 1 & 0 \\ 0 & -2 & 2 & : & 2 & 0 & 1 \end{bmatrix}$$

$\left(\text{Operate } \frac{1}{2}R_2 \text{ and } \frac{1}{2}R_3\right)$

$$\sim\begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & -3 & : & -1/2 & 1/2 & 0 \\ 0 & 1 & 1 & : & 1 & 0 & 1/2 \end{bmatrix}$$

(Operate $R_1 - R_2$ and $R_3 + R_2$)

$$\sim\begin{bmatrix} 1 & 0 & 6 & : & 3/2 & 1/2 & 0 \\ 0 & 1 & -3 & : & -1/2 & 1/2 & 0 \\ 0 & 0 & -2 & : & 1/2 & 1/2 & 1/2 \end{bmatrix}$$

$\left(\text{Operate } R_1 + 3R_3,\ R_2 - \frac{3}{2}R_3 \text{ and } -\frac{1}{2}R_2\right)$

$$\sim\begin{bmatrix} 1 & 0 & 0 & : & 3 & 1 & 3/2 \\ 0 & 1 & 0 & : & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & : & -1/4 & -1/4 & -1/4 \end{bmatrix}$$

Hence the inverse of the given matrix is

$$\begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

Example 4.9

Using Gauss-Jordan method, find the inverse of the matrix

$$\begin{bmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$$

Solution:

Writing the given matrix side by side with the unit matrix of order 3. We have,

$$\begin{bmatrix} 2 & 2 & 3 & : & 1 & 0 & 0 \\ 2 & 1 & 1 & : & 0 & 1 & 0 \\ 1 & 3 & 5 & : & 0 & 0 & 1 \end{bmatrix}$$

$\left(\text{Operate } \frac{1}{2}R_1\right)$

$$\sim\begin{bmatrix} 1 & 1 & 3/2 & : & 1/2 & 0 & 0 \\ 2 & 1 & 1 & : & 0 & 1 & 0 \\ 1 & 3 & 5 & : & 0 & 0 & 1 \end{bmatrix}$$

(Operate $R_2 - 2R_1$, $R_3 - R_1$)

$$\sim\begin{bmatrix} 1 & 1 & 3/2 & : & 1/2 & 0 & 0 \\ 0 & -1 & -2 & : & -1 & 1 & 0 \\ 0 & 2 & 7/2 & : & -1/2 & 0 & 1 \end{bmatrix}$$

(Operate $R_1 + R_2$, $R_3 + 2R_2$)

Scanned with CamScanner

$$\sim\begin{bmatrix} 1 & 0 & -1/2 & : & -1/2 & 1 & 0 \\ 0 & -1 & -2 & : & -1 & 1 & 0 \\ 0 & 0 & -1/2 & : & -5/2 & 2 & 1 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 0 & -1/2 & : & -1/2 & 1 & 0 \\ 0 & 1 & 2 & : & 1 & -1 & 0 \\ 0 & 0 & -1/2 & : & -5/2 & 2 & 1 \end{bmatrix}$$

(Operate $(-2)\,R_3$)

$$\sim\begin{bmatrix} 1 & 0 & -1/2 & : & -1/2 & 1 & 0 \\ 0 & 1 & 2 & : & 1 & -1 & 0 \\ 0 & 0 & 1 & : & 5 & -4 & -2 \end{bmatrix}$$

$\left(\text{Operate } R_1 + \frac{1}{2}R_3,\ R_2 - 2R_3\right)$

$$\sim\begin{bmatrix} 1 & 0 & 0 & : & 2 & -1 & -1 \\ 0 & 1 & 0 & : & -9 & 7 & 4 \\ 0 & 0 & 1 & : & 5 & -4 & -2 \end{bmatrix}$$

Hence the inverse of the given matrix is,

$$\begin{bmatrix} 2 & -1 & -1 \\ -9 & 7 & 4 \\ 5 & -4 & -2 \end{bmatrix}$$

C. Factorization Method

In this method, we factorize the given matrix as $A = LU$ ..... (1)

where, L is a lower triangular matrix with unit diagonal elements and U is an upper triangular matrix.

i.e., $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Now, (1) gives,

$$A^{-1} = (LU)^{-1} = U^{-1}L^{-1} \quad \text{..... (2)}$$

To find L^{-1}, let $L^{-1} = X$, where, X is a lower triangular matrix.

Then, $LX = I$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying the matrices on the LHS and equating the corresponding elements, we have,

$$x_{11} = 1,\ x_{22} = 1,\ x_{33} = 1 \quad \text{..... (3)}$$

$$l_{21}x_{11} + x_{21} = 0,\ l_{31}x_{11} + l_{32}x_{21} + x_{31} = 0$$

and, $l_{32}x_{22} + x_{32} = 0$ ..... (4)

Equation (3) gives,

$$x_{11} = x_{22} = x_{33} = 1$$

Equation (4) gives,

$$x_{21} = -l_{21}x_{11} + x_{21},\ x_{31} = -(l_{31} + l_{32}x_{21}) \text{ and } x_{31} = -l_{32}$$

Scanned with CamScanner

Thus, $L^{-1} = X$ is completely determined.

To find U^{-1}, let $(L^{-1} = Y$, where Y is an upper triangular matrix.

Then, $YU = I$

i.e., $$\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & 0 & y_{33} \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

multiplying the matrices on the LHS and then equating the corresponding elements, we have,

$$y_{11}u_{11} = 1, y_{22}u_{22} = 1, y_{33}u_{33} = 1 \qquad \text{..... (5)}$$

$$\left.\begin{aligned} & y_{11}u_{12} + y_{12}u_{22} = 0 \\ & y_{11}u_{13} + y_{12}u_{23} + y_{13}u_{33} = 0 \\ \text{and, } & y_{22}u_{13} + y_{23}u_{33} = 0 \end{aligned}\right\} \qquad \text{..... (6)}$$

From (5),

$$y_{11} = \frac{1}{u_{11}};\ y_{22} = \frac{1}{u_{22}} \text{ and } y_{33} = \frac{1}{u_{33}}$$

From (6),

$$y_{12} = -y_{11}\frac{u_{12}}{u_{22}}$$

$$y_{13} = -\frac{y_{11}u_{13} + y_{12}u_{23}}{u_{33}}$$

$$y_{23} = -\frac{y_{22}u_{23}}{u_{33}}$$

$\therefore$ We get, $U^{-1} = Y$ completely.

Hence, by (2), we get A^{-1}.

Example 4.10

Using the factorization method, find the inverse of the matrix

$$A = \begin{bmatrix} 50 & 107 & 36 \\ 27 & 54 & 20 \\ 31 & 66 & 21 \end{bmatrix}$$

Solution:

Taking $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$

and, $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Now,

$A = LU$

or, $$\begin{bmatrix} 50 & 107 & 36 \\ 25 & 54 & 20 \\ 31 & 66 & 21 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Scanned with CamScanner

$\therefore \quad 50 = u_{11} \;;\; 107 = u_{12} \;;\; 36 = u_{13}$

$25 = l_{21}u_{11} \;;\; 54 = l_{21}u_{12} \;;\; 20 = l_{21}u_{13} + u_{23}$

$31 = l_{31}u_{11} \;;\; 66 = l_{21}u_{12} + l_{32}u_{22} \;;\; 21 = l_{31}u_{13} + l_{32}u_{23} + u_{33}$

or, $u_{11} = 50 \;;\; u_{12} = 107 \;;\; u_{13} = 36 \;;\; l_{21} = \frac{1}{2} \;;$ [illegible] $= \frac{1}{2} \;;$

$u_{23} = 3 \;;\; l_{31} = \frac{31}{50} \;;\; l_{32} = \frac{-17}{25} \;;\; u_{23} = \frac{1}{25}$

Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 31/50 & 17/25 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 50 & 107 & 36 \\ 0 & 1/2 & 2 \\ 0 & 0 & 1/25 \end{bmatrix}$$

To find L^{-1}, let $L^{-1} = X$. Then $LX = I$

i.e., $$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 31/50 & -17/25 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore \quad x_{11} = 1, \frac{1}{2}x_{11} + x_{21} = 0$

$x_{22} = 1, \frac{31}{50}x_{11} - \frac{17}{25}x_{21} + x_{31} = 0$

$\frac{-17}{25}x_{22} + x_{32} = 0, x_{33} = 1$

or, $x_{11} = x_{22} = x_{33} = 1$

$x_{21} = -\frac{1}{2}, x_{31} = -\frac{24}{25}, x_{32} = \frac{17}{25}$

Thus,

$$L^{-1} = X = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -24/25 & 17/25 & 1 \end{bmatrix}$$

To find U^{-1}, let $U^{-1} = Y$. Then $YU = 1$

i.e., $$\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & 0 & y_{33} \end{bmatrix} \begin{bmatrix} 50 & 107 & 36 \\ 0 & 1/2 & 2 \\ 0 & 0 & 1/25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore \quad 50y_{11} = 1, \; 50y_{12} + 107y_{22} = 0, \; 50y_{13} + 107y_{23} + 36y_{33} = 0$

$\frac{1}{2}y_{22} = 1, \; \frac{1}{2}y_{23} + 2y_{33} = 0, \; \frac{1}{25}y_{33} = 1$

or, $y_{11} = \frac{1}{50}, \; y_{22} = 2, \; y_{33} = 25, \; y_{12} = \frac{-107}{25}, \; y_{23} = -100, \; y_{13} = 196$

Hence, $$U^{-1} = \begin{bmatrix} 1/50 & -107/25 & 196 \\ 0 & 2 & -100 \\ 0 & 0 & 25 \end{bmatrix}$$

Scanned with CamScanner

so, $A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1/50 & -107/25 & 196 \\ 0 & 2 & -100 \\ 0 & 0 & 25 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -24/25 & 17/25 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -186 & 129 & 196 \\ 95 & -66 & -100 \\ -24 & 17 & 25 \end{bmatrix}$$

4.5 ILL-CONDITIONED EQUATIONS

A linear system is said to be ill-conditioned if small changes in the coefficient of the equations result in large changes in the values of the unknowns. On the contrary, a system is well-conditioned if small changes in the coefficients of the system also produce small changes in the solution. We often come across ill-conditioning of a system is usually expected. When the determinant of the coefficient matrix is small. The coefficient matrix of an ill-conditioned system is called an ill-conditioned matrix.

While solving simultaneous equation we also come across two forms of instabilities; Inherent and induced. Inherent instability of a system is a property of the given problem and occurs due to the problem being ill conditioned. It can be avoided by reformulation of the problem suitably. Induced instability occurs because of the incorrect choice of method.

Iterative method to improve accuracy of an ill-conditioned system

Consider the system of equations,

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \quad \text{..... (1)}$$

Let x', y', z' be an approximate solution. Substituting these values on the left hand sides, we get new values of d_1, d_2, d_3 as d_1', d_2', d_3' so that the new system is,

$$\left.\begin{aligned} a_1x' + b_1y' + c_1z' = d_1' \\ a_2x' + b_2y' + c_2z' = d_2' \\ a_3x' + b_3y' + c_3z' = d_3' \end{aligned}\right\} \quad \text{..... (2)}$$

Subtracting each equation in (2), from the corresponding equations in (1), we get,

$$\left.\begin{aligned} a_1x_e + b_1y_e + c_1z_e = k_1 \\ a_2x_e + b_2y_e + c_2z_e = k_2 \\ a_3x_e + b_3y_e + c_3z_e = k_3 \end{aligned}\right\} \quad \text{..... (3)}$$

where, $x_e = x - x'$

$y_e = y - y'$

$z_e = z - z_1'$

$k_1 = d_1 - z_1'$

Scanned with CamScanner

We now solve the system (3) for x_e, y_e, z_e giving $x = x' + x_e$, $y = y' + y_e$ and $z = z' + z_e$ which will be better approximation for x, y, z. We can repeat the procedure for improving the accuracy.

Example 4.11

Establish whether the system 1.01x + 2y = 2.01; x + 2y = 2 is ill conditioned or not?

Solution:

It's solution is $x = 1$ and $y = 0.5$

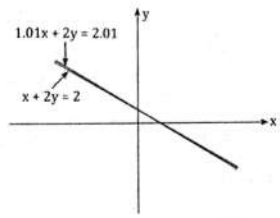
Now, consider the system,

$$x + 2.01y = 2.04$$

and, $x + 2y = 2$

which has the solution $x = -6$ and $y = 4$.

Hence the system is ill-conditioned.



4.6 ITERATIVE METHODS OF SOLUTION

Iterative method is that in which we start from an approximation to the true solution and obtain better and better approximation from a computation cycle repeated as often as may be necessary for achieving a desired accuracy. Thus in an iterative method, the amount of computation depends on the degree of accuracy required.

For large systems, iterative methods may be faster than the direct methods. Even the round-off errors in iterative methods are smaller. In fact, iteration is a self correcting process and any error made at any stage of computation gets automatically in the subsequent steps.

Simple iterative methods can be devised for systems in which the coefficients of the leading diagonal are large as compared to others.

4.6.1 Jacobi's Iteration Method

Consider the equation,

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \quad \text{..... (1)}$$

Scanned with CamScanner

If a_1, b_2, c_3 are large as compared to other coefficients, solve the system can be written as,

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \quad \text{.....(2)}$$

Let us start with the initial approximations x_0, y_0, z_0 for the values of x, y, z respectively. Replacing these on the right sides of (2), the first approximations are given by

$$x = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

$$y = \frac{1}{b_2}(d_2 - a_2x_0 - c_2z_0)$$

$$z = \frac{1}{c_3}(d_3 - a_3x_0 - b_3y_0)$$

Replacing values of x_1, y_1, z_1 on the right sides of (2), the second approximations are given by,

$$x_2 = \frac{1}{a_1}(d_1 - b_1y_1 - c_1z_1)$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

This process is repeated until the difference between the consecutive approximations is negligible.

> **NOTE:**
> In the absence of any better estimates for x_0, y_0, z_0 these may each be taken as zero.

Example 4.12

Solve by Jacobi's iteration method:

$$20x + y - 2x = 17;$$
$$3x + 20y - z = -18;$$
$$2x - 3y + 20z = 25$$

Solution:

We write the given equations in the form,

$$\left.\begin{aligned} x &= \frac{1}{20}(17 - y + 2z) \\ y &= \frac{1}{20}(-18 - 3x + z) \\ z &= \frac{1}{20}(25 - 2x + 3y) \end{aligned}\right\} \quad \text{.....(2)}$$

Scanned with CamScanner

Let, $x_0 = y_0 = z_0 = 0$,

Replacing these on the right sides of the equations (1), we get,

$$x_1 = \frac{17}{20} = 0.85, \quad y_1 = \frac{18}{20} = -0.9, \quad z_1 = \frac{25}{20} = 1.25$$

Putting these values on the right sides of the equation (1), we obtain,

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.02$$

$$y_2 = \frac{1}{20}(-18 - 3x_1 + z_1) = -0.965$$

$$z_2 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.03$$

Replacing values on the right sides of the equations (1), we have,

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.00125$$

$$y_3 = \frac{1}{20}(-18 - 3x_2 + z_2) = 1.0015$$

$$z_3 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 1.00325$$

Replacing values, we get,

$$x_4 = \frac{1}{20}(17 - y_3 + 2z_3) = 1.0004$$

$$y_4 = \frac{1}{20}(-18 - 3x_3 + z_3) = -1.000025$$

$$z_4 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 0.9965$$

Putting these values, we have,

$$x_5 = \frac{1}{20}(-17 - y_4 + 2z_4) = 0.999966$$

$$y_5 = \frac{1}{20}(-18 - 3x_4 + z_4) = -1.000078$$

$$z_5 = \frac{1}{20}(25 - 2x_4 + 3y_4) = 0.999956$$

Again, substituting these values, we get,

$$x_6 = \frac{1}{20}(-17 - y_5 + 2z_5) = 1.0000$$

$$y_6 = \frac{1}{20}(-18 - 3x_5 + z_5) = 0.999997$$

$$z_6 = \frac{1}{20}(25 - 2x_5 + 3y_5) = 0.999992$$

The values in the fifth and sixth iterations being practically the same, we can stop. Hence the solution is,

$x = 1, y = -1$ and $z = 1$

Scanned with CamScanner

4.6.2 Gauss Siedal Iteration Method

This is a modification of Jocobi's method. As before, the system of equations,

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \quad \text{.....(1)}$$

is written as,

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \quad \text{.....(2)}$$

Here, we start with the initial approximations x_0, y_0, z_0 for x, y, z respectively which may each be taken as zero. Replacing $y = y_0$, $z = z_0$ in the first of the equations (2), we get,

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

Then putting $x = x_1$, $z = z_0$ in the second of the equation (2), we have,

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_0)$$

Next substituting $x = x_1$, $y = y_1$ in the third of the equation (2), we have,

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

and so on, *i.e.*, as soon as a new approximations for an unknown is found, it is immediately used in the next step. This process of iteration is repeated until the values of x, y, z are obtained to a desired degree of accuracy.

NOTE:

1. Jacobi and Gauss Siedal methods converge for any choice of the initial approximations if in each equation of the system, the absolute value of the largest coefficient is almost equal to or is atleast one equation greater than the sum of the absolute values of all the remaining coefficients.
2. The convergence in the Gauss-Siedal method is twice as fast as in Jacobi's method.

Example 4.13

Apply the Gauss-Siedal method to solve the equations:

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Solution:

Writing the given equations as,

$$x = \frac{1}{20}(17 - y + 2z) \quad \text{.....(1)}$$

Scanned with CamScanner

$$y = \frac{1}{20}(-18 - 3x + z) \quad \text{..... (2)}$$

$$z = \frac{1}{20}(25 - 2x + 3y) \quad \text{..... (3)}$$

First iteration by putting,

$y = y_0, z = z_0$ in equation (1), we get,

$$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = 0.8500$$

$x = x_1, z = z_0$ in equation (2), we get,

$$y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = -1.0275$$

$x = x_1, y = y_1$ in equation (2), we get,

$$z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.0109$$

Second iteration by putting,

$y = y_1, z = z_1$ in equation (1), we get,

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.0025$$

$x = x_2, z = z_1$ in equation (2), we get,

$$y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$$

$x = x_2, y = y_2$ in equation (2), we get,

$$z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 0.9998$$

Third iteration by putting,

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0000$$

$$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -1.0000$$

$$z_3 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 1.0000$$

The values in the second and third iterations being practically the same, we can stop the iterations. Hence the solution of given equations is,

$x = 1, y = -1$ and $z = 1$

Example 4.14

Solve the equation

$$27x + 6y - z = 85$$
$$x + y + 54z = 110$$
$$6x + 15y + 2z = 72$$

by the Gauss Jacobi and the Gauss Seidal method.

Scanned with CamScanner

Solution:

Writing the given equations as,

$$x = \frac{1}{27}(85 - 6y + z) \quad \text{---(1)}$$

$$y = \frac{1}{15}(72 - 6x - 2z) \quad \text{---(2)}$$

$$z = \frac{1}{54}(110 - x - y) \quad \text{---(3)}$$

a) **Gauss-Siedal's Method**

Starting from an approximation $x_0 = y_0 = z_0 = 0$.

First iteration:

$$x_1 = \frac{85}{27} = 3.148$$

$$y_1 = \frac{72}{15} = 4.8$$

$$z_1 = \frac{110}{54} = 2.037$$

Second iteration:

$$x_2 = \frac{1}{27}(85 - 6y_1 + z_1) = 2.157$$

$$y_2 = \frac{1}{15}(72 - 6x_1 - y_1) = 3.269$$

$$z_2 = \frac{1}{54}(110 - x_1 - y_1) = 1.890$$

Third iteration:

$$x_3 = \frac{1}{27}(85 - 6y_2 + 7z_2) = 2.492$$

$$y_3 = \frac{1}{15}(72 - 6x_2 - 2z_2) = 3.685$$

$$z_3 = \frac{1}{54}(110 - x_2 - y_2) = 1.937$$

Fourth iteration:

$$x_4 = \frac{1}{27}(85 - 6y_3 + z_3) = 2.401$$

$$y_4 = \frac{1}{15}(72 - 6x_3 - 2y_3) = 3.545$$

$$z_4 = \frac{1}{54}(110 - x_3 - y_3) = 1.923$$

Fifth iteration:

$$x_5 = \frac{1}{27}(85 - 6y_4 + z_4) = 2.432$$

Scanned with CamScanner

$$y_5 = \frac{1}{15}(72 - 6x_4 - 2y_4) = 3.583$$

$$z_5 = \frac{1}{54}(110 - x_4 - y_4) = 1.927$$

On repeating this process,

$x_6 = 2.423$,	$y_6 = 3.570$,	$z_6 = 1.926$
$x_7 = 2.426$,	$y_7 = 3.574$,	$z_7 = 1.926$
$x_8 = 2.425$,	$y_8 = 3.573$,	$z_8 = 1.926$
$x_9 = 2.426$,	$y_9 = 3.573$,	$z_9 = 1.926$

Hence, $x = 2.426$, $y = 3.573$ and $z = 1.926$.

b) **Gauss-Jacobi's Method**

First iteration by putting,

$y = y_0 = 0$, $z = z_0 = 0$ in equation (1), we get,

$$x_1 = \frac{1}{27}(85 - 6y_0 + z_0) = 3.14$$

$x = x_1$, $z = z_0$ in equation (2), we get,

$$y_1 = \frac{1}{15}(72 - 6x_1 - 2z_0) = 3.541$$

$x = x_1$, $y = y_1$ in equation (3), we get,

$$z_1 = \frac{1}{54}(110 - x_1 - y_1) = 1.913$$

Second iteration:

$$x_2 = \frac{1}{27}(85 - 6y_1 + z_1) = 2.432$$

$$y_2 = \frac{1}{15}(72 - 6x_2 - 2z_1) = 3.572$$

$$z_2 = \frac{1}{54}(110 - x_2 - y_2) = 1.926$$

Third iteration:

$$x_3 = \frac{1}{27}(85 - 6y_2 + z_2) = 2.426$$

$$y_3 = \frac{1}{15}(72 - 6x_3 - 2z_2) = 3.573$$

$$z_3 = \frac{1}{54}(110 - x_3 - y_3) = 1.926$$

Fourth iteration:

$$x_4 = \frac{1}{27}(85 - 6y_3 + z_3) = 2.426$$

$$y_4 = \frac{1}{15}(72 - 6x_4 - 2z_3) = 3.573$$

Scanned with CamScanner

$$z_4 = \frac{1}{54}(110 - x_4 - y_4) = 1.926$$

Hence, $x = 2.426, y = 3.573$ and $z = 1.926$

4.6.3 Relaxation Method

Consider the system of equations,

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

We define the residuals R_x, R_y and R_z by the relations,

$$\left.\begin{aligned} R_x &= d_1 - a_1x - b_1y - c_1z \\ R_y &= d_2 - a_2x - b_2y - c_2z \\ R_z &= d_3 - a_3x - b_3y - c_3z \end{aligned}\right\} \quad \text{..... (1)}$$

To start with, we assume $x = y = z = 0$ and calculate the initial residuals. The the residuals are reduced step by step, by giving increments to the variables. For this purpose, we construct the following operation table,

	δR_x	δR_y	δR_z
$\delta x = 1$	$-a_1$	$-a_2$	$-a_3$
$\delta y = 1$	$-b_1$	$-b_2$	$-b_3$
$\delta z = 1$	$-c_1$	$-c_2$	$-c_3$

We note from the equations (1) that if x is increased by (1) (Keeping y and z constant), R_x, R_y and R_z decreases by a_1, a_2, a_3 respectively. This is shown in the above table along with the effects on the residuals when y and z are given unit increments. (Table is the transpose of the coefficient matrix).

At each step, the numerically largest residual is reduced to almost zero. To reduce a particular residual, the value of the corresponding variable is changes. *e.g.*, to reduce R_x by p, x should be increased by $\frac{p}{a_1}$.

When all the residuals have been reduced to almost zero, the increments in x, y, z are added separately to give the desired solutions.

NOTE:

1. As a result, the computed values of x, y, z are substituted in (1) and the residuals are calculated. If these residuals are not all negligible, then there is some mistake and the entire process should be rechecked.
2. Relaxation method can be applied successfully only if the diagonal elements of the coefficient matrix dominate the other coefficients in the corresponding row i.e., if in the equations (1),

$$|a_1| \geq |b_1| + |c_1|$$
$$|b_2| \geq |a_2| + |c_2|$$
$$|c_3| \geq |a_3| + |b_3|$$

where, > sign should be valid for atleast one row.

Scanned with CamScanner

Example 4.15

Solve the equation by relaxation method.

$9x - 2y + z = 50$

$x + 5y - 3z = 18$

$-2x + 2y + 7z = 19$

Solution:

$R_x = 50 - 9x + 2y - z$

$R_y = 18 - x - 5y + 3z$

$R_z = 19 + 2x - 2y - 7z$

The operation table is,

	δR_x	δR_y	δR_z
$\delta x = 1$	-9	-1	2
$\delta y = 1$	2	-5	-2
$\delta z = 1$	-1	-3	-7

The relaxation table is,

	R_x	R_y	R_z	
x = y = z = 0	50	18	19	i
$\delta x = 5$	5	13	29	ii
$\delta z = 14$	1	25	1	iii
$\delta y = 5$	11	0	-9	iv
$\delta x = 1$	2	-1	-7	v
$\delta z = -1$	3	-4	0	vi
$\delta y = -0.8$	1.4	0	1.6	vii
$\delta y = 0.23$	1.17	0.69	-0.69	viii
$\delta y = 0.13$	0	0.56	0.17	ix
$\delta y = 0.112$	0.224	0	-0.054	x

$\Sigma\delta x = 6.13$, $\Sigma\delta y = 4.31$, $\Sigma\delta z = 3.23$

Hence, $x = 6.13$, $y = 4.31$ and $z = 3.23$.

In (i), the largest residual is 50. To reduce it, we give an increment $\delta_x = 5$ and the resulting residuals are shown in (ii) of these $R_z = 29$ is the largest and we give an increment $\delta_z = 4$ to get the results in (iii). In (vi), $R_y = -4$ is the numerically) largest value and we give and increment $\delta_y = -\frac{4}{5} = -0.8$ to obtain the results in (vii). Similarly, the other steps have been carried out.

4.7 POWER METHOD

A. Eigen Values and Eigen Vectors

If A is any square matrix of order n with elements a_{ij}, we can find a column matrix X and a constant λ such that $AX = \lambda X$ or $AX - \lambda IX = 0$ or $[A - \lambda I]X = 0$.

Scanned with CamScanner

This matrix equation represents n homogenous linear equations,

$$\left.\begin{array}{l}(a_{11}-\lambda)x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \dots\dots + a_{2n}x_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1} + a_{n2}x_2 + \dots\dots + (a_{nn}-\lambda)x_n = 0\end{array}\right\} \quad \text{---- (1)}$$

which will have a non-trivial solution only if the coefficient determinant vanishes *i.e.*,

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{12}-\lambda & \dots\dots & a_{2n} \\ a_{n1} & a_{n2} & \dots\dots & a_{nn}-\lambda \end{vmatrix} = 0 \quad \text{---- (2)}$$

On expansion, it gives on n^{th} degree equation in λ, called the characteristic equation of the matrix A. If roots λ_i (i = 1, 2, 3, 4, ..., n) are called the Eigen values or latent roots and corresponding to each eigen value, the equation (2) will have a non-zero solution.

$$X = [x_1, x_2, x_3, \dots\dots, x_n]'$$

which is known as the eigen vector. Such an equation can ordinarily be solved easily. However, for larger systems, better methods are to be applied.

Cayley-Hamilton Theorem

Every square matrix satisfies its own characteristic equations. *i.e.*, if the characteristic equation for the nth order square matrix A is,

$$|A - \lambda I| = (-1)^n \lambda^n + k_1\lambda^{n-1} + \dots\dots + k_n = 0$$

Then, $(-1)^n A^n + k_1A^{n-1} + k_n = 0$

Example 4.16

Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Solution:

The characteristic equation is $[A - \lambda I] = 0$

i.e., $\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} = 0$

or, $\lambda^2 - 7\lambda + 6 = 0$

or, $(\lambda - 6)(\lambda - 1) = 0$

$\therefore$ $\lambda = 6, 1$

Hence, the eigen values are 6 and 1.

If, x, y be the components of an eigen vector corresponding to the eigen value λ, then,

$$[A - \lambda I]\, X = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Scanned with CamScanner

Corresponding to $\lambda = 6$, we have,

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

which gives only one independent equation $-x + 4y = 0$

$\therefore \quad \frac{x}{4} = \frac{y}{1}$ giving the eigen vector (4, 1)

Corresponding to $\lambda = 1$, we have $\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

which gives only one independent equation $x + y = 0 \Rightarrow x = -y$

$\therefore \quad \frac{x}{1} = \frac{y}{-1}$ giving the eigen vector (1, –1).

Example 4.17

Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 3 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution:

The characteristic equation is,

$$|A - \lambda I| = \begin{bmatrix} 8-\lambda & -6 & 3 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$$

$$= \lambda^3 + 18\lambda^2 - 45\lambda = 0$$

or, $\quad \lambda(\lambda - 3)(\lambda - 15) = 0$

$\therefore \quad \lambda = 0, 3, 15$

Thus the eigen values of A are 0, 3, 15.

If x, y, z be the components of an eigen vector corresponding to the eigen value λ, we have,

$$(A - \lambda)\,IX = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad \text{---- (1)}$$

Putting $\lambda = 0$, we have,

$8x - 6y + 2z = 0$, $-6x + 7y - 4z = 0$, $2x - 4y + 3z = 0$. These equations determine a single linearly independent solution which may be taken as (1, 2, 2) so that every non-zero multiple of this vector is an eigen vector corresponding to $\lambda = 0$.

Similarly, the eigen vectors corresponding to $\lambda = 3$ and $\lambda = 15$ are the arbitary non-zero multiples of the vectors (2, 1, –2) and (2, –2, 1) which are obtained from (1).

Hence the three eigen vectors may be taken as (1, 2, 2), (2, 1, –2) and (2, –2, 1).

Scanned with CamScanner

B. Properties of Eigen Values

i) The sum of the eigen values of the matrix A is the sum of the elements of its principal diagonal.

ii) If λ is an eigen value of matrix A, then $\frac{1}{\lambda}$ is the eigen value of A^{-1}.

iii) If λ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its eigen value.

iv) If $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ are the eigen values of matrix A, then A^m has the eigen values $\lambda 1^m, \lambda_2 1^m, ..., \lambda_n^m$ (m being a positive integer).

v) Any similarity transformation applied to a matrix leaves its eigen-values unchanged.

vi) If a square matrix A has a linearly independent eigen vectors, then a matrix P can be found such that P^{-1} AP is a diagonal matrix whose diagonal elements are the eigen values of A.

The transformation of A by a non-singular matrix P to P^{-1} AP is called a similarity transformation.

C. Power Method

If $X_1, X_2, ..., X_n$ are the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, ..., \lambda_n$, then an orbitrary column vector can written as,

$$X = k_1X_1 + k_2X_2 + ... + k_nX_n$$

Then, $AX = k_1AX_1 + k_2AX_2 + ... + k_nAX_n$

$$= k_1\lambda_1X_1 + k_2\lambda_2X_2 + ... + k_n\lambda_nX_n$$

Similarly,

$$A^2X = k_1\lambda_1^2X_1 + k_2\lambda_2^2X_2 + ... + k_n\lambda_n^2X_n$$

and, $A^rX = k_1\lambda_1^rX_1 + k_2\lambda_2^rX_2 + ... + k_n\lambda_n^rX_n$

If $|\lambda_1| > |\lambda_2| > .... > |\lambda_n|$, then λ_1 is the largest root and the contribution of the term $k_1\lambda_1^rX_1$ to the sum on the right increases with r and therefore, every time we multiply a column vector by A, it becomes nearer to the eigen vector X_1. Then we make the largest component of the resulting column vector unity to avoid the factor k_1.

Thus, we start with a column vector X which is as near the solution as possible and evaluate AX which is written as λ^1X^1 after normalization. This gives the first approximation λ^1 to the eigen value and X^1 to the eigen vector. Similarly, we evaluate $AX^1 = \lambda^2X^2$ which gives the second approximation. We repeat this process until $[X^r - X^{r-1}]$ becomes negligible. Then λ^r will be the largest eigen value and X^r, the corresponding eigen vector.

This iterative procedure for finding the dominant eigen value of a matrix is known as Rayleigh's power method.

Scanned with CamScanner

NOTE:
We have, $AX = \lambda X$
as $A^{-1}AX = \lambda A^{-1}$
or, $X = \lambda A^{-1}X$
We know,

$$A^{-1}X = \frac{1}{\lambda}X$$

If we use this equation, then the above method yields the smallest eigen value.

If we use this equation, then the above method yields the smallest eigen value.

Example 4.18

Determine the largest eigen value and the corresponding eigen vector of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

Solution:

Let the initial approximations to the eigen vector corresponding to the largest eigen value of A be $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Then, $AX = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \lambda^1 X^1$

So the 1st approximation to the eigen value is $\lambda^1 = 5$ and the corresponding eigen vector is $X^1 = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}$.

Now,

$$AX^1 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.8 \\ 14 \end{bmatrix} = 5.8\begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = \lambda^2 X^2$$

Thus the second approximation to the eigen value is $\lambda^2 = 5.8$ and the corresponding eigen vector is $X^2 = \begin{bmatrix} 1 \\ 0.241 \end{bmatrix}$.

Repeating the above process. We get,

$$AX^2 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = 5.966\begin{bmatrix} 1 \\ 0.248 \end{bmatrix} = \lambda^3 X^3$$

$$AX^3 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.248 \end{bmatrix} = 5.966\begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = \lambda^4 X^4$$

$$AX^4 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = 5.999\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^5 X^5$$

$$AX^5 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^6 X^6$$

Clearly, $\lambda^5 = \lambda^6$ and $X^5 = X^6$ upto 3 decimal places. Hence the largest eigen value is 6 and the corresponding eigen vector is $\begin{bmatrix} 1 \\ 0.25 \end{bmatrix}$.

Scanned with CamScanner

Example 4.19

Find the largest eigen value and the corresponding eigen vector of the matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

using the power method. Take $[1, 0, 0]^T$ as the initial eigen vector.

Solution:

Let the initial approximation to the required eigen vector be X[1, 0, 0]'.

Then,

$$AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^1 X^1$$

So the first approximation to the eigen value is 2 and the corresponding eigen vector

$$X(1) = [1, -0.5, 0]$$

Hence,

$$AX^1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \lambda^2 X^2$$

Repeating the above process, we get,

$$AX^2 = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \lambda^3 X^3$$

$$AX^3 = 3.43\begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = \lambda^4 X^4$$

$$AX^4 = 3.41\begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix} = \lambda^5 X^5$$

$$AX^5 = 3.41\begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = \lambda^6 X^6$$

$$AX^6 = 3.41\begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix} = \lambda^7 X^7$$

Clearly, $\lambda^6 = \lambda^7$ and $X^6 = X^7$ approximately,

Hence, the largest eigen value is 3.41 and the corresponding eigen vector is [0.74, −1, 0.67]'.

Example 4.20

Obtain by the power method, the numerically dominant eigen value and eigen vector of the matrix A.

$$A = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$$

Scanned with CamScanner

Solution:

Let the initial approximation to the required eigen vector be X[1, 1, 1]'.

Then,

$$AX = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ -18 \end{bmatrix} = -18\begin{bmatrix} -0.444 \\ 0.222 \\ \; \end{bmatrix} = \lambda^1 X^1$$

So the first approximation to eigen value is −18 and the corresponding eigen vector is [−0.444, 0.222, 1]'.

Now,

$$AX^1 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} -0.444 \\ 0.222 \\ 1 \end{bmatrix} = -10.548\begin{bmatrix} 1 \\ -0.105 \\ -0.736 \end{bmatrix} = \lambda^2 X^2$$

Therefore, the second approximation to the eigen value is −10.548 and the eigen vector is [1, −0.105, −0.736]'

Repeating the process,

$$AX^2 = -18.948\begin{bmatrix} -0.930 \\ 0.361 \\ 1 \end{bmatrix} = \lambda^3 X^3$$

$$AX^3 = -18.394\begin{bmatrix} 1 \\ -0.415 \\ -0.981 \end{bmatrix} = \lambda^4 X^4$$

$$AX^4 = -19.698\begin{bmatrix} -0.995 \\ 0.462 \\ 1 \end{bmatrix} = \lambda^5 X^5$$

$$AX^5 = -19.773\begin{bmatrix} 1 \\ -480 \\ -0.999 \end{bmatrix} = \lambda^6 X^6$$

$$AX^6 = -19.922\begin{bmatrix} -0.997 \\ 0.490 \\ 1 \end{bmatrix} = \lambda^7 X^7$$

$$AX^7 = -19.956\begin{bmatrix} 1 \\ -495 \\ -0.999 \end{bmatrix} = \lambda^8 X^8$$

Since $\lambda^7 = \lambda^8$ and $X^7 = X^8$ approximately, hence the dominant eigen value and the corresponding eigen vector are given by,

$$\lambda^8 X^8 = 19.956\begin{bmatrix} 1 \\ 495 \\ 0.999 \end{bmatrix} \text{ i.e., } 20\begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix}$$

Hence, the dominant eigen value is 20 and eigen vector is [−1, 0.5, 1]'.

Scanned with CamScanner

BOARD EXAMINATION SOLVED QUESTIONS

1. **Find the inverse of the given matrix by applying Gauss Elimination Method (GEM) with partial pivoting technique.**

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$

[2013/Fall]

Solution: Given that;

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$

Using partial pivoting technique so, arranging the matrix as

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 4 & 1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$$

Now, the augmented matrix is given by

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right]$$

Operate $R_2 \to R_2 - 4R_1$ and $R_3 \to R_3 - 2R_1$

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 9 & -6 & -4 & 1 & 0 \\ 0 & 7 & -5 & -2 & 0 & 1 \end{array}\right]$$

Operate $R_3 \to R_3 - \frac{7}{9} R_2$

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 9 & -6 & -4 & 1 & 0 \\ 0 & 0 & -1/3 & 10/9 & -7/9 & 1 \end{array}\right]$$

Now, $\begin{bmatrix} 1 & -2 & 2 \\ 0 & 9 & -6 \\ 0 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 10/9 \end{bmatrix}$

$$\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 9 & -6 \\ 0 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ 10/9 \end{bmatrix}$$

$\therefore \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 2.33 \\ -2.66 \\ -3.33 \end{bmatrix}$

Also, $\begin{bmatrix} 1 & -2 & 2 \\ 0 & 9 & -6 \\ 0 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -7/9 \end{bmatrix}$

$\therefore \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} -1.33 \\ 1.66 \\ 2.33 \end{bmatrix}$

Scanned with CamScanner

And, $\begin{bmatrix} 1 & -2 & 2 \\ 0 & 9 & -6 \\ 0 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\therefore \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}$

Hence, the inverse of matrix is

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 2.33 & -1.33 & 2 \\ -2.66 & 1.66 & -2 \\ -3.33 & 2.33 & -3 \end{bmatrix}$$

2. Solve the following system of equations by applying Gauss-Seidal iterative method. Carry out the iterations upto 6^th^ stage

$28x + 4y - z = 32$

$x + 3y + 10z = 24$

$2x + 17y + 4z = 35$ **[2013/Fall]**

Solution:

Arranging the equations such that magnitude of all the diagonal element is greater than the sum of magnitude of other two elements in the row i.e.,

$28x + 4y - z = 32$

$2x + 17y + 4z = 35$

$x + 3y + 10z = 24$

$\begin{bmatrix} |28| > |4| + |-1| \\ |17| > |2| + |4| \\ |10| > |1| + |3| \end{bmatrix}$

Forming the equations as

$$x = \frac{32 - 4y + z}{28}$$

$$y = \frac{35 - 2x - 4z}{17}$$

$$z = \frac{24 - x - 3y}{10}$$

Let initial guess be 0 for x, y and z.

Solving the iterations in tabular form.

NOTE: Use the most recent values obtained to find the next one in this method.

Iteration	$x = \frac{32 - 4y + z}{28}$	$y = \frac{35 - 2x - 4z}{17}$	$z = \frac{24 - x - 3y}{10}$
Guess	0	0	0
1	$\frac{32 - 4(0) + 0}{28}$ = 1.142	$\frac{35 - 2(1.142) - 4(0)}{17}$ = 1.924	$\frac{24 - 1.142 - 3(1.924)}{10}$ = 1.708
2	0.929	1.547	1.843
3	1.130	1.492	1.839
4	0.995	1.509	1.847
5	0.993	1.507	1.848
6	1.136	1.490	1.839

Scanned with CamScanner

NOTE:
Procedure to iterate in programmable calculator
Let, A = x, B = y, C = z
Step 1: Set the following in calculator

$$A = \frac{31 - 4B + C}{28};\ B = \frac{35 - 2A - 4C}{17};\ C = \frac{24 - A - 3B}{10}$$

Step 2: Press CALC then
enter the value of B? then press =
enter the value of C? then press =
Step 3: Now press = only, again and again to get the values for respective row for each column.
Step 4: The values are updated automatically so continue pressing = till the required number of iterations.

3. **Solve the following system of equations using Gauss elimination method.**

$$10x_1 - 7x_2 + 3x_3 + 5x_4 = 6$$
$$-6x_1 + 8x_2 - x_3 + 4x_4 = 5$$
$$3x_1 + x_2 + 4x_3 + 11x_4 = 2$$
$$5x_1 - 9x_2 - 2x_3 + 4x_4 = 7$$

[2013/Spring]

Solution:

Writing the given system of equations in matrix form,

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ -6 & 8 & -1 & -4 \\ 3 & 1 & 4 & 11 \\ 5 & -9 & -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 7 \end{bmatrix}$$

Operate $R_2 \to R_2 - \left(-\frac{6}{10}\right)R_1,\ R_3 \to R_3 - \frac{3}{10}R_1,\ R_4 \to R_4 - \frac{5}{10}R_1$

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 3.1 & 3.1 & 9.5 \\ 0 & -5.5 & -3.5 & 1.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8.6 \\ 0.2 \\ 4 \end{bmatrix}$$

Operate $R_3 \to R_3 - \frac{3.1}{3.8}R_2,\ R_4 \to R_4 - \frac{-5.5}{3.8}R_2$

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44 & 10.31 \\ 0 & 0 & -2.34 & 0.05 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8.6 \\ -6.81 \\ 16.44 \end{bmatrix}$$

Operate $R_4 \to R_4 - \left(-\frac{2.34}{2.44}\right)R_3$

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44 & 10.31 \\ 0 & 0 & 0 & 0.993 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8.6 \\ -6.81 \\ 9.90 \end{bmatrix}$$

Scanned with CamScanner

Now, performing back substitution,

$9.93x_4 = 9.90$

∴ $x_4 = 0.99 \approx 1$

$2.44x_3 + 10.31x_4 = -6.81$

or, $2.44x_3 = -6.81 - 10.31 \times -1$

∴ $x_3 = -7.01 \approx -7$

$3.8x_2 + 0.8x_3 - x_4 = 8.6$

or, $3.8x_2 + 0.8(-7) + 1 = 8.6$

∴ $x_2 = 3.47 \approx 3.5$

$10x_1 - 7x_2 + 3x_3 + 5x_4 = 6$

or, $10x_1 - 7(3.5) + 3(-7) + 5(1) = 6$

∴ $x_1 = 4.65$

4. **Determine the highest even value and its corresponding eigen vector for the following matrix using power method.**

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$$ **[2013/Spring]**

Solution:

Let the vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then, the iterations are carried out as,

$$AX_0 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 13 \end{bmatrix}$$

The highest value in AX_0 is 13 so dividing each element by 13.

$$AX_0 = 13\begin{bmatrix} 0.2307 \\ 0.6923 \\ 1 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.2307 \\ 0.6923 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.3076 \\ 6.0767 \\ 12.5385 \end{bmatrix} = 12.5385\begin{bmatrix} 0.1042 \\ 0.4864 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.1042 \\ 0.4864 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5634 \\ 5.2854 \\ 11.8414 \end{bmatrix} = 11.8414\begin{bmatrix} 0.0475 \\ 0.4463 \\ 1 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.0475 \\ 0.4463 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.3864 \\ 5.0351 \\ 11.7377 \end{bmatrix} = 11.7377\begin{bmatrix} 0.0329 \\ 0.4289 \\ 1 \end{bmatrix}$$

Scanned with CamScanner

$$AX_4 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.0329 \\ 0.4289 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.3196 \\ 4.9565 \\ 11.6827 \end{bmatrix} = 11.6827\begin{bmatrix} 0.0273 \\ 0.4242 \\ 1 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.0273 \\ 0.4242 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2999 \\ 4.9303 \\ 11.6695 \end{bmatrix} = 11.6695\begin{bmatrix} 0.0256 \\ 0.4224 \\ 1 \end{bmatrix}$$

Hence, the required eigen value $11.6695 \approx 12$.

And, required eigen vector = $\begin{bmatrix} 0.0256 \\ 0.4224 \\ 1 \end{bmatrix}$.

NOTE:
Procedure to solve in programmable calculator
Step 1: Press MODE then select MATRIX by pressing 6.
Step 2: Select MatA by pressing 1 and select 3 × 3 by pressing 1.
Step 3: Initialize the given matrix from the question.
Step 4: Press SHIFT then 4(MATRIX) and select Dim by pressing 1.
Step 5: Select MatB by pressing 2 and select 3 × 1 by pressing 3.
Step 6: Initialize the initial vector value and press AC.
Step 7: Press SHIFT then 4(MATRIX) and select MatA by pressing 3 and then press Multiply (×).
Step 8: Press SHIFT then 4(MATRIX) and select MatB by pressing 4 and then press =
Step 9: Now find the largest value in matrix and then press Divide (÷) and enter the largest value and then press =
Step 10: Now for next iteration press AC
Step 11: Press SHIFT then 4(MATRIX) and select MatA by pressing 3 then Multiply(×).
Step 12: Press SHIFT then 4(MATRIX) and select MatAns by pressing 6 and then press =
Step 13: Go to step 9.

5. **Using Factorization method, solve the following system of linear equations:**

$$3x + 2y + 7z = 4$$
$$2x + 3y + z = 5$$
$$3x + 4y + z = 7$$

[2013/Spring]

Solution:

In matrix form

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} \quad \text{i.e., } AX = B$$

In factorization method, we represent A as

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Scanned with CamScanner

Solving for unknown values

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{12}u_{11} & u_{12}l_{21} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

$u_{11} = 3$	$u_{12} = 2$	$u_{13} = 7$
$l_{21} = \frac{2}{3} = 0.667$	$2 \times 0.667 + u_{22} = 3$ $\therefore u_{22} = 1.666$	$0.667 \times 7 + u_{23} = 1$ $\therefore u_{23} = -3.669$
$l_{31} = \frac{3}{3} = 1$	$l \times 2 + l_{32}(1.666) = 4$ $\therefore l_{32} = 1.2$	$1 \times 7 + 1.2(-3.669) + u_{33} = 1$ $\therefore u_{33} = -1.597$

Now, substituting obtained coefficients, we have overall system of

LUX = B

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.667 & 1 & 0 \\ 1 & 1.2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 7 \\ 0 & 1.666 & -3.669 \\ 0 & 0 & -1.597 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

Let UX = V then

LV = B

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.667 & 1 & 0 \\ 1 & 1.2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

Using forward substitution

$\therefore$ $v_1 = 4$

or, $0.667v_1 + v_2 = 5$

$\therefore$ $v_2 = 2.332$

or, $1v_1 + 1.2v_2 + v_3 = 7$

$\therefore$ $v_3 = 0.201$

Using the obtained values at UX = V

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 1.666 & -3.669 \\ 0 & 0 & -1.597 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2.332 \\ 0.201 \end{bmatrix}$$

Using backward substitution

$\therefore$ $z = \frac{0.201}{-1.597} = -0.125$

or, $1.666y - 3.669z = 2.332$

$\therefore$ $y = 1.174$

or, $3x + 2y + 7z = 4$

$\therefore$ $x = 0.842$

6. **Solve the following system of equations by applying Gauss Elimination Method (GEM) with partial pivoting technique. And also determine the determinant value.**

Scanned with CamScanner

$$2x + 2y + z = 6$$
$$4x + 2y + 3z = 4$$
$$x - y + z = 0$$

[2014/Fall]

Solution:

By partial pivoting technique, the system of linear equation can be arranged as,

$$4x + 2y + 3z = 4$$
$$2x + 2y + z = 6$$
$$x - y + z = 0$$

The augmented matrix can be written as

$$[A : B] = \begin{bmatrix} 4 & 2 & 3 & : & 4 \\ 2 & 2 & 1 & : & 6 \\ 1 & -1 & 1 & : & 0 \end{bmatrix}$$

Operate $R_1 \rightarrow R_1 - 3R_3$, $R_2 \rightarrow R_2 - 2R_3$

$$= \begin{bmatrix} 1 & 5 & 0 & : & 4 \\ 0 & 4 & -1 & : & 6 \\ 1 & -1 & 1 & : & 0 \end{bmatrix}$$

Operate $R_3 \rightarrow R_3 - R_1$

$$= \begin{bmatrix} 1 & 5 & 0 & : & 4 \\ 0 & 4 & -1 & : & 6 \\ 0 & -6 & 1 & : & -4 \end{bmatrix}$$

Operate $R_2 \rightarrow \frac{R_2}{4}$

$$= \begin{bmatrix} 1 & 5 & 0 & : & 4 \\ 0 & 1 & -1/4 & : & 3/2 \\ 0 & -6 & 1 & : & -4 \end{bmatrix}$$

Operate $R_3 \rightarrow R_3 + 6R_2$

$$[A : B] = \begin{bmatrix} 1 & 5 & 0 & : & 4 \\ 0 & 1 & -1/4 & : & 3/2 \\ 0 & 0 & -1/2 & : & 5 \end{bmatrix}$$

Performing back substitution,

$$-\frac{1}{2}z = 5$$

$\therefore$ $z = -10$

Then, $y - \frac{1}{4}z = \frac{3}{2}$

or, $y + \frac{10}{4} = \frac{3}{2}$

$\therefore$ $y = -1$

and, $x + 5y + 0 = 4$

or, $x + 5(-1) = 4$

$\therefore$ $x = 9$

Scanned with CamScanner

Also, determinant value

$$\begin{vmatrix} 4 & 2 & 3 \\ 2 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= 4\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= 4(2 + 1) - 2(2 - 1) + 3(-2 - 2)$$

$$= -2$$

7. Find the largest eigen value and the corresponding eigen vector correct upto 3 decimal places using power method for the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$ **[2014/Fall, 2017/Fall, 2019/Spring]**

Solution:

Let initial eigen vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Then the iterations are carried out as

$$AX_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} = 4\begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -3.5 \\ 2.5 \end{bmatrix} = 3.5\begin{bmatrix} 0.7142 \\ -1 \\ 0.7142 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.7142 \\ -1 \\ 0.7142 \end{bmatrix} = \begin{bmatrix} 2.4284 \\ -3.4284 \\ 2.4284 \end{bmatrix} = 3.4284\begin{bmatrix} 0.7083 \\ -1 \\ 0.7083 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.7083 \\ -1 \\ 0.7083 \end{bmatrix} = \begin{bmatrix} 2.4166 \\ -3.4166 \\ 2.4166 \end{bmatrix} = 3.4166\begin{bmatrix} 0.7073 \\ -1 \\ 0.7073 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.7073 \\ -1 \\ 0.7073 \end{bmatrix} = \begin{bmatrix} 2.4146 \\ -3.4146 \\ 2.4146 \end{bmatrix} = 3.4146\begin{bmatrix} 0.707 \\ -1 \\ 0.707 \end{bmatrix}$$

$$AX_7 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.707 \\ -1 \\ 0.707 \end{bmatrix} = \begin{bmatrix} 2.414 \\ -3.414 \\ 2.414 \end{bmatrix} = 3.414\begin{bmatrix} 0.707 \\ -1 \\ 0.707 \end{bmatrix}$$

Scanned with CamScanner

Hence the required eigen value is 3.414 correct upto 3 decimal places.

And required eigen vector = $\begin{bmatrix} 0.707 \\ -1 \\ 0.707 \end{bmatrix}$

8. Solve the following system of by using Gauss Seidal method.

$$10x - 5y - 2z = 3$$
$$x + 6y - 10z = -3$$
$$4x - 10y + 3z = -3$$

[2014/Fall]

Solution:

$$10x - 5y - 2z = 3$$
$$x + 6y - 10z = -3$$
$$4x - 10y + 3z = -3$$

Arranging the equations such that magnitude of all the diagonal element is greater than the sum of magnitude of other two elements in the row.

$$10x - 5y - 2z = 3$$
$$4x - 10y + 3z = -3$$
$$x + 6y - 10z = -3$$

Now, forming the equations as,

$$x = \frac{3 + 5y + 2z}{10}$$

$$y = \frac{-3 - 3z - 4x}{-10} = \frac{3 + 3z + 4x}{10}$$

$$z = \frac{-3 - x - 6y}{-10} = \frac{3 + x + 6y}{10}$$

Let initial guess be 0 for x, y and z.

Solving the iterations in tabular form.

Iteration	$x = \frac{3 + 5y + 2z}{10}$	$y = \frac{3 + 3z + 4x}{10}$	$z = \frac{3 + x + 6y}{10}$
Guess	0	0	0
1	0.3	0.42	0.582
2	0.6264	0.7251	0.7977
3	0.8220	0.8681	0.9030
4	0.9146	0.9367	0.9534
5	0.9590	0.9696	0.9776
6	0.9803	0.9854	0.9892
7	0.9905	0.9929	0.9947
8	0.9953	0.9965	0.9974
9	0.9977	0.9983	0.9987
10	0.9988	0.9991	0.9993
11	0.9994	0.9995	0.9996

Scanned with CamScanner

Hence the approximated values of x, y, and z is $0.999 \approx 1$.

NOTE:
Procedure to iterate in programmable calculator:
Let $A = x$, $B = y$, $C = z$
Set the following in calculator:

$$A = \frac{3 + 5B + 2C}{10} : B = \frac{3 + 3C + 4A}{10} : C = \frac{3 + A + 6B}{10}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required no. of iterations.

9. **Use Gauss Elimination Method to solve the equation. Use partial pivoting method where necessary.**

$$4x_1 + 5x_2 - 6x_3 = 28$$
$$2x_1 - 7x_3 = 29$$
$$-5x_1 - 8x_2 = -64$$

[2014/Spring]

Solution:

Writing the given system of equation in matrix form,

$$\begin{bmatrix} 4 & 5 & -6 \\ 2 & 0 & -7 \\ -5 & -8 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 28 \\ 29 \\ -64 \end{bmatrix}$$

Operate $R_2 \rightarrow R_2 - \left(\frac{2}{4}\right) R_1$ and $R_3 \rightarrow R_3 - \frac{(-5)}{4} R_1$

$$\begin{bmatrix} 4 & 5 & -6 \\ 0 & -2.5 & -4 \\ 0 & -1.75 & -7.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 28 \\ 15 \\ -29 \end{bmatrix}$$

Operate $R_3 \rightarrow R_3 - \frac{-1.75}{-2.5} R_2$

$$\begin{bmatrix} 4 & 5 & -6 \\ 0 & -2.5 & -4 \\ 0 & 0 & -4.7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 28 \\ 15 \\ -39.5 \end{bmatrix}$$

Now, performing back substitution

$-4.7x_3 = -39.5$

∴ $x_3 = 8.404$

$-2.5x_2 - 4x_3 = 15$

or, $-2.5x_2 - 4(8.404) = 15$

∴ $x_2 = -19.446$

$4x_1 + 5x_2 - 6x_3 = 28$

or, $4x_1 + 5(-19.446) - 6(8.404) = 28$

∴ $x_1 = 43.913$

10. **Find the largest eigen value λ and the corresponding eigen vector X of the matrix**

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

[2014/Spring]

Scanned with CamScanner

Solution:

Let initial eigen vector be $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

Then the iterations are carried out as

$$AX_0 = \begin{bmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\3\\-1\end{bmatrix} = 3\begin{bmatrix}1\\1\\-0.333\end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\1\\-0.333\end{bmatrix} = \begin{bmatrix}3\\3\\0.333\end{bmatrix} = 3\begin{bmatrix}1\\1\\0.111\end{bmatrix}$$

$$AX_2 = \begin{bmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\1\\0.111\end{bmatrix} = \begin{bmatrix}3\\3\\-0.111\end{bmatrix} = 3\begin{bmatrix}1\\1\\-0.037\end{bmatrix}$$

$$AX_3 = \begin{bmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\1\\-0.037\end{bmatrix} = \begin{bmatrix}3\\3\\0.037\end{bmatrix} = 3\begin{bmatrix}1\\1\\0.012\end{bmatrix}$$

$$AX_4 = \begin{bmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\1\\0.012\end{bmatrix} = \begin{bmatrix}3\\3\\-0.012\end{bmatrix} = 3\begin{bmatrix}1\\1\\-0.004\end{bmatrix}$$

Hence the largest eigen value λ is 3 and largest eigen vector is $\begin{bmatrix}1\\1\\0\end{bmatrix}$.

11. Solve the following by Gauss-Siedal Method

$b + 3c + 2d = 19$
$3b + 2c + 2d = 20$
$a + 4b + 2d = 17$
$-2a + 2b + c + d = 9$

Solution: **[2014/Spring]**

Here, the provided system is not diagonally dominant as the magnitude of all the diagonal element is not greater than the sum of magnitude of other elements in the row.

i.e., |coefficient of a| ≯ |sum coefficient of b, c and d|.

Hence we cannot solve for the convergence from this method.

If it is to be solved from other methods the acquired values a, b, c and d are:

a = 1 b = 2
c = 3 d = 4

12. Solve the following set of equation using LU factorization method.

$3x + 2y + z = 10$
$2x + 3y + 2z = 14$
$x + 2y + 3z = 14$

[2015/Fall, 2017/Fall, 2019/Spring]

Scanned with CamScanner

Solution:

Writing the equation in matrix from AX = B

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

Here, we represent A as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Solving for unknown values,

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & u_{12}l_{21} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$u_{11} = 3$	$u_{12} = 2$	$u_{13} = 1$
$l_{21} = \frac{2}{3} = 0.667$	$u_{12}l_{21} + u_{22} = 3$ $\therefore u_{22} = 1.666$	$0.667 \times 1 + u_{23} = 2$ $\therefore u_{23} = 1.333$
$l_{31} = \frac{1}{3} = 0.333$	$0.333 \times 2 + l_{32}(1.666) = 2$ $\therefore l_{32} = 0.8$	$0.333 \times 1 + 0.8 \times 1.333 + u_{33} = 1$ $\therefore u_{33} = 1.6$

Substituting the values,

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0.667 & 1 & 0 \\ 0.333 & 0.8 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1.666 & 1.333 \\ 0 & 0 & 1.6 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}}_{B}$$

Let LUX = B

$\Rightarrow$ UX = V

so, LV = B

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.667 & 1 & 0 \\ 0.333 & 0.8 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

Now, performing forward substitution,

$\therefore$ $v_1 = 10$

$\Rightarrow$ $0.667v_1 + v_2 = 14$

$\therefore$ $v_2 = 7.33$

$\Rightarrow$ $0.333v_1 + 0.8v_2 + v_3 = 14$

$\therefore$ $v_3 = 4.80$

Then, UX = V becomes

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1.666 & 1.333 \\ 0 & 0 & 1.6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 7.33 \\ 4.80 \end{bmatrix}$$

Scanned with CamScanner

Performing backward substitution,

$z = \frac{4.8}{1.6} = 3$

$\Rightarrow$ $1.666y - 1.333z = 7.33$

$\therefore$ $y = 1.99 \approx 2$

$\Rightarrow$ $3x + 2y + z = 10$

$\therefore$ $x = \frac{3.02}{3} = 1.02 \approx 1$

Hence, $x = 1$;

$y = 2$;

and, $z = 3$.

13. Use Gauss-Seidal iterative method to solve given equations.

$$40x - 20y - 10z = 390$$
$$10x - 60y + 20z = -280$$
$$10x - 30y + 120z = -860$$

[2015/Fall]

Solution:

Here the equations have the dominance of diagonal element so forming the equations as

$$x = \frac{390 + 20y + 10z}{40}$$

$$y = \frac{-280 - 10x - 20z}{-60}$$

$$z = \frac{-860 - 10x - 30y}{120}$$

Let the initial guess be 0 for x, y and z.

Now, solving the iteration in tabular form

Iteration	$x = \frac{390 + 20y + 10z}{40}$	$y = \frac{-280 - 10x - 20z}{-60}$	$z = \frac{-860 - 10x - 30y}{120}$
Guess	0	0	0
1	9.75	6.291	-9.551
2	10.507	3.234	-8.850
3	9.154	3.242	-8.74
4	9.186	3.284	-8.753
5	9.203	3.282	-8.754
6	9.202	.3282	-8.754
7	9.202	3.282	-8.754

Here, the values of x, y and z are correct upto 3 decimal places.

So the approximate values of $x = 9.202$, $y = 3.282$ and $z = -8.754$

Scanned with CamScanner

NOTE:
Procedure to iterate in programmable calculator
Let A = x, B = y, C = z
Set the following in calculator

$A = \frac{390 + 20B + 10C}{40}$: $B = \frac{280 + 10A + 20C}{60}$: $C = \frac{-860 - 10A - 30B}{120}$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required no. of iterations.

14. Find the eigen vlaue and corresponding eigen vector of given matrix

$$\begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

[2015/Fall]

Solution:

Let the initial vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then the iterations are carried out as

$$AX_0 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ -0.333 \\ 1 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -0.333 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0.666 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0.111 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.111 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -0.222 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ -0.037 \\ 1 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -0.037 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0.074 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0.012 \\ 1 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.012 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -0.024 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ -0.004 \\ 1 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -0.004 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0.008 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0.001 \\ 1 \end{bmatrix}$$

Hence the required eigen value = 6.

And the required eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

15. Find the largest eigen value and corresponding eigen vector of the following square matrix using power method.

$$\begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & 3 \\ 4 & 3 & 5 \end{bmatrix}$$

[2015/Spring]

Scanned with CamScanner

Solution:

Let the initial vector be $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

Then the iterations are carried out as

$$AX_0 = \begin{bmatrix}3 & 1 & 4\\1 & 2 & 3\\4 & 3 & 5\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}8\\6\\12\end{bmatrix} = 12\begin{bmatrix}0.667\\0.5\\1\end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix}3 & 1 & 4\\1 & 2 & 3\\4 & 3 & 5\end{bmatrix}\begin{bmatrix}0.667\\0.5\\1\end{bmatrix} = \begin{bmatrix}6.501\\4.667\\9.168\end{bmatrix} = 9.168\begin{bmatrix}0.709\\0.509\\1\end{bmatrix}$$

$$AX_2 = \begin{bmatrix}3 & 1 & 4\\1 & 2 & 3\\4 & 3 & 5\end{bmatrix}\begin{bmatrix}0.709\\0.509\\1\end{bmatrix} = \begin{bmatrix}6.636\\4.727\\9.363\end{bmatrix} = 9.363\begin{bmatrix}0.708\\0.504\\1\end{bmatrix}$$

$$AX_3 = \begin{bmatrix}3 & 1 & 4\\1 & 2 & 3\\4 & 3 & 5\end{bmatrix}\begin{bmatrix}0.708\\0.504\\1\end{bmatrix} = \begin{bmatrix}6.628\\4.716\\9.344\end{bmatrix} = 9.344\begin{bmatrix}0.709\\0.504\\1\end{bmatrix}$$

$$AX_4 = \begin{bmatrix}3 & 1 & 4\\1 & 2 & 3\\4 & 3 & 5\end{bmatrix}\begin{bmatrix}0.709\\0.504\\1\end{bmatrix} = \begin{bmatrix}6.631\\4.717\\9.348\end{bmatrix} = 9.348\begin{bmatrix}0.709\\0.504\\1\end{bmatrix}$$

$$AX_5 = \begin{bmatrix}3 & 1 & 4\\1 & 2 & 3\\4 & 3 & 5\end{bmatrix}\begin{bmatrix}0.709\\0.504\\1\end{bmatrix} = \begin{bmatrix}6.631\\4.717\\9.348\end{bmatrix} = 9.348\begin{bmatrix}0.709\\0.504\\1\end{bmatrix}$$

Hence the required eigen value = 9.348.

And the required eigen vector is $\begin{bmatrix}0.709\\0.504\\1\end{bmatrix}$.

16. Solve the following system of equation by the process of Gauss elimination. (Use partial pivoting if necessary)

$$3x + 2y + z = 10$$
$$2x + 3y + 2z = 14$$
$$x + 2y + 3z = 14$$

[2015/Spring]

Solution:

Writing given equations in matrix form

$$\begin{bmatrix}3 & 2 & 1\\2 & 3 & 2\\1 & 2 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}10\\14\\14\end{bmatrix}$$

Operate $R_2 \to R_2 - \frac{2}{3}R_1$ and $R_3 \to R_3 - \frac{1}{3}R_1$

$$\begin{bmatrix}3 & 2 & 1\\0 & 5/3 & 4/3\\0 & 4/3 & 8/3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}10\\22/3\\32/3\end{bmatrix}$$

Scanned with CamScanner

Operate $R_3 \rightarrow R_3 - \frac{4}{5}R_2$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 8/5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 22/3 \\ 24/5 \end{bmatrix}$$

Now, performing backward substitution,

or, $\frac{8}{5}z = \frac{24}{5}$

$\therefore$ $z = 3$

or, $\frac{5}{3}y + \frac{4}{3}z = \frac{22}{3}$

$\therefore$ $y = 2$

or, $3x + 2y + z = 10$

$x = \frac{10 - 2y - z}{3}$

$\therefore$ $x = 1$

17. **Use Gauss Seidal iteration method to solve**

$2x + y + z = 5$
$3x + 5y + 2z = 15$
$2x + y + 4z = 8$ **[2015/Spring]**

Solution:

Here the equations are in diagonally dominant form.

Forming the equations as

$x = \frac{5 - y - z}{2}$

$y = \frac{15 - 3x - 2z}{5}$

$x = \frac{8 - 2x - y}{4}$

Let initial guess be 0 for x, y and z.

Solving the iterations in tabular form.

Iteration	$x = \frac{5-y-z}{2}$	$y = \frac{15-3x-2z}{5}$	$z = \frac{8-2x-y}{4}$
Guess	0	0	0
1	2.5	1.5	0.375
2	1.562	1.912	0.741
3	1.173	1.999	0.931
4	1.044	2.008	0.976
5	1.008	2.004	0.995
6	1.0005	2.0017	0.9993
7	0.999	2.0008	1.0003
8	0.999	2.0008	1.0003

Scanned with CamScanner

Hence the required valued of x, y and z are 1, 2 and 1 respectively.

> **NOTE:**
> Procedure to iterate in programmable calculator
> Let A = x, B = y, C = z
> Set the following in calculator
> $$A = \frac{5 - B - C}{2};\ B = \frac{15 - 3A - 2C}{5};\ C = \frac{8 - 2A - B}{4}$$
> Now press CALC and enter the initial value of B and C and continue pressing = only for the required no. of iterations.

18. Solve the following system of equations by using Gauss elimination method with partial pivoting technique.

$$x + y + z + w = 2$$
$$x + y + 3z - 2w = -6$$
$$2x + 3y - z + 2w = 7$$
$$x + 2y + z - w = -2$$

[2016/Fall]

Solution:

Writing the system of equations in matrix form,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & -2 \\ 2 & 3 & -1 & 2 \\ 1 & 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 7 \\ -2 \end{bmatrix}$$

Interchanging R_1 and R_3 but not variable x and z as partial pivoting

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 1 & 1 & 3 & -2 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \\ 2 \\ -2 \end{bmatrix}$$

Operate $R_2 \to R_2 - \frac{1}{2}R_1,\ R_3 \to R_3 - \frac{1}{2}R_1,\ R_4 \to R_4 - \frac{1}{2}R_1$

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 0 & -0.5 & 3.5 & -3 \\ 0 & -0.5 & 1.5 & 0 \\ 0 & 0.5 & 1.5 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 7 \\ -9.5 \\ -1.5 \\ -5.5 \end{bmatrix}$$

Operate $R_3 \to R_3 - R_2,\ R_4 \to R_4 + R_2$

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 0 & -0.5 & 3.5 & -3 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 7 \\ -9.5 \\ 8 \\ -15 \end{bmatrix}$$

Interchanging R_3 and R_4 but not the variable z and w as partial pivoting.

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 0 & -0.5 & 3.5 & -3 \\ 0 & 0 & 5 & -5 \\ 0 & 0 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 7 \\ -9.5 \\ -15 \\ 8 \end{bmatrix}$$

Scanned with CamScanner

Operate $R_4 \rightarrow R_4 - \frac{(-2)}{5} R_3$

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 0 & -0.5 & 3.5 & -3 \\ 0 & 0 & 5 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 7 \\ -9.5 \\ -15 \\ 2 \end{bmatrix}$$

Performing backward substitution

or, $1w = 2$

∴ $w = 2$

or, $5z - 5w = -15$

∴ $z = -1$

or, $-0.5y + 3.5yz - 3w = -9.5$

∴ $y = 0$

or, $2x + 3y - z + 2w = 7$

∴ $x = 1$

19. Solve the following system of equations by using Crout's algorithm.

$$2x - 3y + 10z = 3$$
$$-x + 4y + 2z = 20$$
$$5x + 2y + z = -12$$

[2016/Fall]

Solution:

Writing the system of equations in matrix form,

$$\underset{A}{\begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}}$$

Now, using Crout's algorithm, we represent A as

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{22} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

Solving for unknown values,

$l_{11} = 2$	$l_{11}u_{12} = -3$ $\therefore u_{12} = -1.5$	$l_{11}u_{13} = 10$ $\therefore u_{13} = 5$
$l_{21} = -1$	$l_{21}u_{12} + l_{22} = 4$ $\therefore l_{22} = 2.5$	$l_{21}u_{13} + l_{22}u_{23} = 2$ $\therefore u_{23} = 2.8$
$l_{31} = 5$	$l_{31}u_{12} + l_{32} = 2$ $\therefore l_{32} = 9.5$	$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 1$ $\therefore l_{33} = -50.6$

Now, substituting obtained coefficients as LUX = B

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 2.5 & 0 \\ 5 & 9.5 & -50.6 \end{bmatrix}\begin{bmatrix} 1 & -1.5 & 5 \\ 0 & 1 & 2.8 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

Let UX = V, so LV = B then,

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 2.5 & 0 \\ 5 & 9.5 & -50.6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

Using forward substitution,

∴ $v_1 = 1.5$

or $-1v_1 + 2.5v_2 = 20$

∴ $v_2 = 8.6$

or, $5v_1 + 9.5v_2 - 50.6v_3 = -12$

∴ $v_3 = 2$

Then, UX = V

$$\begin{bmatrix} 1 & -1.5 & 5 \\ 0 & 1 & 2.8 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1.5 \\ 8.6 \\ 2 \end{bmatrix}$$

Performing backward substitution,

∴ $z = 2$

or, $y + 2.8z = 8.6$

∴ $y = 3$

or, $x - 1.5y + 5z = 1.5$

∴ $x = -4$

20. Find the largest eigen value and corresponding eigen vector of given matrix using power method.

$$\begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}$$

[2016/Fall]

Solution:

Let the initial vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then performing the iterations as follows,

$$AX_0 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 10 \\ 8 \\ 5 \end{bmatrix}= 10\begin{bmatrix} 1 \\ 0.8 \\ 0.5 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.8 \\ 0.5 \end{bmatrix}=\begin{bmatrix} 8.8 \\ 5.5 \\ 3.5 \end{bmatrix}= 8.8\begin{bmatrix} 1 \\ 0.625 \\ 0.397 \end{bmatrix}$$

Scanned with CamScanner

$$AX_2 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.625 \\ 0.397 \end{bmatrix} = \begin{bmatrix} 7.75 \\ 4.316 \\ 3.191 \end{bmatrix} = 7.75\begin{bmatrix} 1 \\ 0.556 \\ 0.411 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.556 \\ 0.411 \end{bmatrix} = \begin{bmatrix} 7.336 \\ 4.013 \\ 3.233 \end{bmatrix} = 7.336\begin{bmatrix} 1 \\ 0.547 \\ 0.440 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.547 \\ 0.440 \end{bmatrix} = \begin{bmatrix} 7.282 \\ 4.055 \\ 3.320 \end{bmatrix} = 7.282\begin{bmatrix} 1 \\ 0.556 \\ 0.455 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.556 \\ 0.455 \end{bmatrix} = \begin{bmatrix} 7.336 \\ 4.145 \\ 3.365 \end{bmatrix} = 7.336\begin{bmatrix} 1 \\ 0.565 \\ 0.458 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.565 \\ 0.458 \end{bmatrix} = \begin{bmatrix} 7.390 \\ 4.199 \\ 3.374 \end{bmatrix} = 7.390\begin{bmatrix} 1 \\ 0.568 \\ 0.456 \end{bmatrix}$$

$$AX_7 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.568 \\ 0.456 \end{bmatrix} = \begin{bmatrix} 7.408 \\ 4.208 \\ 3.368 \end{bmatrix} = 7.408\begin{bmatrix} 1 \\ 0.568 \\ 0.454 \end{bmatrix}$$

$$AX_8 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.568 \\ 0.454 \end{bmatrix} = \begin{bmatrix} 7.408 \\ 4.202 \\ 3.362 \end{bmatrix} = 7.408\begin{bmatrix} 1 \\ 0.567 \\ 0.453 \end{bmatrix}$$

Hence the required eigen vector is $\begin{bmatrix} 1 \\ 0.567 \\ 0.453 \end{bmatrix}$.

And the required eigen value 7.408.

21. Using Gauss Seidal method solve the following system of liner equations.

$$10x_1 + 6x_2 - 5x_3 = 27$$
$$3x_1 + 8x_2 + 10x_3 = 27$$
$$4x_1 + 10x_2 + 3x_3 = 27$$

[2016/Spring]

Solution:

Arranging the system of liner equations in diagonally dominant forms,

$$10x_1 + 6x_2 - 5x_3 = 27$$
$$4x_1 + 10x_2 + 3x_3 = 27$$
$$3x_1 + 8x_2 + 10x_3 = 27$$

Forming the equations as,

$$x_1 = \frac{27 - 6x_2 + 5x_3}{10}$$

$$x_2 = \frac{27 - 4x_1 - 3x_3}{10}$$

$$x_3 = \frac{27 - 3x_1 - 8x_2}{10}$$

Scanned with CamScanner

Let the initial guess be 0 for x_1, x_2 and x_3

Solving the iterations in tabular form.

Iteration	$x_1 = \frac{27 - 6x_2 + 5x_3}{10}$	$x_2 = \frac{27 - 4x_1 - 3x_3}{10}$	$x_3 = \frac{27 - 3x_1 - 8x_2}{10}$
Guess	0	0	0
1	2.7	1.62	0.594
2	2.025	1.711	0.723
3	2.034	1.669	0.754
4	2.075	1.643	0.763
5	2.095	1.633	0.765
6	2.102	1.629	0.766
7	2.105	1.628	0.766
8	2.106	1.627	0.766
9	2.106	1.627	0.766

Hence the required valued of x_1, x_2 and x_3 are 2.106, 1.627 and 0.766 respectively which are correct upto 3 decimal places.

> **NOTE:**
> Procedure to iterate in programmable calculator
> Let $A = x_1$, $B = x_2$, $C = x_3$
> Set the following in calculator
> $$A = \frac{27 - 6B + 5C}{10} : B = \frac{27 - 4A - 3C}{10} : C = \frac{27 - 3A - 8B}{10}$$
> Now press CALC and enter the initial value of B and C and continue pressing = only for the required no. of iterations.

22. Find the largest eigen value and corresponding eigen vector of the matrix

$$\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$$

[2016/Spring, 2018/Spring]

Solution:

Let the initial vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then performing the iterations as

$$AX_0 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 13 \end{bmatrix} = 13\begin{bmatrix} 0.230 \\ 0.692 \\ 1 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.230 \\ 0.692 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.306 \\ 6.074 \\ 12.538 \end{bmatrix} = 12.538\begin{bmatrix} 0.104 \\ 0.484 \\ 1 \end{bmatrix}$$

Scanned with CamScanner

$$AX_2 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.104 \\ 0.484 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.556 \\ 5.28 \\ 11.832 \end{bmatrix} = 11.832\begin{bmatrix} 0.046 \\ 0.446 \\ 1 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.046 \\ 0.446 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.384 \\ 5.03 \\ 11.738 \end{bmatrix} = 11.738\begin{bmatrix} 0.032 \\ 0.428 \\ 1 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.032 \\ 0.428 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.316 \\ 4.952 \\ 11.68 \end{bmatrix} = 11.68\begin{bmatrix} 0.027 \\ 0.423 \\ 1 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.027 \\ 0.423 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.296 \\ 4.927 \\ 11.665 \end{bmatrix} = 11.668\begin{bmatrix} 0.025 \\ 0.422 \\ 1 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.025 \\ 0.422 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.291 \\ 4.919 \\ 11.663 \end{bmatrix} = 11.663\begin{bmatrix} 0.024 \\ 0.421 \\ 1 \end{bmatrix}$$

Hence the required eigen value 11.663.

And the required eigen vector is $\begin{bmatrix} 0.024 \\ 0.421 \\ 1 \end{bmatrix}$.

21. Find the inverse of the matrix by using Gauss Jordan Method.

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

[2017/Fall]

Solution:

The augmented matrix can be written as

$$[A:I] = \begin{bmatrix} 1 & -1 & 2 & : & 1 & 0 & 0 \\ 3 & 0 & 1 & : & 0 & 1 & 0 \\ 1 & 0 & 2 & : & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - R_1$

$$[A:I] = \begin{bmatrix} 1 & -1 & 2 & : & 1 & 0 & 0 \\ 0 & 3 & -5 & : & -3 & 1 & 0 \\ 0 & 1 & 0 & : & -1 & 0 & 1 \end{bmatrix}$$

Operate $R_2 \to R_2 - 3R_3$

$$[A:I] = \begin{bmatrix} 1 & -1 & 2 & : & 1 & 0 & 0 \\ 0 & 1 & -5 & : & -1 & 1 & -2 \\ 0 & 1 & 0 & : & -1 & 0 & 1 \end{bmatrix}$$

Operate $R_3 \to R_3 - R_2$

$$[A:I] = \begin{bmatrix} 1 & -1 & 2 & : & 1 & 0 & 0 \\ 0 & 1 & -5 & : & -1 & 1 & -2 \\ 0 & 0 & 5 & : & 0 & -1 & 3 \end{bmatrix}$$

Scanned with CamScanner

Operate $R_3 \to \frac{1}{5} R_3$

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -1 & 1 & -2 \\ 0 & 0 & 1 & 0 & -0.2 & 0.6 \end{array}\right]$$

Operate $R_2 \to R_2 + 5R_3$

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -0.2 & 0.6 \end{array}\right]$$

Operate $R_1 \to R_1 + R_2$

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -0.2 & 0.6 \end{array}\right]$$

Operate $R_1 \to R_1 - 2R_3$

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0.4 & -0.2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -0.2 & 0.6 \end{array}\right]$$

Now, inverse of matrix,

$[A : I] = [I : A]$

$[A = A^{-1}]$

$$\text{Hence, } A^{-1} = \begin{bmatrix} 0 & 0.4 & -0.2 \\ -1 & 0 & 1 \\ 1 & -0.2 & 0.6 \end{bmatrix}$$

24. Solve the following set of equation using LU factorization method.

$5x - 2y + z = 4$

$7x + y - 5z = 8$

$3x + 7y + 4z = 10$ **[2017/Spring]**

Solution:

Writing the system of equations in matrix form.

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

In LU factorization method, we represent A as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

Scanned with CamScanner

Solving for unknown values,

$u_{11} = 5$	$u_{12} = -2$	$u_{13} = 1$
$l_{21}u_{11} = 7$ $\therefore l_{21} = 1.4$	$l_{21}u_{12} + u_{22} = 1$ $\therefore u_{22} = 3.8$	$l_{21}u_{13} + u_{23} = -5$ $\therefore u_{23} = -6.4$
$l_{31}u_{11} = 3$ $\therefore l_{31} = 0.6$	$l_{31}u_{12} + l_{32}u_{22} = 7$ $\therefore l_{32} = 2.15$	$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4$ $\therefore u_{33} = 17.16$

Now, substituting obtained coefficient and we have overall system of

$$\underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 1.4 & 1 & 0 \\ 0.6 & 2.15 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 5 & -2 & 1 \\ 0 & 3.8 & -6.4 \\ 0 & 0 & 17.16 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}}$$

Let, $LUX = B$

$UX = V$

$LV = B$, then,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1.4 & 1 & 0 \\ 0.6 & 2.15 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

Now, performing forward substitution,

$\therefore$ $v_1 = 4$

or, $1.4v_1 + v_2 = 8$

$\therefore$ $v_2 = 2.4$

or, $0.6v_1 + 2.15v_2 + v_3 = 10$

$\therefore$ $v_3 = 2.44$

Now,

$UX = V$

$$\begin{bmatrix} 5 & -2 & 1 \\ 0 & 3.8 & -6.4 \\ 0 & 0 & 17.16 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2.4 \\ 2.44 \end{bmatrix}$$

Performing backward substitution,

or, $17.16z = 2.44$

$\therefore$ $z = 0.142$

or, $3.8y - 6.4z = 2.4$

$\therefore$ $y = 0.870$

or, $5x - 2y + z = 4$

$\therefore$ $x = \frac{5.598}{5} = 1.119$

25. Solve the equation by Gauss-Jacobi method.

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

[2017/Spring]

Scanned with CamScanner

Solution:

Given that;

$20x + y - 2z = 17$

$3x + 20y - z = -18$

$2x - 3y + 20z = 25$

The given equations are in diagonally dominant form.

Now, forming the equations as,

$$x = \frac{1}{20}[17 - y + 2z]$$

$$y = \frac{1}{20}[-18 + z - 3x]$$

$$z = \frac{1}{20}[25 + 3y - 2x]$$

Let $x_0 = 0$, $y_0 = 0$ and $z_0 = 0$ be initial guesses.

And solving the iterations in tabular form

Iteration	$x = \frac{1}{20}[17 - y + 2z]$	$y = \frac{1}{20}[-18 + z - 3x]$	$z = \frac{1}{20}[25 + 3y - 2x]$
Guess	0	0	0
1	0.85	−0.9	1.25
2	1.02	−0.965	1.03
3	1.00125	−1.0015	1.00325
4	1.0004	−1.000025	0.99965
5	0.99996	−1.00007	0.99995

Hence the required values of x, y and z are 1, −1 and 1 respectively.

NOTE:

Procedure to iterate in programmable calculator

Let, A = x, B = y, C = z

Step 1: Set the following in calculator

$$A : B : C : D = \frac{17 - B + 2C}{20} : E = \frac{-18 + C - 3A}{20} : F = \frac{25 + 3B - 2A}{20}$$

Step 2: Press CALC then

enter the value of A? then press =

enter the value of B? then press =

enter the value of C? then press =

Step 3: Now press = only, again and again to get the values for respective row for each column.

Step 4: Update the values of A?, B? and C? when asked again.

Step 5: Got to step 3.

Scanned with CamScanner

26. Determine the largest eigen value and the corresponding eigen vector of the matrix using power method.

$$A = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$$

[2017/Spring, 2018/Fall]

Solution:

Let the initial vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Now using power method, the iterations are carried out as

$$AX_0 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ -18 \end{bmatrix} = -18\begin{bmatrix} -0.444 \\ 0.222 \\ 1 \end{bmatrix}$$

NOTE: Here |–18| > 8 and |–4|

Again,

$$AX_1 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} -0.444 \\ 0.222 \\ 1 \end{bmatrix} = \begin{bmatrix} -10.548 \\ 1.104 \\ 7.768 \end{bmatrix} = -10.548\begin{bmatrix} 1 \\ -0.105 \\ -0.736 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.105 \\ -0.736 \end{bmatrix} = -18.948\begin{bmatrix} -0.930 \\ 0.361 \\ 1 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} -0.930 \\ 0.361 \\ 1 \end{bmatrix} = -18.394\begin{bmatrix} 1 \\ -0.415 \\ -0.981 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.415 \\ -0.981 \end{bmatrix} = -19.698\begin{bmatrix} -0.995 \\ 0.462 \\ 1 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} -0.995 \\ 0.462 \\ 1 \end{bmatrix} = -19.773\begin{bmatrix} 1 \\ -0.480 \\ -0.999 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.480 \\ -0.999 \end{bmatrix} = -19.922\begin{bmatrix} -0.997 \\ 0.490 \\ 1 \end{bmatrix}$$

$$AX_7 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} -0.997 \\ 0.490 \\ 1 \end{bmatrix} = -19.956\begin{bmatrix} 1 \\ -0.495 \\ -0.999 \end{bmatrix}$$

$$AX_8 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.495 \\ 0.999 \end{bmatrix} = 19.956\begin{bmatrix} 1 \\ -0.495 \\ -0.999 \end{bmatrix} \approx 20\begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix}$$

Hence the dominant eigen value is 20 and eigen vector is $\begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix}$.

Scanned with CamScanner

27. **Find the inverse of matrix using Gauss Jordan method**

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$ [2018/Fall]

Solution:

The augmented matrix can be written as,

$$[A:I] = \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 3 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

Operate $R_2 \rightarrow R_2 - 3R_1$ and $R_3 \rightarrow R_3 + 2R_1$,

$$[A:I] = \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & -12 & -3 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

Interchanging R_2 and R_3,

$$[A:I] = \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \\ 0 & 0 & -12 & -3 & 1 & 0 \end{array}\right]$$

Operate $R_2 \rightarrow \frac{R_2}{-2}$ and $R_3 \rightarrow \frac{R_3}{-12}$,

$$[A:I] = \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & -0.5 \\ 0 & 0 & 1 & 0.25 & -0.083 & 0 \end{array}\right]$$

Operate $R_2 \rightarrow R_2 + R_3$,

$$[A:I] = \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -0.75 & -0.083 & -0.5 \\ 0 & 0 & 1 & 0.25 & -0.083 & 0 \end{array}\right]$$

Operate $R_1 \rightarrow R_1 - R_2$,

$$[A:I] = \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 1.75 & 0.083 & 0.5 \\ 0 & 1 & 0 & -0.75 & -0.083 & -0.5 \\ 0 & 0 & 1 & 0.25 & -0.083 & 0 \end{array}\right]$$

Operate $R_1 \rightarrow R_1 - 3R_3$,

$$[A:I] = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0.332 & 0.5 \\ 0 & 1 & 0 & -0.075 & -0.083 & -0.5 \\ 0 & 0 & 1 & 0.25 & -0.083 & 0 \end{array}\right]$$

For inverse of matrix,

$[A:I] = [I:A]$

$[A = A^{-1}]$

Hence,

$$A^{-1} = \begin{bmatrix} 1 & 0.332 & 0.5 \\ -0.75 & -0.083 & -0.5 \\ 0.25 & -0.083 & 0 \end{bmatrix}$$

Scanned with CamScanner

28. Solve the following system of equation

$6x_1 - 2x_2 + x_3 = 4$

$-2x_1 + 7x_2 + 2x_3 = 5$

$x_1 + 2x_2 - 5x_3 = -1$

Using Gauss factorization method. **[2018/Fall]**

Solution:

Writing the given system of equation in matrix from AX = B

$$\begin{bmatrix} 6 & -2 & 1 \\ -2 & 7 & 2 \\ 1 & 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}$$

In Gauss factorization method, we decompose matrix A in the following form,

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{12} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 6 & -2 & 1 \\ -2 & 7 & 2 \\ 1 & 2 & -5 \end{bmatrix}$$

Here, A = LU

Solving for unknown values,

$u_{11} = 6$	$u_{12} = -2$	$u_{13} = 1$
$l_{21}u_{11} = -2$	$u_{12}l_{21} + u_{22} = 7$	$l_{21}u_{13} + u_{23} = 2$
$\therefore l_{21} = -0.333$	$\therefore u_{22} = 6.334$	$\therefore u_{23} = 2.333$
$l_{31}u_{11} = 1$	$l_{31}u_{12} + l_{32}u_{22} = 2$	$l_{31}u_{13} + l_{32}u_{23} + u_{33} = -5$
$\therefore l_{31} = 0.167$	$\therefore l_{32} = 0.368$	$\therefore u_{33} = -6.025$

Now, substituting obtained coefficients, we have overall system as,

$$\underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ -0.333 & 1 & 0 \\ 0.167 & 0.368 & 1 \end{bmatrix}}\underset{U}{\begin{bmatrix} 6 & -2 & 1 \\ 0 & 6.334 & 2.333 \\ 0 & 0 & -6.025 \end{bmatrix}}\underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}}$$

Let UX = V,

so, LV = B then

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.333 & 1 & 0 \\ 0.167 & 0.368 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}$$

Using forward substitution

$\therefore \quad v_1 = 4$

or, $\quad -0.333v_1 + v_2 = 5$

$\therefore \quad v_2 = 6.332$

or, $\quad 0.167v_1 + 0.368v_2 + v_3 = -1$

$\therefore \quad v_3 = -3.998$

Now,

UX = V

Scanned with CamScanner

$$\begin{bmatrix} 6 & -2 & 1 \\ 0 & 6.334 & 2.333 \\ 0 & 0 & -6.025 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6.332 \\ -3.998 \end{bmatrix}$$

Using backward substitution

or, $-6.025x_3 = -3.998$

$\therefore$ $x_3 = 0.663$

or, $6.33x_2 + 2.333x_3 = 6.332$

$\therefore$ $x_2 = 0.755$

or, $6x_1 - 2x_2 + x_3 = 4$

$\therefore$ $x_1 = 0.807$

29. Solve the following system of equations using factorization method.

$2x + 3y + z = 9$

$x + 2y + 3z = 6$

$3x + y + 2z = 8$ **[2018/Spring]**

Solution:

Writing the system of equations in matrix form,

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

In factorization method, we decompose matrix in the following form A = LU

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Solving for unknown values

$u_{11} = 2$	$u_{12} = 3$	$u_{13} = 1$
$l_{21}u_{11} = 1$ $\therefore l_{21} = 0.5$	$l_{21}u_{12} + u_{22} = 2$ $\therefore u_{22} = 0.5$	$l_{21}u_{13} + u_{23} = 3$ $\therefore u_{23} = 2.5$
$l_{31}u_{11} = 3$ $\therefore l_{31} = 1.5$	$l_{31}u_{12} + l_{32}u_{22} = 1$ $\therefore l_{32} = -7$	$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 5$ $\therefore u_{33} = 21$

Now, substituting obtained coefficient, we have overall system of LUX = B as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & -7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & 0.5 & 2.5 \\ 0 & 0 & 21 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

UX = V

Then, LV = B

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & -7 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

Now, performing forward substitution,

$\therefore$ $v_1 = 9$

$\rightarrow$ $0.5v_1 + v_2 = 6$

Scanned with CamScanner

$\therefore \quad v_2 = 1.5$

$\rightarrow \quad 1.5v_1 - 7v_2 + v_3 = 8$

$\therefore \quad v_3 = 5$

Then, $UX = V$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 0.5 & 2.5 \\ 0 & 0 & 21 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1.5 \\ 5 \end{bmatrix}$$

Now, performing backward substitution

or, $\quad 12z = 5$

$\therefore \quad z = 0.238$

or, $\quad 05y + 2.5z = 1.5$

$\therefore \quad y = 1.81$

or, $\quad 2x + 3y + 1z = 9$

$\therefore \quad x = 1.66$

30. Find inverse of the matrix, using Gauss Jordan method

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$ **[2019/Fall]**

Solution:

The augumented matrix can be written as

$$[A : I] = \begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 1 & 3 & -3 & : & 0 & 1 & 0 \\ -2 & -4 & -4 & : & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 + 2R_1$

$$[A : I] = \begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 0 & 2 & -6 & : & -1 & 1 & 0 \\ 0 & -2 & 2 & : & 2 & 0 & 1 \end{bmatrix}$$

Operate $R_2 \rightarrow \frac{R_2}{2}$

$$[A : I] = \begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & -3 & : & -1/2 & -1/2 & 0 \\ 0 & -2 & 2 & : & 2 & 0 & 1 \end{bmatrix}$$

Operate $R_3 \rightarrow R_3 + 2R_2$

$$[A : I] = \begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & -3 & : & -1/2 & 1/2 & 0 \\ 0 & 0 & -4 & : & 1 & 1 & 1 \end{bmatrix}$$

Operate $R_3 \rightarrow \frac{R_3}{-4}$

$$[A : I] = \begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & -3 & : & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & : & -1/4 & -1/4 & -1/4 \end{bmatrix}$$

Scanned with CamScanner

Operate $R_1 \rightarrow R_1 - R_2$

$$[A:I] = \left[\begin{array}{ccc:ccc} 1 & 0 & 6 & 1.5 & -0.5 & 0 \\ 1 & 1 & -3 & -0.5 & +0.5 & 0 \\ 0 & 0 & 1 & -0.25 & -0.25 & -0.25 \end{array}\right]$$

Operate $R_1 \rightarrow R_1 - 6R_3$

$$[A:I] = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & 1 & 1.5 \\ 0 & 1 & -3 & -0.5 & +0.5 & 0 \\ 0 & 0 & 1 & -0.25 & -0.25 & -0.25 \end{array}\right]$$

Operate $R_2 \rightarrow R_2 + 3R_3$

$$[A:I] = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & 1 & 1.5 \\ 0 & 1 & 0 & -1.25 & -0.25 & -0.75 \\ 0 & 0 & 1 & -0.25 & -0.25 & -0.25 \end{array}\right]$$

Now, for inversion of matrix

$[A:I] = [I:A]$

$[A = A^{-1}]$

Hence,

$$A^{-1} = \begin{bmatrix} 3 & 1 & 1.5 \\ -1.25 & -0.25 & -0.75 \\ -0.25 & -0.25 & -0.25 \end{bmatrix}$$

31. Determine the largest eigen value and the corresponding eigen vector of the matrix using power method.

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$$ **[2019/Fall]**

Solution:

Let the initial vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$AX_0 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 14 \end{bmatrix} = 14\begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \\ 6.5 \end{bmatrix} = 6.5\begin{bmatrix} 0.076 \\ 0.153 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.076 \\ 0.153 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.617 \\ -0.084 \\ 5.915 \end{bmatrix} = 5.915\begin{bmatrix} 0.273 \\ -0.014 \\ 1 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.273 \\ -0.014 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.429 \\ -0.116 \\ 6.482 \end{bmatrix} = 6.482\begin{bmatrix} 0.374 \\ -0.017 \\ 1 \end{bmatrix}$$

Scanned with CamScanner

$$AX_4 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.374 \\ -0.017 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.425 \\ 0.428 \\ 7.193 \end{bmatrix} = 7.193\begin{bmatrix} 0.337 \\ 0.059 \\ 1 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.337 \\ 0.059 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.160 \\ 0.584 \\ 7.199 \end{bmatrix} = 7.199\begin{bmatrix} 0.300 \\ 0.081 \\ 1 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.300 \\ 0.081 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.057 \\ 0.524 \\ 7.043 \end{bmatrix} = 7.043\begin{bmatrix} 0.292 \\ 0.074 \\ 1 \end{bmatrix}$$

$$AX_7 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.292 \\ 0.074 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.07 \\ 0.464 \\ 6.974 \end{bmatrix} = 6.974\begin{bmatrix} 0.296 \\ 0.066 \\ 1 \end{bmatrix}$$

$$AX_8 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.296 \\ 0.066 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.098 \\ 0.066 \\ 1 \end{bmatrix} = 6.974\begin{bmatrix} 0.3 \\ 0.064 \\ 1 \end{bmatrix}$$

Hence the required largest eigen value is $6.974 \approx 7$

And corresponding eigen vector is $\begin{bmatrix} 0.3 \\ 0.064 \\ 1 \end{bmatrix}$.

32. **Use relaxation method to solve the given systems of equations.**

$20x + y - 2z = 17$

$3x + 20y - z = 18$

$2x - 3y + 20z = 25$

[2019/Fall]

Solution:

The diagonal elements of the coefficient matrix dominate the other coefficients in the corresponding row.

i.e., $|20| \geq |1| + |-2|$

$|20| \geq |3| + |-1|$

$|20| \geq |2| + |-3|$

Now, using relaxation method.

The residuals are given by

$R_x = 17 - 20x - y + 2z$

$R_y = 18 - 3x - 20y + z$

$R_z = 25 - 2x + 3y - 20z$

The operation table is

	δR_x	δR_y	δR_z
$\delta x = 1$	-20	-3	-2
$\delta y = 1$	-1	-20	3
$\delta z = 1$	2	1	-20

Now, the relaxation table is shown below.

Taking $z = y = z = 0$ as initial assumption

Scanned with CamScanner

	R_x	R_y	R_z	
$x = y = z = 0$	17	18	25	
$\delta z = 1$	17+(1×2)=19	18+(1×1)=19	25−(20×1)=5	i
$\delta x = 0.5$	19−(20×0.5)=9	19+(−3×0.5)=17.5	5−(5×0.5)=4	ii
$\delta y = 0.5$	9+(−1×0.5)=8.5	17.5−(20×0.5)=7.5	4+3(0.5)=5.5	iii
$\delta x = 0.5$	8.5+(−20×0.5)=−1.5	7.5−(3×0.5)=6	5.5−2×0.5= 4.5	
$\delta y = 0.33$	−1.83	−0.6	5.49	
$\delta z = 0.28$	−1.27	−0.32	−0.11	
$\delta x = -0.06$	−0.07	−0.14	0.010	
$\delta y = -0.007$	−0.063	0.00	−0.010	
$\delta x = -0.003$	−0.003	0.009	0.006	

Now,

$$\Sigma\delta x = 0.5 + 0.5 - 0.06 - 0.003 = 0.937$$

$$\Sigma\delta y = 0.5 + 0.33 - 0.007 = 0.823$$

$$\Sigma\delta z = 1 + 0.28 = 1.28$$

Thus, $x = 0.937$, $y = 0.823$ and $z = 1.28$

NOTE:
In (i) in the table, the largest residual is 25 so to reduce it, we give an increment in δz at $\delta z = 1$ and the resulting residuals are shown in (ii).
i.e., larger residuals are reduced by assuming suitable increment values.
Similarly the steps are carried out.
Also when increment is done in either δx or δy or δz, use the operation table respectively.

33. Solve the equation by relaxation method

$$9x - y + 2z = 9$$
$$x + 2y - 2z = 15$$
$$2x - 2y - 13z = -17$$

[2020/Fall]

Solution:

$$9x - y + 2z = 9$$
$$x + 2y - 2z = 15$$
$$2x - 2y - 13z = -17$$

Using relaxation method,
The residuals are given by,

$$R_x = 9 - 9x + y - 2z$$
$$R_y = 15 - x - 2y + 2z$$
$$R_z = -17 - 2x + 2y + 13z$$

The operation table is

	δR_x	δR_y	δR_z
$\delta x = 1$	−9	−1	−2
$\delta y = 1$	1	−2	2
$\delta z = 1$	−2	2	13

Scanned with CamScanner

Taking initial guess of x = y = z = 0.

Now, the relaxation table is,

	R_x	R_y	R_z
0	9	15	-17
$\delta z = 1$	7	17	-4
$\delta y = 8$	15	1	12
$\delta z = 2$	-3	-1	8
$\delta z = -0.615$	-1.77	-2.23	0.005
$\delta y = -1.115$	-2.885	0	-0.225
$\delta x = -0.32$	-0.005	0.32	0.415
$\delta z = -0.031$	0.057	0.25	0.012
$\delta y = 0.125$	0.182	0	0.262

Now,

$$\Sigma\delta x = 2 - 0.32 = 1.68$$

$$\Sigma\delta y = 8 - 1.115 + 0.125 = 7.01$$

$$\Sigma\delta z = 1 - 0.615 - 0.031 = 0.354$$

Thus, x = 1.68, y = 7.01 and z = 0.354

34. Determine the largest eigen value and the corresponding eigen vector of the matrix using the power method.

$$A = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}$$

[2020/Fall]

Solution:

$$A = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}$$

Let the initial vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then using power method, performing the iterations as,

$$AX_0 = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 13 \end{bmatrix} = 13\begin{bmatrix} 0.692 \\ 1 \\ 1 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 0.692 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.692 \\ 11.768 \\ 11.768 \end{bmatrix} = 11.768\begin{bmatrix} 0.738 \\ 1 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 0.738 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.738 \\ 11.952 \\ 11.952 \end{bmatrix} = 11.952\begin{bmatrix} 0.731 \\ 1 \\ 1 \end{bmatrix}$$

Scanned with CamScanner

$$AX_3 = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix} \begin{bmatrix} 0.731 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.731 \\ 11.924 \\ 11.924 \end{bmatrix} = 11.924 \begin{bmatrix} 0.732 \\ 1 \\ 1 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix} \begin{bmatrix} 0.732 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.732 \\ 11.928 \\ 11.928 \end{bmatrix} = 11.928 \begin{bmatrix} 0.732 \\ 1 \\ 1 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix} \begin{bmatrix} 0.732 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.732 \\ 11.928 \\ 11.928 \end{bmatrix} = 11.928 \begin{bmatrix} 0.732 \\ 1 \\ 1 \end{bmatrix}$$

Hence the required eigen value is 11.928.

and the eigen vector is $\begin{bmatrix} 0.732 \\ 1 \\ 1 \end{bmatrix}$.

35. Solve the following set of equations by using LU decomposition method.

$3x + 2y + 7z = 32$
$2x + 3y + z = 40$
$3x + 4y + z = 56$ **[2020/Fall]**

Solution:

Writing the system of equations in matrix form AX = B

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 32 \\ 40 \\ 56 \end{bmatrix}$$

In LU factorization method, we represent A as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

Solving for unknown values

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & u_{12}l_{21} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

Solving for unknown values,

$u_{11} = 3$	$u_{12} = 2$	$u_{13} = 7$
$l_{21}u_{11} = 2$ $\therefore l_{21} = 0.667$	$l_{21}u_{12} + u_{22} = 3$ $\therefore u_{22} = 1.666$	$l_{21}u_{13} + u_{23} = 1$ $\therefore u_{23} = -3.669$
$l_{31}u_{11} = 3$ $\therefore l_{31} = 1$	$l_{31}u_{12} + l_{32}u_{22} = 4$ $\therefore l_{32} = 1.2$	$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$ $\therefore u_{33} = -1.597$

Substituting the values

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0.667 & 1 & 0 \\ 1 & 1.2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & 2 & 7 \\ 0 & 1.666 & -3.669 \\ 0 & 0 & -1.597 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 32 \\ 40 \\ 56 \end{bmatrix}}_{B}$$

Scanned with CamScanner

Here, LUX = B

Let UX = V

so, LV = B then,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.667 & 1 & 0 \\ 1 & 1.2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 32 \\ 40 \\ 56 \end{bmatrix}$$

Performing forward substitution

∴ $v_1 = 32$

or, $0.667 v_1 + v_2 = 40$

∴ $v_2 = 18.656$

or, $v_1 + 1.2 v_2 + v_3 = 56$

∴ $v_3 = 1.612$

Now, UX = V

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 1.666 & -3.669 \\ 0 & 0 & -1.597 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 32 \\ 18.656 \\ 1.612 \end{bmatrix}$$

Again, performing backward substitution

or, $-1.597z = 1.612$

∴ $z = -1.009 \approx -1$

or, $1.666y - 3.669z = 18.656$

∴ $y = \frac{14.953}{1.666} = 8.975 \approx 9$

or, $3x + 2y + 7z = 32$

∴ $x = 7.037 \approx 7$

36. Write short notes on: Relaxation method. **[2014/Fall]**

Solution: See the topic 4.6.3.

37. Write short notes on Ill conditioned system. **[2014/Spring, 2016/Spring, 2019/Spring]**

Solution: See the topic 4.5.

38. Write short notes on: Gauss Seidel method of iteration. **[2017/Fall]**

Solution: See the topic 4.6.2.

39. Write a program in any high level language C or C++ to solve a system of linear equation, using gauss elimination method. **[2016/Spring]**

Solution: See the "Appendix", program number 11.

40. Write a program to solve a system of linear equations by Gauss Seidal method. **[2018/Spring]**

Solution: See the "Appendix", program number 16.

Scanned with CamScanner

ADDITIONAL QUESTION SOLUTION

1. **Solve the following system of equation using LU factorization method.**

$$5x_1 + 2x_2 + 3x_3 = 31$$
$$3x_1 + 3x_2 + 2x_3 = 25$$
$$x_1 + 2x_2 + 4x_3 = 25$$

Solution:

Writing the system of equations in matrix form AX = B.

$$\begin{bmatrix} 5 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 31 \\ 25 \\ 25 \end{bmatrix}$$

In LU factorization method, we represent A as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 5 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & u_{12}l_{21} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 5 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

Solving for unknown values,

$u_{11} = 5$	$u_{12} = 2$	$u_{13} = 3$
$l_{21}u_{11} = 3$ $\therefore l_{21} = 0.6$	$l_{21}u_{12} + u_{22} = 3$ $\therefore u_{22} = 1.8$	$l_{21}u_{13} + u_{23} = 2$ $\therefore u_{23} = 0.2$
$l_{31}u_{11} = 1$ $\therefore l_{31} = 0.2$	$l_{31}u_{12} + l_{32}u_{22} = 2$ $\therefore l_{32} = 0.88$	$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4$ $\therefore u_{33} = 3.224$

Now substituting obtained coefficient and we have overall system of

$$\underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 1 & 0 \\ 0.2 & 0.88 & 1 \end{bmatrix}}\underset{U}{\begin{bmatrix} 5 & 2 & 3 \\ 0 & 1.8 & 0.2 \\ 0 & 0 & 3.224 \end{bmatrix}}\underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 31 \\ 25 \\ 25 \end{bmatrix}}$$

Here, LUX = B

Let UX = V

so, LV = B then,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 1 & 0 \\ 0.2 & 0.88 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 31 \\ 25 \\ 25 \end{bmatrix}$$

Now, performing forward substitution

$\therefore$ $v_1 = 31$

or, $0.6v_1 + v_2 = 25$

$\therefore$ $v_2 = 6.4$

or, $0.2v_1 + 0.88v_2 + v_3 = 25$

$\therefore$ $v_3 = 13.168$

Scanned with CamScanner

Then, UX = V becomes

$$\begin{bmatrix} 5 & 2 & 3 \\ 0 & 1.8 & 0.2 \\ 0 & 0 & 3.224 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 31 \\ 6.4 \\ 13.168 \end{bmatrix}$$

Performing backward substitution,

or, $3.224x_3 = 13.168$

∴ $x_3 = 4.084$

or, $1.8x_2 + 0.2x_3 = 6.4$

∴ $x_2 = 3.101$

or, $5x_1 + 2x_2 + 3x_3 = 31$

∴ $x_1 = 2.509$

2. Apply Gauss Seidal Iterative method to solve the linear equations correct to 2 decimal places.

$$10x + y - z = 11.19$$
$$x + 10y + z = 28.08$$
$$-x + y + 10z = 35.61$$

Solution:

Here, the provided equations are in diagonally dominant form, so forming the equations as,

$$x = \frac{11.19 - y + z}{10}$$

$$y = \frac{28.08 - x - z}{10}$$

$$z = \frac{35.61 + x - y}{10}$$

Let the initial guess be 0 for x, y, and z.

Solving the iterations in tabular form,

Iteration	$x = \frac{11.19 - y + z}{10}$	$y = \frac{28.08 - x - z}{10}$	$z = \frac{35.61 + x - y}{10}$
Guess	0	0	0
1	1.119	2.6961	3.4032
2	1.1897	2.3487	3.4451
3	1.2286	2.3406	3.4498
4	1.2299	2.3400	3.4499

Here, the values of x, y and z are correct upto 2 decimal places.

Hence, the required values are;

$x = 1.2299 \approx 1.23, y = 2.34, z = 3.4499 \approx 3.45$

3. Find Inverse of the matrix, using Gauss Jordan method

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

Scanned with CamScanner

Solution:

The augumented matrix can be written as

$$[A:I] = \begin{bmatrix} 3 & 1 & 2 & : & 1 & 0 & 0 \\ 1 & 2 & 3 & : & 0 & 1 & 0 \\ 2 & 3 & 5 & : & 0 & 0 & 1 \end{bmatrix}$$

Operate R_1 and R_2

$$[A:I] = \begin{bmatrix} 1 & 2 & 3 & : & 0 & 1 & 0 \\ 3 & 1 & 2 & : & 1 & 0 & 0 \\ 2 & 3 & 5 & : & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 2R_1$

$$[A:I] = \begin{bmatrix} 1 & 2 & 3 & : & 0 & 1 & 0 \\ 0 & -5 & -7 & : & 1 & -3 & 0 \\ 0 & -1 & -1 & : & 0 & -2 & 1 \end{bmatrix}$$

Operate $R_3 \to R_3 - \frac{1}{5}R_2$

$$[A:I] = \begin{bmatrix} 1 & 2 & 3 & : & 0 & 1 & 0 \\ 0 & -5 & -7 & : & 1 & -3 & 0 \\ 0 & 0 & 2/5 & : & -1/5 & -7/5 & 1 \end{bmatrix}$$

Operate $R_3 \to \frac{5}{2}R_3$

$$[A:I] = \begin{bmatrix} 1 & 2 & 3 & : & 0 & 1 & 0 \\ 0 & -5 & -7 & : & 1 & -3 & 0 \\ 0 & 0 & 1 & : & -1/2 & -7/2 & 5/2 \end{bmatrix}$$

Operate $R_2 \to \frac{R_2}{-5}$

$$[A:I] = \begin{bmatrix} 1 & 2 & 3 & : & 0 & 1 & 0 \\ 0 & 1 & 7/5 & : & -1/5 & 3/5 & 0 \\ 0 & 0 & 1 & : & -1/2 & -7/2 & 5/2 \end{bmatrix}$$

Operate $R_2 \to R_2 - \frac{7}{5}R_3$

$$[A:I] = \begin{bmatrix} 1 & 2 & 3 & : & 0 & 1 & 0 \\ 0 & 1 & 0 & : & 1/2 & 11/2 & -7/2 \\ 0 & 0 & 1 & : & -1/2 & -7/2 & 5/2 \end{bmatrix}$$

Operate $R_1 \to R_1 - 2R_2$

$$[A:I] = \begin{bmatrix} 1 & 0 & 3 & : & -1 & -10 & 7 \\ 0 & 1 & 0 & : & 1/2 & 11/2 & -7/2 \\ 0 & 0 & 1 & : & -1/2 & -7/2 & 5/2 \end{bmatrix}$$

Operate $R_1 \to R_1 - 3R_3$

$$[A:I] = \begin{bmatrix} 1 & 0 & 0 & : & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & : & 1/2 & 11/2 & -7/2 \\ 0 & 0 & 1 & : & -1/2 & -7/2 & 5/2 \end{bmatrix}$$

For inversion of matrix

$[A : I] = [I : A]$

$[A = A^{-1}]$

Scanned with CamScanner

Hence,

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{11}{2} & \frac{-7}{2} \\ \frac{-1}{2} & \frac{-7}{2} & \frac{5}{2} \end{bmatrix}$$

4. Find the largest eigen value and the corresponding eigen vector of the following matrix using the power method with an accuracy of 2 decimal points,

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}.$$

Solution:

Let the initial vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Now, using power method, performing the iterations as

$$AX_0 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} = 5\begin{bmatrix} 0.8 \\ 1 \\ 0.4 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8 \\ 1 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 3.2 \\ 3.4 \\ 2.4 \end{bmatrix} = 3.4\begin{bmatrix} 0.9412 \\ 1 \\ 0.7059 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.9412 \\ 1 \\ 0.7059 \end{bmatrix} = \begin{bmatrix} 3.6471 \\ 4.2942 \\ 2.2353 \end{bmatrix} = 4.2942\begin{bmatrix} 0.8493 \\ 1 \\ 0.5205 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8493 \\ 1 \\ 0.5205 \end{bmatrix} = \begin{bmatrix} 3.3698 \\ 3.7396 \\ 2.3288 \end{bmatrix} = 3.7396\begin{bmatrix} 0.9011 \\ 1 \\ 0.6227 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.9011 \\ 1 \\ 0.6227 \end{bmatrix} = \begin{bmatrix} 3.5238 \\ 4.0476 \\ 2.2784 \end{bmatrix} = 4.0476\begin{bmatrix} 0.8706 \\ 1 \\ 0.5629 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8706 \\ 1 \\ 0.5629 \end{bmatrix} = \begin{bmatrix} 3.4335 \\ 3.8670 \\ 2.3077 \end{bmatrix} = 3.8670\begin{bmatrix} 0.8879 \\ 1 \\ 0.5968 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8879 \\ 1 \\ 0.5968 \end{bmatrix} = \begin{bmatrix} 3.4847 \\ 3.9694 \\ 2.2911 \end{bmatrix} = 3.9694\begin{bmatrix} 0.8779 \\ 1 \\ 0.5772 \end{bmatrix}$$

$$AX_7 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8779 \\ 1 \\ 0.5772 \end{bmatrix} = \begin{bmatrix} 3.4551 \\ 3.9102 \\ 2.3007 \end{bmatrix} = 3.9102\begin{bmatrix} 0.8836 \\ 1 \\ 0.5884 \end{bmatrix}$$

Scanned with CamScanner

$$AX_8 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8836 \\ 1 \\ 0.5884 \end{bmatrix} = \begin{bmatrix} 3.4720 \\ 3.9440 \\ 2.2952 \end{bmatrix} = 3.9440\begin{bmatrix} 0.8803 \\ 1 \\ 0.5819 \end{bmatrix}$$

$$AX_9 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8803 \\ 1 \\ 0.5819 \end{bmatrix} = \begin{bmatrix} 3.4622 \\ 3.9244 \\ 2.2984 \end{bmatrix} = 3.9244\begin{bmatrix} 0.8822 \\ 1 \\ 0.5857 \end{bmatrix}$$

$$AX_{10} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8822 \\ 1 \\ 0.5857 \end{bmatrix} = \begin{bmatrix} 3.4679 \\ 3.9358 \\ 2.2965 \end{bmatrix} = 3.9358\begin{bmatrix} 0.8811 \\ 1 \\ 0.5835 \end{bmatrix}$$

$$AX_{11} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8811 \\ 1 \\ 0.5835 \end{bmatrix} = \begin{bmatrix} 3.4646 \\ 3.9292 \\ 2.2976 \end{bmatrix} = 3.9292\begin{bmatrix} 0.8818 \\ 1 \\ 0.5848 \end{bmatrix}$$

$$AX_{12} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8818 \\ 1 \\ 0.5848 \end{bmatrix} = \begin{bmatrix} 3.4666 \\ 3.9332 \\ 2.2970 \end{bmatrix} = 3.9332\begin{bmatrix} 0.8814 \\ 1 \\ 0.5840 \end{bmatrix}$$

$$AX_{13} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.8814 \\ 1 \\ 0.5840 \end{bmatrix} = \begin{bmatrix} 3.4654 \\ 3.9308 \\ 2.2974 \end{bmatrix} = 3.9308\begin{bmatrix} 0.8816 \\ 1 \\ 0.5845 \end{bmatrix}$$

Here the values are correct upto 2 decimal places.

Hence the required eigen values is 3.9308.

And the required eigen vector is $\begin{bmatrix} 0.8816 \\ 1 \\ 0.5845 \end{bmatrix}$.

5. Solve the following linear equations using Guass elimination method using partial pivoting.

$$2x + 3y + 2z = 2$$
$$10x + 3y + 4z = 16$$
$$3x + 6y + z = 6$$

Solution:

Writing the given system of equations in matrix form

$$\begin{bmatrix} 2 & 3 & 2 \\ 10 & 3 & 4 \\ 3 & 6 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 16 \\ 6 \end{bmatrix}$$

Interchanging R_1 and R_2 but not x and y as partial pivoting.

$$\begin{bmatrix} 10 & 3 & 4 \\ 2 & 3 & 2 \\ 3 & 6 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 2 \\ 6 \end{bmatrix}$$

Operate $R_2 \to R_2 - \frac{2}{10}R_1$ and $R_3 \to R_3 - \frac{3}{10}R_1$

$$\begin{bmatrix} 10 & 3 & 4 \\ 0 & 2.4 & 1.2 \\ 0 & 5.1 & -0.2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ -1.2 \\ 1.2 \end{bmatrix}$$

Scanned with CamScanner

Interchanging R_2 and R_3 but not y and z variable

$$\begin{bmatrix} 10 & 3 & 4 \\ 0 & 5.1 & -0.2 \\ 0 & 2.4 & 1.2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 16 \\ 1.2 \\ -1.2 \end{bmatrix}$$

Operate $R_3 \to R_3 - \frac{2.4}{5.1} R_2$

$$\begin{bmatrix} 10 & 3 & 4 \\ 0 & 5.1 & -0.2 \\ 0 & 0 & 1.2941 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 16 \\ 1.2 \\ -1.7647 \end{bmatrix}$$

Now performing backward substitution,

or, $1.2941z = -1.7647$

∴ $z = -1.3637$

or, $5.1y - 0.2z = 1.2$

∴ $y = 0.1818$

or, $10x + 3y + 4z = 16$

∴ $x = 2.0909$

6. Solve the following system of linear algebraic equations using the Gauss elimination method.

$$2x_1 + 3x_2 + 2x_3 + 5x_4 = 11$$
$$4x_1 + 2x_2 + 2x_3 + 4x_4 = 11$$
$$4x_1 + x_2 + 4x_3 + 5x_4 = 11$$
$$5x_1 - 5x_2 + 3x_3 + x_4 = 11$$

Solution:

Writing the given system of equations in matrix form

$$\begin{bmatrix} 2 & 3 & 2 & 5 \\ 4 & 2 & 2 & 4 \\ 4 & 1 & 4 & 5 \\ 5 & -5 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 11 \\ 11 \\ 11 \\ 11 \end{bmatrix}$$

Operate $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - \frac{5}{2} R_1$

$$\begin{bmatrix} 2 & 3 & 2 & 5 \\ 0 & -4 & -2 & -6 \\ 0 & -5 & 0 & -5 \\ 0 & -12.5 & -2 & -11.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 11 \\ -11 \\ -11 \\ -16.5 \end{bmatrix}$$

Operate $R_3 \to R_3 - \frac{5}{4} R_2$ and $R_4 \to R_4 - \frac{12.5}{4} R_2$

$$\begin{bmatrix} 2 & 3 & 2 & 5 \\ 0 & -4 & -2 & -6 \\ 0 & 0 & 2.5 & 2.5 \\ 0 & 0 & 4.25 & 7.25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 11 \\ -11 \\ 2.75 \\ 17.875 \end{bmatrix}$$

Scanned with CamScanner

Operate $R_4 \rightarrow R_4 - \frac{4.25}{2.5} R_3$

$$\begin{bmatrix} 2 & 3 & 2 & 5 \\ 0 & -4 & -2 & -6 \\ 0 & 0 & 2.5 & 2.5 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11 \\ -11 \\ 2.75 \\ 13.2 \end{bmatrix}$$

Now, performing backward substitution,

or, $3x_4 = 13.2$

∴ $x_4 = 4.4$

or, $2.5x_3 + 2.5x_4 = 2.75$

∴ $x_3 = -3.3$

or, $-4x_2 - 2x_3 - 6x_4 = -11$

∴ $x_2 = -2.2$

or, $2x_1 + 3x_2 + 2x_3 + 5x_4 = 11$

∴ $x_1 = 1.1$

Hence, the required values of the equation are;

$x_1 = 1.1, x_2 = -2.2, x_3 = -3.3, x_4 = 4.4$

7. Solve the following system of linear equations using the Gauss Seidal iteration method.

$$x_1 + 3x_2 - x_3 + 7x_4 = 19$$
$$2x_1 + 8x_2 + x_3 - x_4 = 17$$
$$3x_1 + x_2 + 9x_3 - x_4 = 15$$
$$9x_1 - x_2 - x_3 + 2x_4 = 13$$

Solution:

Arranging the given linear equations in diagonally dominant form

$$9x_1 - x_2 - x_3 + 2x_4 = 13$$
$$2x_1 + 8x_2 + x_3 - x_4 = 17$$
$$3x_1 + x_2 + 9x_3 - x_4 = 15$$
$$x_1 + 3x_2 - x_3 + 7x_4 = 19$$

Now, forming the equations as

$$x_1 = \frac{13 + x_2 + x_3 - 2x_4}{9}$$

$$x_2 = \frac{17 - 2x_1 - x_3 + x_4}{8}$$

$$x_3 = \frac{15 - 3x_1 - x_2 + x_4}{9}$$

$$x_4 = \frac{19 - x_1 - 3x_2 + x_3}{7}$$

Let the initial guess be 0 for x_1, x_2, x_3 and x_4.

Solving the iterations in tabulator form

Iteration	$x_1 = \frac{13+x_2+x_3-2x_4}{9}$	$x_2 = \frac{17-2x_1-x_3+x_4}{8}$	$x_3 = \frac{15-3x_1-x_2+x_4}{9}$	$x_4 = \frac{19-x_1-3x_2+x_3}{7}$
Guess	0	0	0	0
1	1.4444	1.7639	0.9892	1.8933
2	1.3296	1.9056	1.2221	1.8822
3	1.3737	1.8641	1.2108	1.8921
4	1.3656	1.8688	1.2141	1.8917
5	1.3666	1.8681	1.2138	1.8918
6	1.3665	1.8681	1.2138	1.8919

Here, the values of x_1, x_2, x_3 and x_4 are correct upto 3 decimal places.

So, the approximated values of $x_1 = 1.3665$, $x_2 = 1.8681$, $x_3 = 1.2138$ and $x_4 = 1.8919$.

8. Find the largest eigen value and the corresponding vector of the following matrix using the power method.

$$\begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}$$

Solution:

Let the initial vector be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Now, using power method, performing the iterations as

$$AX_0 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 14 \end{bmatrix} = 14\begin{bmatrix} 0.5714 \\ 0.4286 \\ 1 \end{bmatrix}$$

Again,

$$AX_1 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.5714 \\ 0.4286 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.2858 \\ 4.9998 \\ 11.8572 \end{bmatrix} = 11.8572\begin{bmatrix} 0.3615 \\ 0.4217 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.3615 \\ 0.4217 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.8315 \\ 3.9641 \\ 11.6266 \end{bmatrix} = 11.6266\begin{bmatrix} 0.3295 \\ 0.3410 \\ 1 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.3295 \\ 0.3410 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.3640 \\ 3.9655 \\ 11.3525 \end{bmatrix} = 11.3525\begin{bmatrix} 0.2963 \\ 0.3493 \\ 1 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.2963 \\ 0.3493 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.3391 \\ 3.7829 \\ 11.3442 \end{bmatrix} = 11.3442\begin{bmatrix} 0.2943 \\ 0.3335 \\ 1 \end{bmatrix}$$

Scanned with CamScanner

$$AX_5 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.2943 \\ 0.3335 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2561 \\ 3.8045 \\ 11.2948 \end{bmatrix} = 11.2948\begin{bmatrix} 0.2883 \\ 0.3368 \\ 1 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.2883 \\ 0.3368 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2606 \\ 3.7679 \\ 11.2987 \end{bmatrix} = 11.2987\begin{bmatrix} 0.2886 \\ 0.3335 \\ 1 \end{bmatrix}$$

$$AX_7 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.2886 \\ 0.3335 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2447 \\ 3.7760 \\ 11.2891 \end{bmatrix} = 11.2891\begin{bmatrix} 0.2874 \\ 0.3345 \\ 1 \end{bmatrix}$$

$$AX_8 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.2874 \\ 0.3345 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2473 \\ 3.7680 \\ 11.2909 \end{bmatrix} = 11.2909\begin{bmatrix} 0.2876 \\ 0.3337 \\ 1 \end{bmatrix}$$

$$AX_9 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.2876 \\ 0.3337 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2437 \\ 3.7706 \\ 11.2887 \end{bmatrix} = 11.2887\begin{bmatrix} 0.2873 \\ 0.3340 \\ 1 \end{bmatrix}$$

$$AX_{10} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.2873 \\ 0.3340 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2446 \\ 3.7685 \\ 11.2893 \end{bmatrix} = 11.2893\begin{bmatrix} 0.2874 \\ 0.3338 \\ 1 \end{bmatrix}$$

$$AX_{11} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.2874 \\ 0.3338 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2438 \\ 3.7694 \\ 11.2888 \end{bmatrix} = 11.2888\begin{bmatrix} 0.2873 \\ 0.3339 \\ 1 \end{bmatrix}$$

Hence, the required eivgen value is $11.2888 \approx 11.29$.

And the corresponding vector is $\begin{bmatrix} 0.2873 \\ 0.3339 \\ 1 \end{bmatrix}$.

9. Solve the following set of linear equations using LU factorization method.

$$x - 3y + 10z = 3$$
$$-x + 4y + 2z = 20$$
$$5x + 2y + z = -12$$

Solution:

Writing the given set of equations in matrix form AX = B

$$\begin{bmatrix} 1 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

In LU factorization method, we represent A as,

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

Solving for unknown values,

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & u_{12}l_{21} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

Scanned with CamScanner

$u_{11} = 1$	$u_{12} = -3$	$u_{13} = 10$
$l_{21}u_{11} = -1$ $\therefore l_{21} = -1$	$l_{21}u_{12} + u_{22} = 4$ $\therefore u_{22} = 1$	$l_{21}u_{13} + u_{23} = 2$ $\therefore u_{23} = 12$
$l_{31}u_{11} = 5$ $\therefore l_{31} = 5$	$l_{31}u_{12} + l_{32}u_{22} = 2$ $\therefore l_{32} = 17$	$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$ $\therefore u_{33} = -253$

Substituting the values,

$$\underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 17 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & -3 & 10 \\ 0 & 1 & 12 \\ 0 & 0 & -253 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}}$$

Here, LUX = B

Let, UX = V

so, LV = B then,

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 17 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

Performing forward substitution,

$\therefore$ $v_1 = 3$

or, $-v_1 + v_2 = 20$

$\therefore$ $v_2 = 23$

or, $5v_1 + 17v_2 + v_3 = -12$

$\therefore$ $V_3 = -418$

Then, UX = V becomes

$$\begin{bmatrix} 1 & -3 & 10 \\ 0 & 1 & 12 \\ 0 & 0 & -253 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 23 \\ -418 \end{bmatrix}$$

Performing backward substitution,

or, $-253z = -418$

$\therefore$ $z = 1.6522$

or, $y + 12z = 23$

$\therefore$ $y = 3.1736$

or, $x - 3y + 10z = 3$

$\therefore$ $x = -4.0012$

10. Find the inverse of the matrix, using Gauss Jordan elimination method

$$A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & 5 & 3 \end{bmatrix}$$

Solution:

The augumented matrix can be written as

$$[A : I] = \begin{bmatrix} 4 & 3 & -1 & : & 1 & 0 & 0 \\ 1 & 1 & 1 & : & 0 & 1 & 0 \\ 3 & 5 & 3 & : & 0 & 0 & 1 \end{bmatrix}$$

Scanned with CamScanner

Interchanging R_1 and R_2

$$[A:I]=\begin{bmatrix} 1 & 1 & 1 & : & 0 & 1 & 0 \\ 4 & 3 & -1 & : & 1 & 0 & 0 \\ 3 & 5 & 3 & : & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_2 \to R_2 - 4R_1$ and $R_3 \to R_3 - 3R_1$

$$[A:I]=\begin{bmatrix} 1 & 1 & 1 & : & 0 & 1 & 0 \\ 0 & -1 & -5 & : & 1 & -4 & 0 \\ 0 & 2 & 0 & : & 0 & -3 & 1 \end{bmatrix}$$

Operate $R_3 \to R_3 + 2R_2$

$$[A:I]=\begin{bmatrix} 1 & 1 & 1 & : & 0 & 1 & 0 \\ 0 & -1 & -5 & : & 1 & -4 & 0 \\ 0 & 0 & -10 & : & 2 & -11 & 1 \end{bmatrix}$$

Operate $R_3 \to \frac{R_3}{-10}$

$$[A:I]=\begin{bmatrix} 1 & 1 & 1 & : & 0 & 1 & 0 \\ 0 & -1 & -5 & : & 1 & -4 & 0 \\ 0 & 0 & -1 & : & -1/5 & 11/10 & -1/10 \end{bmatrix}$$

Operate $R_2 \to \frac{R_2}{-1}$

$$[A:I]=\begin{bmatrix} 1 & 1 & 1 & : & 0 & 1 & 0 \\ 0 & 1 & 5 & : & -1 & 4 & 0 \\ 0 & 0 & 1 & : & -1/5 & 11/10 & -1/10 \end{bmatrix}$$

Operate $R_1 \to R_1 - R_2$

$$[A:I]=\begin{bmatrix} 1 & 0 & -4 & : & 1 & -3 & 0 \\ 0 & 1 & 5 & : & -1 & 4 & 0 \\ 0 & 0 & 1 & : & -1/5 & 11/10 & -1/10 \end{bmatrix}$$

Operate $R_2 \to R_2 - 5R_3$

$$[A:I]=\begin{bmatrix} 1 & 0 & -4 & : & 1 & -3 & 0 \\ 0 & 1 & 0 & : & 0 & -1.5 & 0.5 \\ 0 & 0 & 1 & : & -1/5 & 1.1 & -0.1 \end{bmatrix}$$

Operate $R_1 \to R_1 + 4R_3$

$$[A:I]=\begin{bmatrix} 1 & 0 & 0 & : & 0.2 & 1.4 & -0.4 \\ 0 & 1 & 0 & : & 0 & -1.5 & 0.5 \\ 0 & 0 & 1 & : & -0.2 & 1.1 & -0.1 \end{bmatrix}$$

For inversion of matrix

$$[A:I] = [I:A]$$

$$[A = A^{-1}]$$

Hence,

$$A^{-1} = \begin{bmatrix} 0.2 & 1.4 & -0.4 \\ 0 & -1.5 & 0.5 \\ -0.2 & 1.1 & -0.1 \end{bmatrix}$$

Scanned with CamScanner

Chapter

5

SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

♠♠♠

5.1 INTRODUCTION OF INITIAL AND BOUNDARY VALUE PROBLEMS

The solution of an ordinary differential equation means finding an explicit expression for y in terms of a finite number of elementary functions of x. Such a solution of a differential equation is known as the closed or finite form of solution. In the absence of such a solution, we have recourse to numerical methods of solution.

let us consider the first order differential equation,

$$\frac{dy}{dx} = f(x, y) \text{ given } y(x_0) = y_0 \qquad \text{..... (1)}$$

to study the various numerical methods of solving such equations. In most of these methods, we replace the differential equations by a difference equation and then solve it. These methods yields solutions either as a power series in x from which the values of y can be found by direct substitution or a set of values of x and y. The methods of Picard and Taylor series belong to the former class of solution. In these methods, y in equation (1) is approximated by a truncated series, each term, of which is a function of x. The information about the curve at one point is utilized and the solution is not iterated. As such, these are referred to as single-step methods.

Scanned with CamScanner

The methods of Euler, Range-Kutta, Milne, etc belong to the latter class of solutions. In these methods, the next point on the curve is evaluated in short steps ahead, by performing iterations until sufficient accuracy is achieved. As such, these methods are called step-by-step methods.

Euler and Runge-Kutta methods are used for computing y over a limited range of x-values whereas Milne and Adams methods may be applied for finding y over a wider range of x-values which are found by Picard's Taylor series or Runge-Kutta methods.

Initial and Boundary Conditions

An ordinary differential equation of the n^{th} order is of the form,

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{d^2x}, \dots\dots\dots, \frac{d^ny}{d^nx}\right) = 0 \quad \text{..... (2)}$$

Its general solution contains n arbitrary constants and is of the form,

$$\phi\,(x, y, c_1, c_2, \dots\dots\dots, c_n) = 0 \quad \text{..... (3)}$$

To obtain its particular solution, n conditions must be given so that the constants $c_1, c_2, \dots\dots\dots, c_n$ can be determined.

If these conditions are prescribed at one point only (say: x_0), then the differential equation together with the conditions constitute an initial value problem of the n^{th} order.

If the conditions are prescribed at two or more points, then the problem is termed as boundary value problem.

5.2 PICARD'S METHOD

Consider the first order equation,

$$\frac{dy}{dx} = f(x, y) \quad \text{..... (1)}$$

It is required to find that particular solution of (1) which assumes the value y_0 when $x = x_0$.

On integrating (1) between limits, we get,

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx$$

or, $$y = y_0 + \int_{x_0}^{x} f(x, y)\, dx \quad \text{..... (2)}$$

This is an integral equation equivalent to (1), for it contains the unknown y under the integral sign.

As a first approximation y_1 to the solution, we put,

$y = y_0$ in $f(x, y)$ and integrate (2), giving,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

Scanned with CamScanner

For a second approximation y_2, we put $y = y_1$ in $f(x, y)$ and integrate (2) giving,

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y)\, dx$$

Similarly, a third approximation is,

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$$

Continuing this process, we get, $y_4, y_5, y_6, \ldots\ldots y_n$, where,

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

Hence this method gives a sequence of approximations $y_1, y_2, y_3, \ldots\ldots\ldots$, each giving a better result than the preceding one.

Example 5.1

Find the value of y for x = 0.1 by Picard's method, given that,

$$\frac{dy}{dx} = \frac{y - x}{y + x}, \quad y(0) = 1$$

Solution:

We have,

$$y = 1 + \int_0^x \frac{y - x}{y + x}\, dx$$

1st approximation:

Put y =1 in the integrand giving

$$y_1 = 1 + \int_0^x \frac{y - x}{y + x}\, dx = 1 + \int_0^x \left(-1 + \frac{2}{1 + x}\right) dx$$

$$= 1 + \left[-x + 2 \log (1 + x)\right]_0^x$$

$$= 1 - x + 2 \log (1 + x)$$

2nd approximation:

Put $y = 1 - x + 2 \log (1 + x)$ in the integrand giving,

$$y_2 = 1 + \int_0^x \frac{1 - x + 2 \log (1 + x) - x}{1 - x + 2 \log (1 + x) - x}\, dx$$

$$= 1 + \int_0^x \left[1 - \frac{2x}{1 + 2\log (1 + x)}\right] dx$$

which is very difficult to integrate.

Hence we use the first approximation and taking x = 0.1, we get,

$$y(0.1) = 1 - 0.1 + 2 \log (1.1) = 0.9828$$

5.3 TAYLOR'S SERIES METHOD

Consider the first order equation,

$$\frac{dy}{dx} = f(x, y) \quad \ldots\ldots (1)$$

Scanned with CamScanner

Differentiating (1) with respect to x, we get,

$$\frac{d_y^2}{d_x^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

i.e., $y'' = f_x + f_y f'$ ---- (2)

Differentiating this successively, we can get y''', y^{iv} etc

Putting $x = x_0$ and $y = 0$, the values of $(y')_0$, $(y'')_0$ and $= y_0$ can be obtained. Hence the Taylor series

$$y = y_0 + (x - x_0)(y') + (x - x_0)^2 (y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \ldots \quad \text{---- (3)}$$

Gives the values of y for every value of x for which (3) converges.

On finding the value y_1 for $x = x_1$ from (3), y', y" etc can be evaluated at $x = x_1$ by means of (1), (2) etc. Then y can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series (3).

> **NOTE:**
> This is a single step method and works well so long as the successive derivatives can be calculated easily. If (x, y) is somewhat complicated and the calculation of higher order derivatives becomes tedious, the Taylor's method cannot be used significantly. This is the main drawback of this method. However, it is useful for finding starting values for the application of powerful methods like Runge-Kutta, Milne method.

Example 5.2

Solve $y' = x + y$, $y(0) = 1$ by Taylor's series method. Hence find the values of y at $x = 0.1$ and $x = 0.2$.

Solution:

Differentiating successively, we get,

$y' = x + y$ $\quad$ $y'(0) = 1$ $\quad$ $[\because y(0) = 1]$

$y'' = 1 + y'$ $\quad$ $y''(0) = 2$

$y''' = y''$ $\quad$ $y'''(0) = 2$, etc

Now, Taylor's series is,

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x + x_0)^2}{2!}(y'')_0 + \frac{(x + x_0)^3}{3!}(y''')_0 + \ldots$$

Here, $x_0 = 0, y_0 = 1$

$$\therefore \quad y = 1 + x(1) + \frac{x^2}{2} \times 2 + \frac{x^3}{3!} \times 2 + \frac{x^4}{4!} \times 4 \ldots$$

$$\text{Hence, } y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{3!} + \ldots$$

$$= 1.1103$$

$$\text{and, } y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{6} + \ldots$$

$$= 1.2427$$

Scanned with CamScanner

Example 5.3

Find by Taylor's method, the values of the y at x = 0.1 and x = 0.2 to five places of decimals from $\frac{dy}{dx} = x^2y - 1$, y(0) = 1

Solution:

$$\frac{dy}{dx} = x^2 y - 1$$

Differentiating successively, we get,

$y' = x^2y - 1$, $(y')_0 = -1$ $[\because y(0) = 1]$

$y'' = 2xy + x^2y'$, $(y'')_0 = 0$

$y''' = 2y + 4xy' + x\,2y''$, $(y''')_0 = 2$

$y^{iv} = 6y' + 6xy'' + x\,2y'''$, $(y^{iv})_0 = -6$ etc

Replacing these values in the Taylor series, we get,

$$y = 1 + x(-1) + \frac{x^2}{2}(0) + \frac{(x^3)}{3!} \times 2 + \frac{x^4}{4!}(-6) + \dots\dots$$

$$= 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots$$

$$y(0.1) = 1 - 0.1 + \frac{0.1^3}{3} - \frac{0.1^4}{4} = 0.90033$$

Hence, y(0.1) = 0.90033 and y(0.2) = 0.80227

Example 5.4

Solve by Taylor series method of third order equation $\frac{dy}{dx} = \frac{x^3 + xy^2}{e^x}$, y(0) = 1 for y at x = 0.1 x = 0.2 and x = 0.3.

Solution:

We have,

$$y' = (x^3 + xy^2)\, e^{-x},\; y'(0) = 0$$

Differentiating successively and replacing x = 0 and y = 1

$$y'' = (x^3 + xy^2)(-e^{-x}) + (3x^2 + y^2 + x2y \cdot y')\, e^{-x}$$

$$= (-x^3 - xy^2 + 3x^2 + y^2 + 2xyy')\, e^{-x};\; y''(0) = 1$$

$$y''' = (-x^3 - xy^2 + 3x^2 + y^2 + 2xyy')(-e^{-x}) + \{-3x^2 - (y^2 + x\,2y\,y') + 6x + 2yy + 2[yy' + x(y'^2 + yy'')]\}\, e^{-x},\quad y'''(0) = -2$$

Replacing these values in the Taylor's series, we have,

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \dots\dots$$

$$= 1 + x(0) + \frac{x^2}{2}(1) + \frac{x^3}{6}(-2) + \dots\dots = 1 + \frac{x^2}{2} - \frac{x^3}{3} + \dots\dots$$

Hence, $y(0.1) = 1 + \frac{1}{2}(0.1)^2 - \frac{1}{3}(0.1)^3 = 1.005$

$$y(0.2) = 1 + \frac{1}{2}(0.2)^2 - \frac{1}{3}(0.2)^3 = 1.017$$

$$y(0.3) = 1 + \frac{1}{2}(0.3)^2 - \frac{1}{3}(0.3)^3 = 1.036$$

Scanned with CamScanner

5.4 THE EULER METHOD

Consider the equation,

$$\frac{dy}{dx} = f(x, y) \quad \text{..... (1)}$$

given that $y(x_0) = y_0$. Its curve of solution through $P(x_0, y_0)$ is shown dotted in figure 5.1. Now, we have to find the ordinate of any other point Q on this curve.

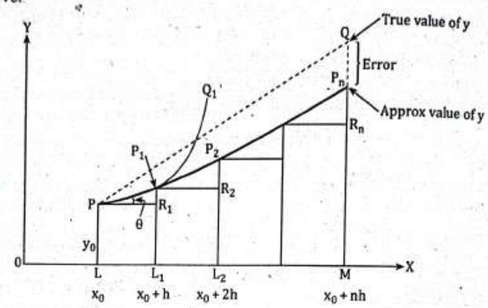


Figure 5.1

Let us divide LM into n-sub-intervals each of width h at $L_1, L_2, \ldots\ldots$, so that h is quite small. In the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in P_1 $(x_0 + h, y_1)$ then,

$$y_1 = L_1P_1 = LP + R_1P_1 = y_0 + PR_1 \tan\theta$$

$$= y_0 + h\left(\frac{dy}{dx}\right)_p = y_0 + hf(x_0, y_0)$$

Let P_1Q_1 be the curve of solution (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in P_2 $(x_0 + 2h, y_2)$. Then,

$$y_2 = y_1 + hf(x_0 + h, y_1) \quad \text{..... (2)}$$

Repeating this process n times, we finally reach on an approximation MP_n of MQ given by,

$$y_n = y_{n-1} + hf(x_0 + \overline{n-1}h, y_{n-1})$$

This is Fuller's method of finding an approximate solution of (1).

NOTE:
In Euller's method, we approximate the curve of solution by the tangent in each interval *i.e.*, by a sequence of short lines. Unless h is small, the error is bound to be quite significant. This sequence of lines may also deviate considerably from the curve of solution. As such, the method is very slow and hence there is a modification of this method which is given in next section.

Scanned with CamScanner

Example 5.5

Using Euler's method, find an approximate value of y corresponding to x = 1 given that $\frac{dy}{dx} = x + y$ and y = 1 when x = 0.

Solution:

Given that;

$$\frac{dy}{dx} = x + y$$

We take n = 10 and h = 0.1 which is sufficiently small. The various calculations are arranged as follows.

x	y	$x + y = \frac{dy}{dx}$	Old y + 0.1 $\left(\frac{dy}{dx}\right)$	New y
0.1	1.00	1.00	1.00 + 0.1 (1.00)	1.10
0.1	1.10	1.20	1.10 + 0.1 (1.20)	1.22
0.2	1.22	1.42	1.22 + 0.1 (1.42)	1.36
0.3	1.36	1.66	1.36 + 0.1 (1.66)	1.53
0.4	1.53	1.93	1.53 + 0.1 (1.93)	1.72
0.5	1.72	2.22	1.72 + 0.1 (2.22)	1.94
0.6	1.94	2.54	1.94 + 0.1 (2.54)	2.19
0.7	2.19	2.89	2.19 + 0.1 (2.89)	2.48
0.8	2.48	3.29	2.48 + 0.1 (3.29)	2.81
0.9	2.81	3.71	2.81 + 0.1 (3.71)	3.18
1.0	3.18			

Thus the required approximation value of y = 3.18

5.5 MODIFIED EULER'S METHOD OR HUEN'S METHOD

In Euler's method, the curve of solution in the interval LL_1 is approximated by the tangent at P (Figure 5.1) such that at P_1, we have,

$$y_1 = y_0 + h\, f(x_0, y_0) \quad \text{..... (1)}$$

Then the slope of the curve of solution through P_1,

$$\left[i.e., \left(\frac{dy}{dx}\right) P_1 = f(x_0 + h, y_1)\right] \quad \text{..... (5)}$$

is computed and the tangent at P_1 to P_1Q_1 is drawn meeting the ordinate through L_2 in,

$$P_2(x_0 + 2h, y_2)$$

Now, we find a better approximation y_1^1 of $y(x_0 + h)$ by taking the slope of the curve as the mean of the slopes of the tangents at P and P_1,

Scanned with CamScanner

i.e., $y_1^1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1)]$

As the slope of the tangent at P_1 is not known, we take y_1 as found in (1) by Euler's method and insert it on RHS of (2) to obtain the first modified value y_1. Again (2) is applied and we find a still better value $y_{1(2)}$ corresponding to L_1 as,

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1^{(1)})]$$

We repeat this step, until two consecutive values of y agree. This is then taken as the starting point for the next interval L_1L_2. Once y_1 is obtained to a desired degree of accuracy, y corresponding to L_2 is found from (1).

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

and a better approximation $y_2^{(1)}$ is obtained from (2)

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)]$$

We repeat this step until y_2 becomes stationary. Then we proceed to calculate $y_3^{(2)}$, as above and so on. This is the modified Euler's method which gives great improvement in accuracy over the original method.

Example 5.6

Using modified Euler's method, find an approximate value of y when x = 0.3 given that $\frac{dy}{dx} = x + y$ and y = 1 when x = 0.

Solution:

The various calculations are arranged as followings taking h = 0.1

x	x + y = y'	Mean slope	Old y + 0.1 (mean slope) = New y
0.0	0 + 1	–	1.00 + 0.1 × 1.00 = 1.10
0.1	0.1 + 1.1	$\frac{1}{2}(1 + 1.2)$	1.00 + 0.1 (1.1) = 1.11
0.1	0.1 + 1.11	$\frac{1}{2}(1 + 1.21)$	1.00 + 0.1 (1.105) = 1.1105
0.1	0.1 + 1.1105	$\frac{1}{2}(1 + 1.2105)$	1.00 + 0.1 (1.1052) = 1.1105

Since the last two values are equal, we take y(0.1) = 1.1105.

x	x + y = y'	Mean slope	Old y + 0.1 (mean slope) = New y
0.1	1.2105	–	1.1105 + 0.1 (1.2105) = 1.2316
0.2	0.2 + 1.2316	$\frac{1}{2}(1.12105+1.4316)$	1.1105 + 0.1 (1.3211) = 1.2426
0.2	0.2 + 1.2426	$\frac{1}{2}(1.2105+1.4426)$	1.1105 + 0.1 (1.3266) = 1.2432
0.2	0.2 + 1.2432	$\frac{1}{2}(1.2105+1.4432)$	1.1105 + 0.1 (1.3268) = 1.2432

Scanned with CamScanner

Since the last two values are equal, we take y(0.2) = 1.2432

x	x + y = y'	Mean slope	Old y + 0.1 (mean slope) = New y
0.2	0.3 + 1.3875	-	1.2432 + 0.1 (1.4432) = 1.3875
0.3	0.3 + 1.3875	$\frac{1}{2}$(1.4432 + 1.6875)	1.2432 + 0.1 (1.5654) = 1.3997
0.3	0.3 + 1.3997	$\frac{1}{2}$(1.4432 + 1.6997)	1.2432 + 0.1 (1.5715) = 1.4003
0.3	0.3 + 1.4003	$\frac{1}{2}$(1.4432 + 1.7003)	1.2432 + 0.1 (1.5718) = 1.4004
0.3	0.3 + 1.4004	$\frac{1}{2}$(1.4432 + 1,7004)	1.2432 + 0.1(1.5718) = 1.4004

Since the last two values are equal, we take, y(0.3) = 1.4004

Hence, y(0.3) = 1.4004 approximately.

Example 5.7

Solve the following by Euler's modified method

$$\frac{dy}{dx} = \log(x + y),\ y(0) = 2$$

at x = 1.2 and 1.4 with h = 0.2

Solution:

The various calculations are arranged as follows

x	log(x + y) = y'	Mean slope	Old y + 0.2 (mean slope) = New y
0.0	log(0+2)	-	2+0.2(0.301)=2.0602
0.2	log(0.2+2.0602)	$\frac{1}{2}$(0.310+0.3541)	2+0.2(0.3276)=2.0655
0.2	log(0.2+2.0655)	$\frac{1}{2}$(0.301+0.3552)	2+0.2(0.3281)=2.0656

x	log(x + y) = y'	Mean slope	Old y + 0.2 (mean slope) = New y
0.2	0.3552	-	2.0656+0.2(0.3552)=2.1366
0.4	log(0.4+2.1366)	$\frac{1}{2}$(0.3552+0.4042)	2.056+0.2(0.3797)=2.1415
0.4	log(0.4+2.1415)	$\frac{1}{2}$(0.3552+0.4051)	2.0656+0.2(0.3801)=2.1416

x	log(x + y) = y'	Mean slope	Old y + 0.2 (mean slope) = New y
0.4	0.4051	-	2.1416+0.2(0.4051)=2.2226
0.6	log(0.6+2.2226)	$\frac{1}{2}$(0.4051+0.4506)	2.1416+0.2(0.4279)=2.2272
0.6	log(0.6+2.2272)	$\frac{1}{2}$(0.4051+0.4514)	2.1416+0.2(0.4282)=2.2272
0.6	0.4514	-	2.2272+0.2(0.4514)=2.3175
0.8	log(0.8+2.3175)	$\frac{1}{2}$(0.4514+0,4938)	2.2272+0.2(0.4726)=2.3217

Scanned with CamScanner

x	log(x + y) = y'	Mean slope	Old y + 0.2 (mean slope) = New y
0.8	log(0.8+2.3217)	$\frac{1}{2}$(0.4514+0.4913)	2.2272+0.2(0.4727)=2.3217
0.8	0.4943	–	2.3217+0.2(0.4943)=2.4206
1.0	log(1+2.4206)	$\frac{1}{2}$(0.4943+0.5341)	2.3217+0.2(0.5142)=2.4245

x	log(x + y) = y'	Mean slope	Old y + 0.2 (mean slope) = New y
1.0	log(1+2.4245)	$\frac{1}{2}$(0.4943+0.5346)	2.3217+0.2(0.5144)=2.4245
1.0	0.5346	–	2.425+0.2(0.5346)=2.5314
1.2	log(1.2+2.5314)	$\frac{1}{2}$(0.5346+0.5719)	2.4245+0.2(0.5532)=2.5351

x	log(x + y) = y'	Mean slope	Old y + 0.2 (mean slope) = New y
1.2	log(1.2+2.5351)	$\frac{1}{2}$(0.5346+0.5123)	2.4245+0.2(0.5534)=2.5351
1.2	0.5723	–	2.5351+0.2(0.5723)=2.6496

x	log(x + y) = y'	Mean slope	Old y + 0.2 (mean slope) = New y
1.4	log(1.4+2.6496)	$\frac{1}{2}$(0.5723+0.6074)	2.5351+0.2(0.5898)=2.6531
1.4	log(1.4+2.6531)	$\frac{1}{2}$(0.5723+0.6078)	2.5351+0.2(0.5900)=2.6531

Hence, y (1.2) = 2.5351 and dy (1.4) = 2.6531 approximately.

5.6 RUNGE'S METHOD

Consider the differential equation,

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \quad \text{.... (1)}$$

Clearly the slope of the curve through $P(x_0, y_0)$ is $f(x_0, y_0)$

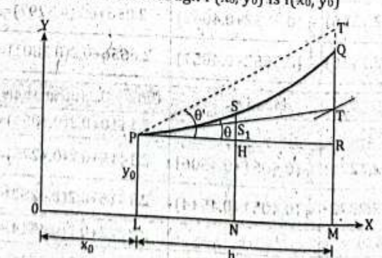


Figure 5.2

Scanned with CamScanner

Integrating both sides of equation (1) from (x_0, y_0) to $(x_0 + h, y_0 + k)$, we have,

$$\int_0^{y_0+k} d_y = \int_{x_0}^{x_0+h} f(x, y)\, dx \qquad \text{..... (2)}$$

To evaluate the integral on the right, we take N as the midpoint of LM and find the values of f(x, y) $\left(i.e., \frac{dy}{dx}\right)$ at the point x_0, $x_0 + \frac{h}{2}$, $x_0 + h$. For this purpose, we first determine the values of y at these points.

Let the ordinate through N cut the curve PQ in S and the tangent PT in S_1.

The value of y_s is given by the point S_1,

$$\therefore \quad y_s = NS_1 = LP + HS_1 = y_0 + PH \tan\theta$$

$$= y_0 + h\left(\frac{dy}{dx}\right)_p$$

$$= y_0 + \frac{h}{2} f(x_0, y_0) \qquad \text{..... (3)}$$

Also, $y_T = MT = LP + RT = y_0 + PR \tan\theta = y_0 + hf(x_0 + y_0)$

Now the value of y_Q at $x_0 + h$ is given by the point T" where the line through P drawn with slope at $T(x_0 + h_1, y_T)$ meets MQ.

$$\therefore \quad \text{Slope at } T = \tan\theta' = f(x_0 + h_1, y_T)$$

$$= f[x_0 + h, y_0 + hf(x_0, y_0)]$$

$$\therefore \quad y_Q = R + RT = y_0 + PR \tan\theta'$$

$$= y_0 + hf[x_0 + h, y_0 + hf(x_0, y_0)] \qquad \text{..... (4)}$$

Thus, the value of f(x, y) at P = $f(x_0, y_0)$

the value of f(x, y) at S = $f\left(x_0 + \frac{h}{2}, y_s\right)$

the value of (x, y) at Q = $(x_0 + h_1, y_0)$

where, y_s and y_Q are given by (3) and (4)

Hence from (2), we get,

$$k = \int_{x_0}^{x_0+h} f(x, y)\, dx = \frac{h}{6}[f_p + 4f_s + f_Q] \qquad \text{[By Simpson's rule]}$$

$$= \frac{h}{6}\left[f(x_0 + y_0) + f\left(x_0 + \frac{h}{2}, y_s\right) + (x_0 + h, y_Q)\right] \qquad \text{..... (5)}$$

which gives a sufficiently accurately value of k and also $y = y_0 + k$.

The repeated application of (5) gives the values of y for equispaced points.

Working Rule to Solve by Runge's Method

Calculate successively;

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k' = hf(x_0 + h, y_0 + k_1)$$

Scanned with CamScanner

and, $k_3 = hf(x_0 + h, y_0 + k')$

Finally compute,

$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

which gives the required approximate value as $y_1 = y_0 + k$.

Note that k is the weighted mean of k_1, k_2 and k_3.

Example 5.8

Apply Runge's method to find an approximate value of y when x = 0.2, given that $\frac{dy}{dx} = x + 1$ and y = 1 when x = 0.

Solution:

Given that;

$$\frac{dy}{dx} = x + 1$$

$$y = 1 \text{ when } x = 0$$

We have,

$$x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$$

$\therefore \quad k_1 = hf(x_0, y_0) = 0.2(1) = 0.20$

$\therefore \quad k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2\, f(0.1, 1.1) = 0.240$

$\therefore \quad k' = hf(x_0 + h, y_0 + k_1) = 0.2f(0.2, 1.2) = 0.280$

$\therefore \quad k_3 = hf(x_0 + h, y_0 + k') = 0.2f(0.1, 1.28) = 0.296$

$\therefore \quad k = \frac{1}{6}(k_1 + 4k_2 + k_3) = \frac{1}{6}(0.20 + 0.960 + 0.296) = 0.2426$

Hence the required approximate value is 1.2426

5.7 RUNGE-KUTTA METHOD

Runge-Kutta method do not require the calculations of higher order derivatives and give greater accuracy. The Runge-Kutta formulae possess the advantage of requiring only the function values at some selected points. These methods agree with Taylor's series solution up to the term in h^r where r differs from method to method and is called the order of that method.

A. First order R-K Method

From Euler's method,

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0' \qquad [\because y' = f(x,y)]$$

Expanding by Taylor's series,

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \ldots\ldots$$

Scanned with CamScanner

It follows that the Euler's method agrees with the Taylor's series solution up to the term in h.

Hence Euler's method is the Runge-Kutta method of the first order.

B. Second order R-K Method

The modified Euler's method gives,

$$y_1 = y + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1)] \quad ..... (1)$$

Replacing $y_1 = y_0 + hf(x_0, y_0)$ on the right hand side of (1), we get,

$$y_1 = y_0 + \frac{h}{2}[f_0 + f(x_0 + h), y_0 + hf_0]$$

where, $f_0 = (x_0, y_0)$ ..... (2)

Expanding L.H.S. by Taylor's series, we get,

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + ... \quad ..... (3)$$

Expanding $f(x_0 + h + y_0, hf_0)$ by Taylor's series for a function of two various, (2) gives,

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + \left\{f_0 = (x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + hf_0\left(\frac{\partial f}{\partial y}\right)_0 + O(h^2)\right\}\right]$$

$$= y_0 + \frac{1}{2}\left[hf_0 + hf_0 + h^2\left\{\left(\frac{\partial f}{\partial x}\right)_0 + \left(\frac{\partial f}{\partial y}\right)_0\right\} + O(h^3)\right]$$

$$= y_0 + hf_0 + \frac{h^2}{2}f_0' + O(h^3) \qquad \left[\because \frac{df(x, y)}{dx} = \frac{\partial f}{\partial x} + \frac{f\partial f}{\partial y}\right]$$

$$= y_0 + hy_0' + \frac{h^2}{2!}y_0'' + O(h^3) \quad ..... (4)$$

where, $O(h^2)$ means terms contianing second and higher powers of h" and is read as order of h^2.

Comparing (3) and (4), it follows that the modified Euler's method agrees with the Taylor's series solution up to the term in h^2.

Hence the modified Euler's method is the Runge-Kutta method of the second order.

The second order Rung-Kutta formula is,

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where, $k_1 = hf(x_0, y_0)$ and $k_2 = hf(x_0 + h, y_0 + k)$

C. Third order R-K Method

Runge's method is the Runge-Kutta method of the third order.

∴ The third order Runge-kutta formula is,

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where, $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{1}{2}h_0, y_0 + \frac{1}{2}k_1\right)$$

and, $k_3 = hf(x_0 + h, y_0 + k')$

where, $k' = k_3 = hf(x_0 + h, y_0 + k_1)$

D. Fourth order R-K Method

This method is most commonly used and is often referred to as the Runge-Kutta method only.

Working rule for finding the increment of k of y corresponding to an increment h of x. By Runge-Kutta method from,

$\frac{dy}{dx} = f(x, y), y(x_0)$ is as follows;

Calculate successively, $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

and, $k_4 = hf(x_0 + h, y_0 + k_3)$

Finally,

$$\text{Compute } k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

which gives the required approximate value as $y_1 = y_0 + k$

NOTE: One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.

Example 5.9

Apply the Runge-Kutta fourth order method to find an approximate value of y when x = 0.2 given that $\frac{dy}{dx} = x + y$ and y = 1 when x = 0.

Solution:

$$\frac{dy}{dx} = x + y$$

Here;

$x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$

$\therefore$ $k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.20$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.240$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2 \times f(0.1, 1.12) = 0.2440$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.244) = 0.2888$$

Scanned with CamScanner

$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.20 + 0.480 + 0.4880 + 0.2888)$

$= \frac{1}{6} \times 1.4568$

$= 0.2428$

Hence the required approximate value of $y = 1.2428$

Example 5.10

Apply the Runge-Kutta method to find the approximate value of y for x = 0.2, in steps 0.1, if $\frac{dy}{dx} = x + y^2$, y = 1 where, x = 0.

Solution:

Given that;

$f(x, y) = x + y^2$

Here, we take h = 0.1 and carry out the calculations in two steps

Step I:

$x_0 = 0, y_0 = 0, h = 0.1$

$\therefore \quad k_1 = hf.(x_0, y_0) = 0.1\, f(0, 1) = 0.10$

$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.1\, f(0.05, 1.1) = 0.1152$

$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.1\, f(0.05, 1.1152) = 0.1168$

$k_4 = hf\,(x_0 + h, y_0 + k_3) = 0.1\, f(0.1, 1.1168) = 0.1347$

$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.10 + 0.2304 + 0.2336 + 0.1347)$

$= 0.1165$

giving $(0.1) = y_0 + k = 1.1165$

Step II:

$x_1 = x_0 + h = 0.1, y_1 = 1.1165, h = 0.1$

$\therefore \quad k_1 = hf\,(x_1, y_2) = 0.1\, f(0.1, 1.1165) = 0.1347$

$k_2 = hf\left(x_0 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.1\, f\,(0.15, 1.1838) = 0.1551$

$k_3 = hf\left(x_0 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.1\, f\,(0.15, 1.194) = 0.1576$

$k_4 = hf\,(x_1 + h, y_2 + k_3) = 0.1\, f(0.2, 1.1576) = 0.1823$

$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1571$

Hence, $y(0.2) = y_1 + k = 1.2736$

Scanned with CamScanner

5.8 SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS

The simultaneous differential equations of the type,

$$\frac{dy}{dx} = f(x, y, z) \quad \text{---- (1)}$$

$$\frac{dz}{dx} = \phi(x, y, z) \quad \text{---- (2)}$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by the methods discussed in the preceding sections, especially Picard's or Runge-Kutta methods.

I) Picard's method gives

$$y_1 = y_0 + \int f(x, y_0, z_0)\,dx,\ z_1 = z_0 + \int \phi(x, y_0, z_0)\,dx$$

$$y_2 = y_0 + \int f(x, y_1, z_1)\,dx,\ z_2 = z_0 + \int \phi(x, y_1, z_1)\,dx$$

$$y_3 = y_0 + \int f(x, y_2, z_2)\,dx,\ z_3 = z_0 + \int \phi(x, y_2, z_2)\,dx$$

and so on.

II) Taylor's method is used as follows

If h be the step-size, $y_1 = y(x_0 + h)$ and $z_1 = z(x_0 + h)$. Then Taylor's algorithm for (1) and (2) gives,

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \quad \text{..... (3)}$$

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \quad \text{..... (4)}$$

Differentiating (1) and (2) successively, we get y", z". So the values y_0', y_0'', y_0''' and z_0', z_0'', z_0''' .... are known.

Replacing these values in (3) and (4), we obtain y_1, z_1 for the next step.

Similarly, we have the algorithms,

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \quad \text{..... (5)}$$

$$z_2 = z_0 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \quad \text{..... (6)}$$

Since y_1 and z_1 are known, we can calculate y_1', y_1'', ... and z_1', z_1'', ------ Replacing these in (5) and (6), we get y_2 and z_2.

Proceeding further, we can calculate the other values of y and z step by step.

III) Runge-Kutta method is applied as follows

Starting at (x_0, y_0, z_0) and taking the step-sizes for x, y, z to be h, k, l respectively, the Runge-kutta method gives,

$$k_1 = hf(x_0, y_0, z_0)$$

Scanned with CamScanner

$l_1 = h\phi(x_0, y_0, z_0)$

$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$

$l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$

$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$

$l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$

$k_4 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_3, z_0 + \frac{1}{2}l_3\right)$

$l_4 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_3, z_0 + \frac{1}{2}l_3\right)$

Hence, $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

To compute y_2 and z_2, we simply replace x_0, y_0, z_0 by x_1, y_1, z_1 in above formula.

Example 5.11

Solve the differential equations

$$\frac{dy}{dx} = 1 + xz, \frac{dz}{dx} = -xy \text{ for } x = 0.3$$

Using the fourth order Runge Kutta method. Initial values are x = 0, y = 0 and z = 1.

Solution:

Here;

$f(x, y, z) = 1 + xz, \phi(x, y, z) = -xy$

$x_0 = 0, y_0 = 0, z_0 = 1$

Let us take h = 0.3,

$\therefore \quad k_1 = hf(x_0, y_0, z_0) = 0.3 f(0, 0, 1) = 0.3(1 + 0) = 0.3$

$l_1 = h\phi(x_0, y_0, z_0) = 0.3(-0 \times 0) = 0$

$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$

$= 0.3f(0.15, 0.15, 1)$

$= 0.3(1 + 0.15) = 0.345$

$l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$

$= 0.3[-(0.15(0.15)]$

$= -0.00675$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{k_2}{2}k_1, z_0 + \frac{1}{2}\right)$$

$= 0.3\,f(0.15, 0.1725, 0.996625)$

$= 0.3\,[1 + 0.996625 \times 0.15]$

$= 0.34485$

$$l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{k}{2}, z_0 + \frac{1}{2}\right)$$

$= 0.3\,[-(0.15)(0.1725)]$

$= -0.007762$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3]$

$= 0.3f(0.3, 0.34485, 0.99224)$

$= 0.3893$

$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= 0.3\,[-(0.3)(0.34485)]$

$= -0.03104$

Hence, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

i.e., $y(0.3) = 0 + \frac{1}{6}[0.3 + 2 \times (0.345) + 2 \times (0.34485) + 0.3893]$

$= 0.34483$

and, $z(x + h) = y_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$z(0.3) = 1 + \frac{1}{6}[0 + 2(-0.00675) + (0.0077625) + (-0.03104)]$

$= 0.98999$

5.9 SECOND ORDER DIFFERENTIAL EQUATIONS

Consider the second order differential equation,

$$\frac{d^2y}{dx^2} = \left(x, y, \frac{dy}{dx}\right)$$

By writing $\frac{dy}{dx} = z$, it can be reduced to two first order simultaneous differential equations.

$$\frac{dy}{dx} = z, \frac{dz}{dx} f(x, y, z)$$

These equations can be solved as explained above.

Example 5.12

Using the Runge-Kutta method, solve $y'' = xy'^2 - y^2$ for $x = 0.2$ correct to 4 decimal places. Initial conditions are $x = 0$, $y = 1$, $y' = 0$.

Scanned with CamScanner

Solution:

Let, $\frac{dy}{dx} = f(x, y, z) = z$

Then, $\frac{dy}{dx} = xz^2 - y^2 = \phi(x, y, z)$

We have,

$x_0 = 0, y_0 = 1, z_0 = 0, h = 0.2$

∴ Runge-Kutta formulae becomes,

$k_1 = hf(x_0, y_0, z_0) = 0.2 \times 0 = 0$

$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$

$= 0.2(-0.1) = -0.02$

$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$

$= 0.2(-0.0999) = -0.02$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= 0.2(-0.1958) = -0.0392$

∴ $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.0199$

Now,

$l_1 = hf(x_0, y_0, z_0) = 0.2(-1) = -0.2$

$l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$

$= -0.2(-0.999) = -0.1998$

$l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$

$= 0.2(-0.9791) = -0.1958$

$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= 0.2(0.9527)$

$= -0.1905$

$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$= \frac{1}{6}(-0.2 - 0.1998 \times 2 - 2 \times 0.1958 - 0.1905)$

$= -0.1970$

Hence at $x = 0.2$,

$y = y_0 + k = 1 - 0.0199 = 0.9801$

$y' = z = z_0 + l = 0 - 0.1970 = -0.1970$

Scanned with CamScanner

5.10 BOUNDARY VALUE PROBLEMS

Such a problem requires the solution of a differential equation in a region R subject to the various conditions on the boundary of R. Practical applications give rise to many such problems.

A. Shooting Method or Marching Method

In this method, the given boundary value problem is first transformed to an initial value problem. Then this initial value problem is solved by Taylor's series method or Runge-Kutta method etc. Finally, the given boundary value problem is solved. The approach in this method is quite simple.

Consider the boundary value problem,

$$y''(x) = y(x);\ y(a) = A,\ y(b) = B \qquad \text{---- (1)}$$

One condition is $y(a) = A$ and let us assume that $y'(a) = m$ which represents the slope. We start with two initial guesses for m, then find the corresponding value of y(b) using any initial value method.

Let the two guesses be m_0, m_1 so that the corresponding values of y(b) are $y(m_0, b)$ and $y(m_1, b)$. Assuming that the values of m and y (b) are linearly related, we obtain the better approximation m_2 for m from the relation.

$$\frac{m_2 - m_1}{y(b) - y(m_1, b)} = \frac{m_1 - m_0}{y(m_1, b) - y(m_0, b)}$$

This gives,

$$m_2 = m_1 - (m_1 - m_0)\frac{y(m, b) - y(b)}{y(m_1, b) - y(m_0, b)} \qquad \text{---- (2)}$$

Now, we solve the initial value problem,

$$y''(x) = y(x),\ y(a) = A,\ y'(a) = m_2$$

and obtain the solution $y(m_2, b)$

To obtain a better approximation m_3 for m, we again use the linear relation (2) with $[m_1, y(m_1, b)$ and $[m_2, y(m_2, b)]$. This process is repeated until the value of $y(m_i, b)$ agrees with y(b) to desired accuracy.

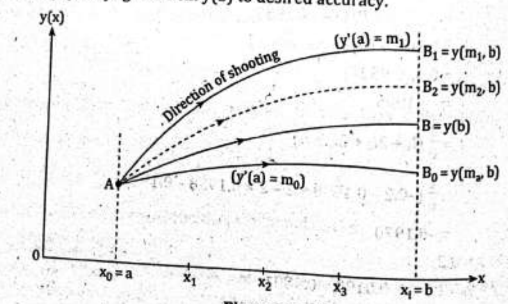


Figure 5.3

NOTE:
Shooting method is quite slow in practice. Also, this method is quite tedious to apply to higher order boundary value problems.

Example 5.13

Using the shooting method, solve the boundary value problem.

$y''(x) = y(x)$; $y(0) = 0$ and $y(1) = 1.17$

Solution:

Let the initial guesses for $y'(0) = m$ be $m_0 = 0.8$

and, $m_1 = 0.9$. Then $y''(x) = y(x)$; $y(0) = 0$ gives,

$$y'(0) = m$$
$$y''(0) = y(0) = 0$$
$$y'''(0) = y'(0) = m$$
$$y^{iv}(0) = y''(0) = 0$$
$$y^{v}(0) = y'''(0) = m$$
$$y^{vi}(0) = y^{iv}(0) = 0$$

and so on.

Replacing these values in the Taylor's series, we have,

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \ldots\ldots$$
$$= m\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \ldots\ldots\right)$$
$$\therefore \quad y(1) = m(1 + 0.1667 + 0.0083 + 0.0002 + \ldots\ldots)$$
$$= m(1.175)$$

For $m_0 = 0.8$, $y(m_0,1) = 0.8 \times 1.175 = 0.94$

For $m_1 = 0.9$, $y(m_1, 1) = 0.9 \times 1.175 = 1.057$

Hence a better approximation for m, *i.e.*, m_2 is given by,

$$m_2 = m_1 - (m_1 - m_0)\frac{y(m_1, 1) - y(1)}{y(m_1, 1) - y(m_0, 1)}$$
$$= 0.9 - (0.1)\left(\frac{1.057 - 1.175}{1.057 - 0.94}\right)$$
$$= 0.9 + 0.10085 = 1.00085$$

Which is closer to the exact value of $y'(0) = 0.996$

Solving the initial value problem

$$y''(x) = y(x),\ y(0) = 0,\ y'(0) = m_2$$

Taylor's series solution is given by,

$$y(m_2, 1) = m_2(1.175) = 1.1759$$

Hence the solution at $x = 1$ is $y = 1.176$ which is close to the exact value of $y(1) = 1.17$.

Scanned with CamScanner

B. Finite Difference Method

In this method, the derivatives appearing in the differential equation and the boundary conditions are replaced by their finite-difference approximation and the resulting linear system of equations are solved by any standard procedure. These roots are the values of the required solution at the pivotal points.

The finite difference approximations to the various derivatives are derived as under.

If y(x) and its derivatives are single-valued continuous functions of x then by Taylor's expansion.

We have,

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \ldots\ldots \quad \text{.... (1)}$$

and, $$y(x-h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \ldots\ldots \quad \text{.... (2)}$$

Equation (1) gives,

$$y'(x) = \frac{1}{h}[y(x+h) - y(x)] - \frac{h}{2}y''(x) - \ldots\ldots$$

i.e., $$y'(x) = \frac{1}{h}[y(x+h) - y(x)] + O(h)$$

which is the forward difference approximation of y'(x) with an error of the order h.

Similarly, (2) gives,

$$y'(x) = \frac{1}{h}[y(x) - y(x-h)] + O(h)$$

Which is the backward difference approximation of y'(x) with an error of the order h.

Subtracting (2) from (1), we get,

$$y'(x) = \frac{1}{2h}[y(x+h) - y(x-h)] + O(h^2)$$

which is the central-difference approximation of y'(x) with an error of the order h^2. Clearly, this central difference approximations and hence should be preferred.

Adding (1) and (2), we get,

$$y''(x) = \frac{1}{h^2}[y(x+h) - 2y(x) + y(x-h)] + O(h^2)$$

which is the central difference approximation to higher derivatives.

Hence the working expressions for the central difference approximations to the first four derivatives of y_i are as under;

$$y_i' = \frac{1}{2h}(y_{i+1} - y_{i-1}) \quad ..... (3)$$

$$y_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \quad ..... (4)$$

$$y_i''' = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}) \quad ..... (5)$$

$$y_i^{iv} = \frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) \quad ..... (6)$$

NOTE: The accuracy of this method depends on the size of the sub-interval h and also on the order of approximation. As we reduce h, the accuracy improves but the number of equations to be solved also increases.

Example 5.14

Solve the equation y" = x + y with the boundary conditions y(0) = y (1) = 0.

Solution:

Divide the interval (0, 1) into four sub-intervals so that $h = \frac{1}{4}$ and the pivot points are at $x_0 = 0$,

$$x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4} \text{ and } x_4 = 1$$

Then the differential equation is approximated as,

$$\frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] = x_i + y_i$$

or, $16\,y_{i+1} - 33y_i + 16\,y_{i-1} = x_i \;;\; i = 1, 2, 3 \,.........$

Using $y_0 = y_4 = 0$, we get the system of equations

or, $16\,y_2 - 33\,y_1 = \frac{1}{4}$

or, $16\,y_3 - 33\,y_2 + 16\,y_1 = \frac{1}{2}$

or, $-33\,y_3 + 16\,y_2 = \frac{3}{4}$

Their solution gives,

$y_1 = -0.03488$

$y_2 = -0.05632$

$y_3 = -0.05003$

Scanned with CamScanner

BOARD EXAMINATION SOLVED QUESTIONS

1. Solve $\frac{dy}{dx} = y - \frac{2x}{y}$, y(0) = 1 in the range $0 \le x \le 0.2$ by using (i) Euler's method and (ii) Huen's method. Comment on the results. Take h_1 = 0.2. [2013/Fall]

Solution:

$$\frac{dy}{dx} = y - \frac{2x}{y} \text{ and } y(0) = 1$$

$$\Rightarrow x_0 = 0 \text{ and } y_0 = 1$$

Also, h = 0.2, $0 \le x \le 0.2$

i) From Euler's method,

$$f(x_0, y_0) = y_0 - \frac{2x_0}{y_0} = 1 - \frac{2(0)}{1} = 1$$

Now,

$$y_1 = y_{new} = y_0 + hf(x_0, y_0)$$

or, $y_{new} = y_{old} + h\frac{dy}{dx} = 1 + 0.2(1)$

∴ $y_1 = 1.2$

iii) From Huen's method or modified Euler's method

h = 0.2

Solving in tabular form

S.N.	x	$\frac{dy}{dx} = y - \frac{2x}{y}$	Mean slope	$y_{new} = y_{old}$ + h (mean slope)
1	0	$1 - \frac{2(0)}{1} = 1$	-	1 + 0.2 × 1 = 1.2
2	0.2	$1.2 - \frac{2(0.2)}{1.2}$ = 0.8667	$\frac{1}{2}(1 + 0.8667)$ = 0.9333	1 + 0.2 × 0.9333 = 1.1866
3	0.2	$1.1866 - \frac{2(0.2)}{1.1866}$ = 0.8495	$\frac{1}{2}(1 + 0.8495)$ = 0.9247	1 + 0.2 × 0.9247 = 1.1849

Here the last two values are equal at $y_1 = 1.1849$.

The result from Euler's method is 1.2 and from Huen's method is 1.1849 which shows better result and we prefer Huen's method or modified Euler's method.

2. Using Runge Kutta method of order 4, solve the equation $\frac{d^2y}{dx^2} = 6xy^2 + y$, y(0) = 1 and y'(0) = 0 to find y(0.2) and y'(0.2). Take h = 0.2. [2013/Fall]

Scanned with CamScanner

Solution:

$$\frac{d^2y}{dx^2} - 6xy^2 - y = 0$$

or, $y'' - 6xy^2 - y = 0$ ..... (1)

Also, $y(0) = 1$ and $y'(0) = 0$ and $h = 0.2$

Let, $y' = z = f_1(x, y, z)$ ..... (A)

so, $y'' = z'$, then equation (1) becomes

$z' = 6xy^2 + y = f_2(x, y, z)$ ..... (B)

Given that;

$y(0) = 1 \quad \Rightarrow x_0 = 0, \quad y_0 = 1$

and, $y'(0) = 0 = z_0$

Now, using RK method to find increment value of k and l

$k_1 = hf_1(x_0, y_0, z_0)$ at equation (A)

$= hf_1(0, 1, 0)$

$= 0$

$l_1 = hf_2(x_0, y_0, z_0)$ at equation (B)

$= hf_2(0, 1, 0)$

$= (6(0)(1)^2 + 1)\, 0.2$

$= 0.2$

Likewise,

$$k_2 = hf_1\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2},\ z_0 + \frac{l_1}{2}\right)$$

$$= 0.2f_1\left(0 + \frac{0.2}{2},\ 1 + \frac{0}{2},\ 0 + \frac{0.2}{2}\right)$$

$= 0.2 \times 0.1$

$= 0.02$

$l_2 = hf_2(0.1, 1, 0.1)$

$= 0.2\,[6(0.1)(1)^2 + 1]$

$= 0.32$

$$k_3 = hf_1\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right)$$

$= 0.2f_1(0.1, 1.01, 0.16)$

$= 0.2 \times 0.16$

$= 0.032$

$l_3 = hf_2(0.1, 1.01, 0.16)$

$= 0.2\,[6(0.1)(1.01)^2 + 1.01] = 0.324$

$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= 0.2f_1(0.2, 1.032, 0.324)$

$= 0.2 \times 0.324 = 0.064$

$l_4 = hf_2(0.2, 1.032, 0.324) = 0.462$

Scanned with CamScanner

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0 + 0.064 + 2(0.02 + 0.032)]$$

$$= 0.028$$

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$

$$= \frac{1}{6}[0.2 + 0.462 + 2(0.32 + 0.324)]$$

$$= 0.325$$

Now,

$y_1 = y_0 + k = 1 + 0.028 = 1.028$

and, $z_1 = z_0 + l = 0 + 0.325 = 0.325$

are the required answer for y'(0.2) and y(0.2).

3. Use the Runge-Kutta 4^th^ order method to estimate y(0.2) of the following equation with h = 0.1

$$y'(x) = 3x + \frac{1}{2}y,\ y(0) = 1$$ **[2013/Spring]**

Solution:

Given that,

$y'(x) = 3x + 0.5y$

and, $y(0) = 1$

→ $x_0 = 0,\ y_0 = 1,\ h = 0.1$

Now, using RK method to find increment on k

$k_1 = hf(x_0, y_0)$
$= 0.1f(0, 1)$
$= 0.1[3(0) + 0.5(1)]$
$= 0.05$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$

$$= 0.1f\left(0 + \frac{0.1}{2},\ 1 + \frac{0.05}{2}\right)$$

$= 0.1f(0.05, 1.025)$
$= 0.0662$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$

$= 0.1f(0.05, 1.033)$
$= 0.0666$

$k_4 = hf(x_0 + h, y_0 + k_3)$
$= 0.1f(0.1, 1.0666)$
$= 0.0833$

Scanned with CamScanner

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0.05 + 0.0833 + 2(0.0662 + 0.0666)]$$

$$= 0.0664$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$\therefore\ y_1 = y_0 + k = 1 + 0.0664 = 1.0664$$

Again,

$$x_1 = 0.1,\ y_1 = 1.0664,\ h = 0.1$$

Then,

$$k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.0664) = 0.0833$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= 0.1f(0.15, 1.1080)$$

$$= 0.1004$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1f(0.15, 1.1166)$$

$$= 0.1008$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= 0.1f(0.2, 1.1672)$$

$$= 0.1183$$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0.0833 + 0.1183 + 2(0.1004 + 0.1008)]$$

$$= 0.1006$$

so,

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$

$$y_2 = y_1 + k = 1.0664 + 0.1006 = 1.167$$

is the required estimated value of $y(0.2)$.

4. Solve the following equation by Picard's method.
$y'(x) = x^2 + y^2$, $y(0) = 0$ and estimate $y(0.1)$, $y(0.2)$ and $y(1)$.
[2013/Spring]

Solution:

Given that;

$$y'(x) = x^2 + y^2, \qquad y(0) = 0$$

$$\rightarrow \quad x_0 = 0, \quad y_0 = 0$$

Scanned with CamScanner

Now, using Picard's method

$$y = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx = 0 + \int_0^x (x^2 + y^2)\, dx$$

First approximation, put y = 0 in the integrand

$$y_1 = 0 + \int_0^x (x^2 + 0^2)\, dx$$

$$= \int_0^x (x^2)\, dx = \left[\frac{x^3}{3}\right]_0^x = \frac{x^3}{3}$$

Second approximation, put $y = \frac{x^3}{3}$ in the integrand

$$y_2 = 0 + \int_0^x \left[x^2 + \left(\frac{x^3}{3}\right)^2\right] dx$$

$$= \int_0^x \left(x^2 + \frac{x^6}{9}\right) dx$$

$$= \left[\frac{x^3}{3} + \frac{x^7}{63}\right]_0^x$$

$$= \frac{x^3}{3} + \frac{x^7}{63}$$

Further processing of this task is difficult from here so we stop at

$$y_2 = \frac{x^3}{3} + \frac{x^7}{63}$$

Now, using the second approximation and taking

x = 0.1, 0.2 and 1

We have,

$$y(0.1) = \frac{(0.1)^3}{3} + \frac{(0.1)^7}{63} = 0.000033$$

$$y(0.2) = \frac{(0.2)^3}{3} + \frac{(0.2)^7}{63} = 0.0026$$

$$y(1) = \frac{(1)^3}{3} + \frac{(1)^7}{63} = 0.3492$$

5. **Given: $\frac{dy}{dx} = \frac{2x + e^x}{x^2 + xe^x}$; y(1) = 0. Solve for y at x = 1.04, by using Euler's method (take h = 0.01)** **[2014/Fall]**

Solution:

Given that;

$$\frac{dy}{dx} = \frac{2x + e^x}{x^2 + xe^x}$$

$y(1) = 0$, $h = 0.01$

$x_0 = 1$, $y_0 = 0$

Scanned with CamScanner

Using Euler's method, in tabular form

S.N.	x	y	$\frac{dy}{dx} = \frac{2x + e^x}{x^2 + xe^x}$	$y_{new} = y_{old} + h\frac{dy}{dx}$
1	1	0	1.268	0 + 0.01 (1.268) = 0.0126
2	1.01	0.0126	1.256	0.0126 + 0.01 (1.256) = 0.0251
3	1.02	0.0251	1.244	0.0375
4	1.03	0.0375	1.231	0.0498
5	1.04	0.0498		

Hence the required solution at x = 1.04 for y is 0.0498.

6. **Solve $\frac{dy}{dx} = 1 + xz$, $\frac{dz}{dx} = -xy$ for y(0.6) and z(0.6), given that y = 0, z = 1 at x = 0 by using Heun's method. Assume, h = 0.3.** **[2014/Fall]**

Solution:

$\frac{dy}{dx} = 1 + xz$, $x_0 = 0$, $y_0 = 0$, $h = 0.3$

and, $\frac{dz}{dx} = -xy$, $x_0 = 0$, $y_0 = 0$, $h = 0.3$

Using Heun's method, solving in tabular form

S.N.	x	$\frac{dy}{dx} + 1 + xz$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
1	0	1 + (0) (1)	–	0 + 0.3 × 1 = 0.3
2	0.3	1 + (0.3) (1) = 1.3	$\frac{1 + 1.3}{2} = 1.15$	0 + 0.3 × 1.15 = 0.345
3	0.3	1 + (0.3) (0.9865) = 1.295	$\frac{1 + 1.295}{2} = 1.147$	0 + 0.3 × 1.147 = 0.344
4	0.3	1 + (0.3) (0.9847) =1.295	$\frac{1 + 1.295}{2} = 1.147$	0 + 0.3 × 1.147 = 0.344

Here, the last two values are equal at $y_1 = 0.344$

S.N.	x	$\frac{dy}{dx} = -xy$	Mean slope	$z_{new} = z_{old} + h$ (mean slope)
1	0	(−0) (0)	–	1 + 0.3 × 0 = 1
2	0.3	−(0.3) (0.3) = −0.09	$\frac{0 - 0.09}{2} = -0.045$	1 + 0.3 × −0.045 = 0.9865
3	0.3	−(0.3) (0.344) = −0.103	$\frac{0 - 0.103}{2} = -0.051$	1 + 0.3 × −0.051 = 0.9847
4	0.3	−(0.3) (0.344) = −0.103	$\frac{0 - 0.103}{2} = -0.051$	1 + 0.3 × −0.051 = 0.9847

Scanned with CamScanner

Here, the last two values are equal at $z_1 = 0.9847$.

NOTE:
Use both tables to use the value of y_{new} and z_{new} to calculate $\frac{dy}{dx}$ and $\frac{dz}{dx}$

Again,

S.N.	x	$\frac{dy}{dx} = 1 + xz$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
1	0.3	1 + (0.3) (0.9847) =1.295	–	0.344 + 0.3 (1.295) = 0.732
2	0.6	1 + (0.6) (0.9538) = 1.572	$\frac{1.295 + 1.572}{2}$ = 1.433	0.344 + 0.3 (1.433) = 0.773
3	0.6	1 + (0.6) (0.899) = 1.539	$\frac{1.295 + 1.539}{2}$ = 1.417	0.344 + 0.3 (1.417) = 0.769
4	0.6	1 + (0.6) (0.9) = 1.54	$\frac{1.295 + 1.54}{2}$ = 1.417	0.344 + 0.3 (1.417) = 0.769

Here, the last two values are equal at $y_2 = 0.769$.

S.N.	x	$\frac{dy}{dx} = -xy$	Mean slope	$z_{new} = z_{old} + h$ (mean slope)
1	0.3	−(0.3) (0.344) = −0.103	–	0.9847 + 0.3 (−0.103) = 0.9538
2	0.6	−(0.6) (0.773) = −0.463	$\frac{-0.103 - 0.463}{2}$ = −0.283	0.9847 + 0.3 (−0.283) = 0.899
3	0.6	−(0.6) (0.769) = −0.461	$\frac{-0.103 - 0.461}{2}$ = −0.282	0.9847 + 0.3 (−0.282) = 0.900
4	0.6	−(0.6) (0.769) = −0.461	$\frac{-0.103 - 0.461}{2}$ = −0.282	0.9847 + 0.3 (−0.282) = 0.9

Here, the last two values are equal at $z_2 = 0.9$.

Hence, the required values of y(0.6) = 0.769 and z(0.6) = 0.9.

7. **Using R-K fourth order method, solve the given differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6$, y(0) = 0, y'(0) = 1 with h = 0.2 for y(0.4)?**

[2014/Spring]

Scanned with CamScanner

Solution:

Given that;

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6$$

or, $y'' + 2y' - 3y = 6$ ..... (1)

Also, $y(0) = 0,\ y'(0) = 1,\ h = 0.2$

→ $x_0 = 0,\quad y_0 = 0$

Let, $y' = z = f_1(x, y, z)$

so, $y'' = z'$ then (1) becomes

$z' = 6 + 3y - 2z = f_2(x, y, z)$

Subject to

$y'(0) = 1$

→ $z_0 = 1$

Now, using RK method to find increments,

$k_1 = hf_1(x_0, y_0, z_0)$

$= 0.2f_1(0, 0, 1) = 0.2$

$l_1 = hf_2(x_0, y_0, z_0)$

$= 0.2[6 + 3(0) - 2(1)] = 0.8$

$k_2 = hf_1\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2},\ z_0 + \frac{l_1}{2}\right)$

$= 0.2f_1(0.1, 0.1, 1.4) = 0.28$

$l_2 = hf_2(0.1, 0.1, 1.4)$

$= 0.2[6 + 3(0.1) - 2(1.4)] = 0.7$

$k_3 = hf_1\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right)$

$= 0.2f_1(0.1, 0.14, 1.35) = 0.27$

$l_3 = hf_2(0.1, 0.14, 1.35) = 0.744$

$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= 0.2f_1(0.2, 0.27, 1.744) = 0.348$

$l_4 = hf_2(0.2, 0.27, 1.744) = 0.664$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0.2 + 0.348 + 2(0.28 + 0.27)] = 0.274$$

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$

$$= \frac{1}{6}[0.8 + 0.664 + 2(0.7 + 0.744)] = 0.725$$

Then, $y_1 = y_0 + k = 0 + 0.274 = 0.274$

$z_1 = z_0 + l = 1 + 0.725 = 1.725$

Scanned with CamScanner

Again, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$y_1 = 0.274$

$z_1 = 1.725$

Using RK method to find increment on k and *l*:

$$k_1 = hf_1(x_1, y_1, z_1) = 0.2f_1(0.2, 0.274, 1.725) = 0.345$$

$$l_1 = hf_2(x_1, y_1, z_1) = 0.2\,(0.2, 0.274, 1.725) = 0.674$$

$$k_2 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right) = 0.2f_1(0.3, 0.4465, 2.062) = 0.412$$

$$l_2 = hf_2(0.3, 0.4465, 2.062) = 0.643$$

$$k_3 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right) = 0.2f_1(0.3, 0.48, 2.046) = 0.409$$

$$l_3 = hf_2(0.3, 0.48, 2.046) = 0.669$$

$$k_4 = hf_1(x_1 + h, y_1 + k_3, z_1 + l_3) = 0.2f_1(0.4, 0.683, 2.394) = 0.478$$

$$l_4 = hf_2(0.4, 0.683, 2.394) = 0.652$$

Then,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)] = \frac{1}{6}[0.345 + 0.478 + 2(0.412 + 0.409)] = 0.410$$

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)] = \frac{1}{6}[0.674 + 0.652 + 2(0.643 + 0.669)] = 0.658$$

Now,

$x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$\therefore \quad y_2 = y(0.4) = y_1 + k = 0.274 + 0.410 = 0.684$

Scanned with CamScanner

8. **Given the boundary value problem: y" = 6x with y(1) = 2 and y(2) = 9. Solve it in the interval (1, 2) by using RK method of second order (take, h = 0.5 and guess value = 3.25)** **[2014/Spring]**

Solution:

Given that;

$y'' = 6x$ ..... (1)

$y(1) = 2$

$y(2) = 9$

$h = 0.5$

Let, $y' = z = f_1(x, y, z)$

$y'' = z'$

So equation (1) becomes,

$z' = 6x = f_2(x, y, z)$

Subjected to

$y(1) = 2 = z(1)$

Initial guess value = 3.25

Now, from RK method of second order

Iteration 1:

$x_0 = 1,\ \ y_0 = 2,\ \ z_0 = 2$

$k_1 = hf_1(x_0, y_0, z_0)$
$= 0.5f_1(1, 2, 2)$
$= 0.5 \times 2$
$= 1$

$l_1 = hf_2(x_0, y_0, z_0)$
$= 0.5 \times 6 \times 1$
$= 3$

$k_2 = hf_1(x_0 + h, y_0 + k_1, z_0 + l_1)$
$= 0.5f_1(1.5, 3, 5)$
$= 0.5 \times 5$
$= 2.5$

$l_2 = hf_2(1.5, 3, 5)$
$= 0.5 \times 6 \times 1.5$
$= 4.5$

Then, $k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(1 + 2.5) = 1.75$

$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(4.5 + 3) = 3.75$

so, $y_1 = y_0 + k = 2 + 1.75 = 3.75$

$z_1 = z_0 + l = 2 + 3.75 = 5.75$

$x_1 = x_0 + h = 1 + 0.5 = 1.5$

Scanned with CamScanner

Again, iteration 2:

$$k_1 = hf_1(x_1, y_1, z_1) = 0.5f_1(1.5, 3.75, 5.75) = 0.5 \times 5.75 = 2.875$$

$$l_1 = hf_2(1.5, 3.75, 5.75) = 0.5 \times 6 \times 1.5 = 4.5$$

$$k_2 = hf_1(1.5 + 0.5, 3.75 + 2.875, 5.75 + 4.5) = 0.5f_1(2, 6.62, 10.25) = 0.5 \times 10.25 = 5.125$$

$$l_2 = hf_2(2, 6.62, 10.25) = 0.5 \times 6 \times 2 = 6$$

so, $k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(2.875 + 5.125) = 4$

$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(4.5 + 6) = 5.25$

Then, $x_2 = x_1 + h = 1.5 + 0.5 = 2$

$y_2 = y_1 + k = 3.75 + 4 = 7.75$

$z_2 = z_1 + l = 5.75 + 5.25 = 11$

Thus, we obtain, $y(2) = 7.75 < y(2) = 9$ and can be further denoted as $y_B = B = 9$ giving $B_1 = 7.75$.

9. Using Euler's method solve the given differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6$, $y(0) = 0$, $y'(0) = 1$ with $h = 0.2$ for $y(0.4) = ?$ [2015/Fall]

Solution:

Given that;

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6 \quad \text{.... (1)}$$

or, $y'' + 2y' - 3y = 6$

Let, $y' = \frac{dy}{dx} = z$ .... (A)

Then, $y'' = z'$

So, equation (1) becomes

$z' + 2z - 3y = 6$

or, $z' = 6 + 3y - 2z$ .... (B)

Subject to

$y(0) = 0 \rightarrow x_0 = 0,\ y_0 = 0$

$y'(0) = 1 \rightarrow z_0 = 1$ at $h = 0.2$

Scanned with CamScanner

Now, using Euler's method

$y_1 = y(0.2) = y_0 + h\dfrac{dy_0}{dx_0} = 0 + 0.2\,(z_0) = 0.2 \times 1 = 0.2$

$z_1 = z(0.2) = z_0 + hz'(x_0)$ from equation (B)

$= 1 + 0.2\,(6 + 3y_0 - 2z_0)$

$= 1 + 0.2\,[6 + 3(0) - 2(1)] = 1.8$

Again, $y(0.4) = y_1 + hy'(x_1) = y_1 + h\dfrac{dy_1}{dx_1} = y_1 + h(z_1)$

$= 0.2 + 0.2(1.8) = 0.56$

∴ $y(0.4) = 0.56$ is the required solution.

10. Solve the following differential equation within $0 \le x \le 0.5$ using RK 4th order method. $20\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 4y = 5$, $y(0) = 0$, $y'(0) = 0$.

Take h = 0.25. **[2015/Fall]**

Solution:

Given that;

$$20\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 4y = 5$$

Let, $\dfrac{dy}{dx} = y' = z = f_1(x, y, z)$

Then, $\dfrac{d^2y}{dx^2} = y'' = z' = f_2(x, y, z)$

or, $20z' + 2z - 4y = 5$

or, $z' = \dfrac{5 - 2z + 4y}{20} = f_2(x, y, z)$

Subject to

$y(0) = 0 \rightarrow x_0 = 0, y_0 = 0$

$y'(0) = 0 \rightarrow z_0 = 0$

and, $h = 0.25$

Now, by RK 4th order method

$k_1 = hf_1(x_0, y_0, z_0)$

$= 0.25f_1(0, 0, 0)$

$= 0$

$l_1 = hf_2(x_0, y_0, z_0)$

$= 0.25\left(\dfrac{5 - 0 + 0}{20}\right)$

$= 0.0625$

$k_2 = hf_1\left(x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_1}{2}, z_0 + \dfrac{l_1}{2}\right)$

$= hf_1(0.125, 0, 0.03125)$

$= 0.25 \times 0.03125$

$= 0.00781$

Scanned with CamScanner

$l_2 = hf_2\,(0.125, 0, 0.03125)$

$= 0.25 \times \left(\frac{5 - 2(0.03125) + 4(0)}{20}\right)$

$= 0.06171$

$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= 0.25\,(0.125, 0.0039, 0.0308)$

$= 0.0077$

$l_3 = hf_2\,(0.125, 0.0039, 0.0308)$

$= 0.0619$

$k_4 = hf_1\,(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= 0.25f_1\,(0.25, 0.0077, 0.0619)$

$= 0.0154$

$l_4 = hf_2\,(0.25, 0.0077, 0.0619)$

$= 0.0613$

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$

$= \frac{1}{6}[0 + 0.0154 + 2(0.00781 + 0.0077)]$

$= 0.0077$

$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$

$= \frac{1}{6}[0.0625 + 0.0613 + 2(0.06171 + 0.0619)]$

$= 0.0618$

so, $x_1 = x_0 + h = 0 + 0.25 = 0.25$

$y_1 = y_0 + k = 0 + 0.0077 = 0.0077$

$z_1 = z_0 + l = 0 + 0.0618 = 0.0618$

Again, $k_1 = hf_1\,(x_1, y_1, z_1)$

$= 0.25f_1\,(0.25, 0.0077, 0.0618)$

$= 0.25 \times 0.0618$

$= 0.0154$

$l_1 = hf_2\,(x_1, y_1, z_1)$

$= 0.613$

$k_2 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$

$= 0.25f_1\,(0.375, 0.0154, 0.0924)$

$= 0.0231$

$l_2 = hf_2\,(0.375, 0.0154, 0.0924)$

$= 0.609$

Scanned with CamScanner

$k_3 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right)$

$= 0.25f_1\,(0.375, 0.0192, 0.0922)$

$= 0.0230$

$l_3 = hf_2\,(0.375, 0.0192, 0.0922)$

$= 0.0611$

$k_4 = hf_1\,(x_1 + h, y_1 + k_3, z_1 + l_3)$

$= 0.25f_1\,(0.5, 0.0307, 0.1229)$

$= 0.0307$

$l_4 = hf_2\,(0.5, 0.0307, 0.1229)$

$= 0.0609$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0.0154 + 0.0307 + 2(0.0231 + 0.0230)]$$

$$= 0.0229$$

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$

$$= \frac{1}{6}[0.0613 + 0.0609 + 2(0.0609 + 0.0611)]$$

$$= 0.0610$$

Then, $x_2 = x_1 + h = 0.25 + 0.25 = 0.5$

$y_2 = y_1 + k = 0.0077 + 0.0229 = 0.0306$

$z_2 = z_1 + l = 0.0618 + 0.0610 = 0.1228$

11. Solve the following differential equation within $0 \le x \le 0.5$ using RK 4th order method. $10\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 5$, $y(0) = 0$, $y'(0) = 0$. Take $h = 0.25$. **[2015/Spring]**

Solution:

Given that;

$$10\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 4y = 5$$

Let, $\frac{dy}{dx} = y' = z = f_1\,(x, y, z)$

Then, $\frac{d^2y}{dx^2} = y'' = z' = f_2\,(x, y, z)$

or, $z' = \frac{5 - 2z + 4y}{10} = f_2\,(x, y, z)$

Scanned with CamScanner

Subject to

$y(0) = 0 \rightarrow x_0 = 0, y_0 = 0$

$y'(0) = 0 \rightarrow z_0 = 0$

and, $h = 0.25$

Now, by RK 4th order method,

$k_1 = hf_1(x_0, y_0, z_0)$
$= 0.25f_1(0, 0, 0)$
$= 0$

$l_1 = hf_2(x_0, y_0, z_0)$

$$= 0.25\left(\frac{5 - 2(0)0 + 4(0)}{10}\right)$$

$= 0.125$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$= 0.25f_1(0.125, 0, 0.0625)$
$= 0.25 \times 0.0625$
$= 0.0156$

$l_2 = hf_2(0.125, 0, 0.0625)$

$$= 0.25 \times \left(\frac{5 - 2(0.0625) + 4(0)}{10}\right)$$

$= 0.1218$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$= 0.25f_1(0.125, 0.0078, 0.0609)$
$= 0.0152$

$l_3 = hf_2(0.125, 0.0078, 0.0609)$
$= 0.1227$

$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$
$= 0.25f_1(0.25, 0.0152, 0.1227)$
$= 0.0306$

$l_4 = hf_2(0.25, 0.0152, 0.1227)$
$= 0.1203$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0 + 0.0306 + 2(0.0156 + 0.0152)]$$

$= 0.0153$

Scanned with CamScanner

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$

$$= \frac{1}{6}[0.125 + 0.1203 + 2(0.1218 + 0.1227)]$$

$$= 0.1223$$

so, $x_1 = x_0 + h = 0 + 0.25 = 0.25$

$y_1 = y_0 + k = 0 + 0.0153 = 0.0153$

$z_1 = z_0 + l = 0 + 0.01223 = 0.1223$

Again, $k_1 = hf_1(x_1, y_1, z_1)$

$= 0.25f_1(0.25, 0.0153, 0.1223)$

$= 0.0305$

$l_1 = hf_2(0.25, 0.0153, 0.1223)$

$= 0.1204$

$$k_2 = hf_1\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2},\ z_1 + \frac{l_1}{2}\right)$$

$= 0.25f_1(0.375, 0.0305, 0.1825)$

$= 0.0456$

$l_2 = hf_2(0.375, 0.0305, 0.1825)$

$= 0.1189$

$$k_3 = hf_1\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2},\ z_1 + \frac{l_2}{2}\right)$$

$= 0.25f_1(0.375, 0.0381, 0.1817)$

$= 0.0454$

$l_3 = hf_2(0.375, 0.0381, 0.1817)$

$= 0.1197$

$k_4 = hf_1(x_1 + h, y_1 + k_3, z_1 + l_3)$

$= 0.25f_1(0.5, 0.0607, 0.242)$

$= 0.0605$

$l_4 = hf_2(0.5, 0.0607, 0.242)$

$= 0.1189$

Then, $$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0.0305 + 0.0605 + 2(0.0456 + 0.0454)]$$

$$= 0.0455$$

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$

$$= \frac{1}{6}[0.1204 + 0.1189 + 2(0.1189 + 0.1197)]$$

$$= 0.1194$$

Scanned with CamScanner

Now,

$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$
$y_2 = y_1 + k = 0.0153 + 0.0455 = 0.0608$
$z_2 = z_1 + l = 0.1223 + 0.1194 = 0.2417$

12. Solve the given differential equation by RK 4th order method y" – xy' + y = 0 with initial condition y(0) = 3, y'(0) = 0 for y(0.2) taking h = 0.2. [2016/Fall]

Solution:

Given that;

$y'' - xy' + y = 0$ ..... (1)

Let $y'' = z'$ and $y' = z$

So equation (1) becomes

$z' - xz + y = 0$

or, $z' = xz - y$

We have,

$y' = z = f_1(x, y, z)$

and, $z' = xz - y = f_2(x, y, z)$

Subject to

$y(0) = 3 \rightarrow x_0 = 0, y_0 = 3$

$y'(0) = 0 \rightarrow z_0 = 0$

Taking h = 0.2.

Now, using RK 4th order method,

$k_1 = hf_1(x_0, y_0, z_0)$
$= 0.2f_1(0, 3, 0)$
$= 0.2 \times 0$
$= 0$

$l_1 = hf_2(x_0, y_0, z_0)$
$= 0.2f_2(0, 3, 0)$
$= 0.2(0 \times 0 - 3)$
$= -0.6$

$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$
$= 0.2f_1(0.1, 3, -0.3)$
$= -0.06$

$l_2 = hf_2(0.1, 3, -0.3)$
$= -0.606$

$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$
$= 0.2f_1(0.1, 2.97, -0.303)$
$= -0.0606$

$l_3 = hf_2(0.1, 2.97, -0.303)$
$= -0.6$
$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$
$= 0.2f_1(0.2, 2.9394, -0.6)$
$= -0.12$
$l_4 = hf_2(0.2, 2.9394, -0.6)$
$= -0.6118$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0 + (-0.12) + 2(-0.06 - 0.0606)]$$
$$= -0.0602$$
$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$
$$= \frac{1}{6}[-0.6 - 0.6118 + 2(-0.606 - 0.6)]$$
$$= -0.6039$$

Then, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$\therefore$ $y_1 = y(0.2) = y_0 + k = 3 - 0.0602 = 2.9398$

13. Solve the differential equation y' = x + y using approximate method within $0 \le x \le 0.2$ with initial condition y(0) = 1 and stepsize h = 0.1. **[2016/Fall]**

Solution:

Given that;

$y' = x + y, \quad 0 \le x \le 0.2$

Subject to

$y(0) = 1$ at $h = 0.1$

$\rightarrow$ $x_0 = 0, y_0 = 1$

Now, using modified Euler's method

Solving in tabular form

S.N.	x	$\frac{dy}{dx} = x + y$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
1	0	0 + 1	–	1 + 0.1 × 1 = 1.1
2	0.1	0.1 + 1.1 = 1.2	$\frac{1+1.2}{2} = 1.1$	1 + 0.1 × 1.1 = 1.11
3	0.1	0.1 + 1.11 = 1.21	$\frac{1+1.21}{2} = 1.105$	1 + 0.1 × 1.105 = 1.1105
4	0.1	0.1 + 1.1105 = 1.2105	$\frac{1+1.2105}{2} = 1.1052$	1 + 0.1 × 1.1052 = 1.1105

Scanned with CamScanner

Here, last two values are equal at $y_1 = 1.1105$.

S.N.	x	$\frac{dy}{dx} = x + y$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
5	0.1	$0.1 + 1.1105 = 1.2105$	-	$1.1105 + 0.1 \times 1.2105 = 1.2315$
6	0.2	$0.2 + 1.2315 = 1.4315$	$\frac{1.2105 + 1.4315}{2} = 1.3210$	1.2426
7	0.2	1.4426	1.3265	1.2431
8	0.2	1.4431	1.3268	1.2431

Here, last two values are equal at $y_2 = 1.2431$.

Hence the required solution within $0 \le x \le 0.2$ are,

$x_0 = 0$, $y_0 = 1$

$x_1 = 0.1$, $y_1 = 1.1105$

and, $x_2 = 0.2$, $y_2 = 1.2431$

14. Employ Taylor's method to obtain approximate value of y at x = 0.2 for the differential equation.

$$y' = 2y + e^x, \; y(0) = 0$$

[2016/Spring]

Solution:

We have,

$$y' = 2y + e^x \quad \text{and} \quad y(0) = 0$$

Then, $y'(0) = 2y(0) + e^0 = 2(0) + 1 = 1$

Now, differentiating successively and substituting $x_0 = 0$ and $y_0 = 0$ we get,

$y'' = 2y' + e^x$, $y''(0) = 2y'(0) + e^0 = 2(1) + 1 = 3$

$y''' = 2y'' + e^x$, $y'''(0) = 2y''(0) + 1 = 2(3) + 1 = 7$

$y^{iv} = 2y''' + e^x$, $y^{iv}(0) = 2y'''(0) + 1 = 2(7) + 1 = 15$

and so on.

Now, putting these values in the Taylor's series. We have,

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \ldots$$

$$= 0 + x(1) + \frac{x^2}{2}(3) + \frac{x^3}{6}(7) + \frac{x^4}{24}(15) + \ldots$$

$$= x + \frac{3x^2}{2} + \frac{7x^2}{6} + \frac{5}{8}x^4 + \ldots$$

Hence, $y(0.2) = 0.2 + \frac{3(0.2)^2}{2} + \frac{7(0.2)^3}{6} + \frac{5(0.2)^4}{8} + \ldots$

$\therefore \quad y(0.2) = 0.2703$

Scanned with CamScanner

15. Using Runge-Kutta second order method, solve the differential equation $y'' = xy' - y$; $y(0) = 3$, $y'(0) = 0$ for $x = 0, 0.2, 0.4$. **[2016/Spring]**

Solution:

Given that;

$y'' = xy' - y$ ..... (1)

Let $y' = z$

Then, $y'' = z'$

So equation (1) becomes

$z' = xz - y = f_2(x, y, z)$

and, $y' = z = f_1(x, y, z)$

Subject to

$y(0) = 3 \rightarrow x_0 = 0, \quad y_0 = 3$

$y'(0) = 0 \rightarrow z_0 = 0$

Taking $h = 0.2$

Now, using Runge-Kutta second order method

$$k_1 = hf_1(x_0, y_0, z_0) = 0.2f_1(0, 3, 0) = 0.2(0) = 0$$

$$l_1 = hf_2(x_0, y_0, z_0) = 0.2f_2(0, 3, 0) = 0.2[0(0) - 3] = -0.6$$

$$k_2 = hf_1(x_0 + h, y_0 + k_1, z_0 + l_1) = 0.2f_1(0.2, 3, -0.6) = -0.12$$

$$l_2 = 0.2f_2(0.2, 3, -0.6) = -0.624$$

Then, $k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}[0 + (-0.12)] = -0.06$

$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(-0.6 - 0.624) = -0.612$

and, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$y_1 = y_0 + k = 3 + (-0.06) = 2.94$

$z_1 = z_0 + l = 0 - 0.612 = -0.612$

Again,

$$k_1 = hf_1(x_1, y_1, z_1) = 0.2f_1(0.2, 2.94, -0.612) = -0.1224$$

$$l_1 = hf_2(0.2, 2.94, -0.612) = -0.6124$$

Scanned with CamScanner

$k_2 = hf_1(x_1 + h, y_1 + k_1, z_1 + l_1)$
$= 0.2f_1(0.4, 2.8176, -1.2244)$
$= -0.2448$
$l_2 = hf_2(0.4, 2.8176, -1.2244)$
$= -0.6614$

Then, $k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(-0.1224 - 0.2448) = -0.1836$

$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(-0.6124 - 0.6614) = -0.6369$

and, $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$
$y_2 = y_1 + k = 2.94 - 0.1836 = 2.7564$
$z_2 = z_1 + l = -0.612 - 0.6369 = -1.2489$

16. Solve the differential equation $y' = y + \sin x$ using appropriate method within $0 \le x \le 0.2$ with initial condition $y(0) = 2$ and step size = 0.1. **[2017/Fall]**

Solution:

Given that;

$y' = y + \sin x \quad , \quad 0 \le x \le 0.2$

and, $y(0) = 2$

$\rightarrow \quad x_0 = 0, \quad y_0 = 2$

Taking step size h = 0.1

Now, using Euler's method for solving the differential equation. We have,

$$y_{new} = y_{old} + h\frac{dy}{dx} = y_{old} + hf(x, y)$$

Then, $y_1 = y_0 + hf(x_0, y_0)$
$= 2 + 0.1[2 + \sin(0)]$

$\therefore \quad y_1 = 2.2$

$y_2 = y_1 + hf(x_1, y_1)$
$= 2.2 + 0.1[2.2 + \sin(0.1)]$

$\therefore \quad y_2 = 2.429$

and, $y_3 = y_2 + hf(x_2, y_2)$
$= 2.429 + 0.1[2.429 + \sin(0.2)]$

$\therefore \quad y_3 = 2.691$

17. Apply RK-4 method to solve y(0.2) for the equation $\frac{d^2y}{dx^2} = x\frac{dy}{dx} - y$ given that y = 1 and $\frac{dy}{dx} = 0$ when x = 0. (Assume h = 0.2) **[2017/Fall, 2017/Spring]**

Solution:

Given that;

$$\frac{d^2y}{dx^2} = x\frac{dy}{dx} - y$$

Scanned with CamScanner

or, $y'' = xy' - y = 0$ ..... (1)

Let, $y' = z$

Then, $y'' = z'$

So, equation (1) becomes

$z' = xz - y = f_2(x, y, z)$

and, $y' = z = f_1(x, y, z)$

Also,

$x_0 = 0, \quad y_0 = 1, \quad z_0 = 0$

At $h = 0.2$

Now, using RK-4 method

$$k_1 = hf_1(x_0, y_0, z_0) = 0.2f_1(0, 1, 0) = 0.2 \times 0 = 0$$

$$l_1 = hf_2(0, 1, 0) = 0.2[0(0) - 1] = -0.2$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = 0.2f_1(0.1, 1, -0.1) = -0.02$$

$$l_2 = hf_2(0.1, 1, -0.1) = 0.2[0.1(-0.1) - 1] = -0.202$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = 0.2f_1(0.1, 0.99, -0.101) = -0.0202$$

$$l_3 = hf_2(0.1, 0.99, -0.101) = -0.2$$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2f_1(0.2, 0.979, -0.2) = -0.04$$

$$l_4 = hf_2(0.2, 0.979, -0.2) = -0.203$$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)] = \frac{1}{6}[0 - 0.04 + 2(-0.02 - 0.0202)] = -0.02006$$

Scanned with CamScanner

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$

$$= \frac{1}{6}[-0.2 - 0.203 + 2(-0.202 - 0.2)]$$

$$= -0.2011$$

Then, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$y_1 = y_0 + k = 1 - 0.02006 = 0.9799$

$z_1 = z_0 + l = 0 - 0.2011 = -0.2011$

Hence, y(0.2) = 0.9799 is the required solution.

18. Solve the given differential equation by RK 4th order method $y'' - x^2y' - 2xy = 0$ with initial condition $y(0) = 1$ $y'(0) = 0$, for $y(0.1)$ taking $h = 0.1$. **[2018/Fall]**

Solution:

Given that;

$$y'' - x^2y' - 2xy = 0 \quad \text{.... (1)}$$

Let, $y' = z$

Then, $y'' = z'$

So, equation (1) becomes

$z' = x^2z + 2xy = f_2(x, y, z)$

and, $y' = z = f_1(x, y, z)$

Subject to

$y(0) = 1 \rightarrow x_0 = 0, \quad y_0 = 1$

$y'(0) = 0 \rightarrow z_0 = 0$

Taking h = 0.1

Now, using RK-4th method

$k_1 = hf_1(x_0, y_0, z_0)$

$= 0.1f_1(0, 1, 0)$

$= 0.1 \times 0$

$= 0$

$l_1 = hf_2(0, 1, 0)$

$= 0.1[0^2(0) + 2(0)(1)]$

$= 0$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$= 0.1f_1(0.05, 1, 0)$

$= 0$

$l_2 = hf_2(0.05, 1, 0)$

$= 0.01$

$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= 0.1f_1 (0.05, 1, 0.005)$

$= 0.0005$

$l_3 = hf_2 (0.05, 1, 0.005)$

$= 0.010$

$k_4 = hf_1 (x_0 + h, y_0 + k_3, z_0 + l_3)$

$= 0.1f_1 (0.1, 1.0005, 0.010)$

$= 0.001$

$l_4 = hf_2 (0.1, 1.0005, 0.010)$

$= 0.020$

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$

$= \frac{1}{6}[0 + 0.001 + 2(0 + 0.0005)]$

$= 0.00033$

$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$

$= \frac{1}{6}[0 + 0.02 + 2(0.01 + 0.01)]$

$= 0.01$

Now,

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$\therefore$ $y_1 = y(0.1) = y_0 + k = 1 + 0.00033 = 1.00033$

$\therefore$ $z_1 = z_0 + l = 0 + 0.01 = 0.01$

19. Solve the differential equation $y' = y - \frac{2x}{y}$ using appropriate method within $0 \le x \le 0.2$ with initial conditions $y(0) = 1$ and step size $h = 0.1$.
[2018/Fall]

Solution:

Given that;

$y' = y - \frac{2x}{y}, \quad 0 \le x \le 0.2$

and, $y(0) = 1$

$\rightarrow$ $x_0 = 0, \quad y_0 = 1$

Step size = h = 0.1

Now, using Euler's method

$f(x_0, y_0) = y_0 - \frac{2x_0}{y_0} = 1 - \frac{2(0)}{1} = 1$

$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1 \times (1) = 1.1$

Scanned with CamScanner

Again, $f(x_1, y_1) = y_1 - \frac{2x_1}{y_1} = 1.1 - \frac{2(0.1)}{1.1} = 0.918$

$y_2 = y_1 + hf(x_1, y_1) = 1.1 + 0.1 \times 0.918 = 1.1918$

Hence, the required solutions are

$\therefore \quad x_0 = 0 \quad , \quad y_0 = 1$

$\therefore \quad x_1 = 0.1 \quad , \quad y_1 = y(0.1) = 1.1$

$\therefore \quad x_2 = 0.2 \quad , \quad y_2 = y(0.2) = 1.1918$

20. Use the Runge-Kutta 4th order to solve $10\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$ for the interval $0 \le x \le 0.4$ with $h = 0.1$. **[2018/Spring]**

Solution:

Given that;

$$10\frac{dy}{dx} = x^2 + y^2 \ , \ 0 \le x \le 0.4$$

or, $$y' = \frac{x^2 + y^2}{10}$$

Subjected to

$y(0) = 1 \rightarrow x_0 = 0, \quad y_0 = 1$

Taking $h = 0.1$

Now, using Runge-Kutta 4th order method

$$k_1 = hf(x_0, y_0) = 0.1f(0, 1) = 0.1\left(\frac{0^2 + 1^2}{10}\right) = 0.01$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f(0.05, 1.005) = 0.0101$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1f(0.05, 1.00505) = 0.0101$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1.0101) = 0.0103$$

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$

$= \frac{1}{6}[0.01 + 0.0103 + 2(0.0101 + 0.0101)]$

$= 0.01011$

so, $x_1 = x_0 + h = 0 + 0.1 = 0.1$

$\therefore \quad y_1 = y_0 + k = 1 + 0.01011 = 1.01011$

Again, $k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.01011) = 0.0103$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1f(0.15, 1.0152) = 0.0105$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1f(0.15, 1.0153) = 0.01053$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1f(0.2, 1.0206) = 0.0108$$

Scanned with CamScanner

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)] = 0.0105$

so, $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

∴ $y_2 = y_1 + k = 1.01011 + 0.0105 = 1.0206$

Again,

$k_1 = hf(x_2, y_2) = 0.1f(0.2, 1.0206) = 0.0108$

$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.1f(0.25, 1.026) = 0.0111$

$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0.1f(0.25, 1.0261) = 0.0111$

$k_4 = hf(x_2 + h, y_2 + k_3) = 0.1f(0.3, 1.0317) = 0.0115$

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)] = 0.0111$

so, $x_3 = x_2 + h = 0.3$

∴ $y_3 = y_2 + k = 1.0206 + 0.0111 = 1.0317$

Again,

$k_1 = hf(x_3, y_3) = 0.1f(0.3, 1.0317) = 0.0115$

$k_2 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}\right) = 0.1f(0.35, 1.0374) = 0.0119$

$k_3 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}\right) = 0.1f(0.35, 1.0376) = 0.0119$

$k_4 = hf(x_3 + h, y_3 + k_3) = 0.1f(0.4, 1.0436) = 0.0124$

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)] = 0.0119$

so, $x_4 = x_3 + h = 0.4$

∴ $y_4 = y_3 + k = 1.0317 + 0.0119 = 1.0436.$

21. Solve the boundary value problem

$y''(x) = y(x)$,

$y(0) = 0$ and $y(1) = 1.1752$ by shooting method,

taking $m_0 = 0.8$ and $m_1 = 0.9$ **[2018/Spring]**

Solution:

Given that;

$m_0 = 0.8$ and $m_1 = 0.9$ be initial guess for $y'(0) = m$

Then, using shooting method,

$y'' = y(x)$, $y(0) = 0$ gives

$y'(0) = m$, $y''(0) = y(0) = 0$

$y'''(0) = y'(0) = m$, $y^{iv}(0) = y''(0) = 0$

$y^{v}(0) = y'''(0) = m$, $y^{vi}(0) = y^{iv}(0) = 0$

and so on.

Scanned with CamScanner

Putting these values in the Taylor's series. We have,

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \ldots\ldots$$

$$= m\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \ldots\ldots\right)$$

$\therefore$ $y(1) = m(1 + 0.1667 + 0.0083 + 0.0002 + \ldots\ldots$

$= m\,(1.175)$

For $m_0 = 0.8$, $y(m_0, 1) = 0.85 \times 1.175 = 0.94$

For $m_1 = 0.9$, $y(m_1, 1) = 0.9 \times 1.175 = 1.057$

So, for better approximation of m,

$$m_2 = m_1 - (m_1 - m_0)\frac{y(m_1, 1) - y(1)}{y(m_1, 1) - y(m_0, 1)}$$

$$= 0.9 - (0.1)\frac{1.057 - 1.175}{1.057 - 0.94}$$

$$= 0.9 + 0.10085$$

$$= 1.00085$$

Here, $m_2 = 1.00085$ is closer to the exact value of $y'(0) = 0.996$.

We know solve the initial value problem

$y''(x) = y(x), y(0) = 0, y'(0) = m_2$

Taylor's series solution is given by

$y(m_2, 1) = m_2\,(1.175) = 1.00085 \times 1.175 = 1.17599$

Hence, the solution at $x = 1$ is $y = 1.176$ which is close to the exact value of $y(1) = 1.1752$.

22. Use Picard's method to approximate the value of y when x = 0.1, x = 0.2 and x = 0.4, given that y = 1 at x = 0 and $\frac{dy}{dx} = 1 + xy$ correct to three decimal places. (Use upto second approximation) [2019/Fall]

Solution:

Given that;

$$\frac{dy}{dx} = 1 + xy = f(x, y)$$

and, $x_0 = 0$, $y_0 = 1$?

Using Picard's method, we have,

$$y = y_0 + \int_{x_0}^{x} f(x, y)\,dx$$

First approximation, put y = 1 in the integrand

$$y_1 = 1 + \int_0^x [1 + x(1)]\,dx = 1 + \left[x + \frac{x^2}{2}\right]_0^x = 1 + x + \frac{x^2}{2}$$

Second approximation, put $y = 1 + x + \frac{x^2}{2}$ in the integrand

Scanned with CamScanner

$$y_2 = 1 + \int_0^x \left[1 + x\left(1 + x + \frac{x^2}{2}\right)\right] dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2}\right) dx$$

$$= 1 + \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right]_0^x$$

$$\therefore \quad y_2 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

Now, using the first approximation and taking

$x = 0.1, 0.2, 0.4$

We have,

$$y_1(0.1) = 1 + x + \frac{x^2}{2} = 1 + 0.1 + \frac{(0.1)^2}{2} = 1.105$$

$y_1(0.2) = 1.06$

$y_1(0.4) = 1.24$

Now, using the second approximation and taking

$x = 0.1, 0.2, 0.4$

We have,

$$y_2(0.1) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} = 1.1053$$

$y_2(0.2) = 1.2228$

$y_2(0.4) = 1.5045$

Also, the exact solution of $y' = 1 + xy$ is e^x

$y(0) = e^0 = 1$

$y(0.1) = e^{0.1} = 1.1051$

$y(0.2) = e^{0.2} = 1.221$

$y(0.4) = e^{0.4} = 1.492$

Here, $y(0.1) = 1.105$ is correct upto 3 decimal places.

For $y(0.2)$ using $y(0.1) = 1.105$ as initial value.

First approximation, put $y = 1.105$ in the integrand

$$y_1 = 1.105 + \int_{0.1}^x [1 + x(1.105)]dx$$

$$= 1.105 + \left[x + \frac{x^2}{2}(1.105)\right]_{0.1}^x$$

$$= 1.105 + x + 0.5525x^2 - 0.1 - 0.0055$$

$$= 0.999 + x + 0.5525x^2$$

Second approximation, put $y = 0.999 + x + 0.5525x^2$ in the integrand

$$y_2 = 1.105 + \int_{0.1}^x [1 + x(0.999 + x + 0.5525x^2)]dx$$

Scanned with CamScanner

$$= 1.105 + \int_{0.1}^{x} [1 + 0.999x + x^2 + 0.5525x^3]dx$$

$$= 1.105 + \left[x + \frac{0.999x^2}{2} + \frac{x^3}{3} + \frac{0.5525x^4}{4}\right]_{0.1}^{x}$$

$$= 1.105 + x + 0.499x^2 + 0.333x^3 + 0.1381x^4 - 0.1$$

$$- \frac{0.999(0.1)^2}{2} - \frac{(0.1)^3}{3} - \frac{0.5525(0.1)^4}{4}$$

$$= 0.999 + x + 0.499x^2 + 0.333x^3 + 0.1381x^4$$

Now, using the second approximation and taking x = 0.2, 0.4

We have,

$\therefore$ $y(0.2) = 0.999 + 0.2 + 0.499\,(0.2)^2 + 0.333\,(0.2)^3 + 0.1381\,(0.2)^4$

$= 1.2218$

$\therefore$ $y(0.4) = 1.5036$

Here, y(0.2) = 1.2218 is correct upto three decimal places compared to exact solution.

For, y(0.4), using y(0.2) = 1.2218 as initial value.

First approximation, put y = 1.2218 in the integrand.

$$y_1 = 1.2218 + \int_{0.2}^{x} [1 + x(1.2218)]dx$$

$$= 1.2218 + \left[x + \frac{1.2218x^2}{2}\right]_{0.2}^{x}$$

$$= 1.2218 + x + 0.6109x^2 - 0.2 - 0.0244$$

$$= 0.9974 + x + 0.6109x^2$$

Second approximation, put $y = 0.9974 + x + 0.6109x^2$ in the integrand.

$$y_2 = 1.2218 + \int_{0.2}^{x} [1 + x(0.9974 + x + 0.6109x^2)]\,dx$$

$$= 1.2218 + \left[x + \frac{0.9974x^2}{2} + \frac{x^3}{3} + \frac{0.6109x^4}{4}\right]_{0.2}^{x}$$

$$= 0.9989 + x + \frac{0.9974x^2}{2} + \frac{x^3}{3} + \frac{0.6109x^4}{4}$$

Now, using the second approximation and taking

x = 0.4

We have,

$$y(0.4) = 0.9989 + 0.4 + \frac{0.9974}{2}(0.4)^2 + \frac{(0.4)^3}{3} + \frac{0.6109}{4}(0.4)^4$$

$\therefore$ $y(0.4) = 1.5039$

Here, y(0.4) = 1.5039 is correct upto 3 decimal places.

Thus, y(0.1) = 1.105

y(0.2) = 1.221

y(0.4) = 1.503

Scanned with CamScanner

23. **Using Runge-Kutta method of second order (RK-2), obtain a solution of the equation $y'' = y + xy'$ with initial condition $y(0) = 1$, $y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$, taking $h = 0.1$.** **[2019/Fall]**

Solution:

Given that;

$$y'' = xy' + y \qquad \text{..... (1)}$$

Let, $y' = z$

Then, $y'' = z'$

So, equation (1) becomes

$$z' = xz + y = f_2(x, y, z)$$

and, $y' = z + f_1(x, y, z)$

Subject to

$$y(0) = 1 \rightarrow x_0 = 0, \quad y_0 = 1$$

$$y'(0) = 0 \rightarrow z_0 = 0$$

Taking $h = 0.1$

Now, using Runge-Kutta method of second order,

$$k_1 = hf_1(x_0, y_0, z_0) = 0.1f_1(0, 1, 0) = 0.1 \times 0 = 0$$

$$l_1 = hf_2(0, 1, 0) = 0.1(0(0) + 1) = 0.1$$

$$k_2 = hf_1(x_0 + h, y_0 + k_1, z_0 + l_1) = 0.1f_1(0.1, 1, 0.1) = 0.01$$

$$l_2 = hf_2(0.1, 1, 0.1) = 0.101$$

Then,

$$k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0 + 0.01) = 0.005$$

$$l = \frac{1}{2}(l_1 + l_4) = \frac{1}{2}(0.1 + 0.101) = 0.1005$$

so,

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = y_0 + k = 1 + 0.005 = 1.005$$

$$z_1 = z_0 + l = 0 + 0.1005 = 0.1005$$

Again,

$$k_1 = hf_1(x_1, y_1, z_1) = 0.1f_1(0.1, 1.005, 0.1005) = 0.01$$

Scanned with CamScanner

$l_1 = hf_2 (0.1, 1.005, 0.1005)$
$= 0.1015$
$k_2 = hf_1 (x_1 + h, y_1 + k_1, z_1 + l_1)$
$= 0.1f_1 (0.2, 1.015, 0.202)$
$= 0.020$
$l_2 = hf_2 (0.2, 1.015, 0.202)$
$= 0.1055$

Then, $k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.01 + 0.02) = 0.015$

$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(0.1015 + 0.1055) = 0.1035$

Hence,
$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$
$\therefore$ $y_2 = y_1 + k = 1.005 + 0.015 = 1.02$
$\therefore$ $z_2 = z_1 + l = 0.1005 + 0.1035 = 0.204$

24. Solve the given differential equation by Heun's method $y'' - y' - 2y = 3e^{2x}$ with initial condition $y(0) = 0$, $y'(0) = -2$ for $y(0.2)$ taking $h = 0.1$ **[2019/Spring]**

Solution:
Given that;
$y'' - y' - 2y = 3e^{2x}$ ..... (1)
Let, $y' = z$
Then, $y'' = z'$
So, equation (1) becomes
$z' - z - 2y = 3e^{2x}$
and, $z' = z + 2y + 3e^{2x}$
Subject to
$y(0) = 1 \quad \rightarrow x_0 = 0, \quad y_0 = 0$
$y'(0) = -2 \quad \rightarrow z_0 = -2$
Taking $h = 0.1$
Now, using Heun's method or modified Euler's method solving in tabular form.

S.N.	x	y' = z	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
1	0	-2	–	$0 + 0.1 \times (-2) = -0.2$
2	0.1	-1.9	$\frac{-2-1.9}{2} = -1.95$	$0 + 0.1 \times (-1.95) = -0.195$
3	0.1	-1.882	$\frac{-2-1.882}{2} = -1.94$	$0 + 0.1 \times (-1.94) = -0.194$
4	0.1	-1.881	$\frac{-2-1.882}{2} = -1.94$	$0 + 0.1 \times (-1.94) = -0.194$

Scanned with CamScanner

Here, the last two values are equal at $y_1 = -0.194$.

S.N.	x	$z' = z + 2y + 3e^{2x}$	Mean slope	$z_{new} = z_{old} + h$ (mean slope)
1	0	$-2 + 2(0) + 3e^{2(0)} = 1$	–	$-2 + 0.1 \times 1 = -1.9$
2	0.1	$-1.9 + 2(-0.2) + 3e^{2(0.1)} = 1.364$	$\frac{1 + 1.364}{2} = 1.18$	$-2 + 0.1 \times 1.18 = -1.882$
3	0.1	$-1.88 + 2(-0.195) + 3e^{2(0.1)} = 1.394$	$\frac{1 + 1.394}{2} = 1.19$	$-2 + 0.1 \times 1.19 = -1.881$
4	0.1	$-1.88 + 2(-0.194) + 3e^{2(0.1)} = 1.396$	$\frac{1 + 1.396}{2} = 1.19$	$-2 + 0.1 \times 1.19 = -1.881$

Here, the last two values are equal at $z_1 = -1.881$.

Again,

S.N.	x	$y' = z$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
1	0.1	-1.881	–	$-0.194 + 0.1 \times (-1.881) = -0.382$
2	0.2	-1.741	$\frac{-1.881 - 1.741}{2} = -1.811$	$-0.194 + 0.1 \times (-1.811) = -0.375$
3	0.2	-1.712	$\frac{-1.881 - 1.712}{2} = -1.796$	$-0.194 + 0.1 \times (-1.796) = -0.373$
4	0.2	-1.710	$\frac{-1.881 - 1.710}{2} = -1.795$	$-0.194 + 0.1 \times (-1.795) = -0.373$

Here, the last two values are equal at $y_2 = -0.373$

S.N.	x	$z' = z + 2y + 3e^{2x}$	Mean slope	$z_{new} = z_{old} + h$ (mean slope)
1	0	$-1.881 + 2(-0.194) + 3e^{2(0.1)} = 1.395$	–	$-1.881 + 0.1(1.395) = -1.741$
2	0.1	$-1.741 + 2(-0.382) + 3e^{2(0.2)} = 1.970$	$\frac{1.395 + 1.970}{2} = 1.682$	$-1.881 + 0.1(1.682) = -1.712$
3	0.1	$-1.712 + 2(-0.375) + 3e^{2(0.2)} = 2.013$	$\frac{1.395 + 2.013}{2} = 1.704$	$-1.881 + 0.1(1.704) = -1.710$
4	0.1	$-1.71 + 2(-0.373) + 3e^{2(0.2)} = 2.019$	$\frac{1.395 + 2.019}{2} = 1.707$	$-1.881 + 0.1(1.707) = -1.710$

Here, the last two values are equal at $z_2 = -1.710$.

Hence, the required solution of $y(0.2) = -0.373$.

25. **Solve $y' = y + e^x$, $y(0) = 0$ for $y(0.2)$ and $y(0.4)$ by RK-4th order method.** **[2019/Spring]**

Scanned with CamScanner

Solution:

Given that;

$y' = y + e^x$

$y(0) = 0 \quad \rightarrow x_0 = 0, \quad y_0 = 0$

Taking h = 0.2

Now, using RK-4[th] order method

$k_1 = hf(x_0, y_0) = 0.2f(0, 0) = 0.2(0 + e^0) = 0.2$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2f(0.1, 0.1) = 0.241$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2f(0.1, 0.120) = 0.245$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 0.245) = 0.293$

Then,

$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$

$= \frac{1}{6}[0.2 + 0.293 + 2(0.241 + 0.245)]$

$= 0.244$

so, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$\therefore \quad y_1 = y_0 + k = 0 + 0.244 = 0.244$

Again,

$k_1 = hf(x_1, y_1) = 0.2f(0.2, 0.244) = 0.293$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2f(0.3, 0.39) = 0.347$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2f(0.3, 0.417) = 0.353$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 0.597) = 0.417$

Then,

$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$

$= \frac{1}{6}[0.293 + 0.417 + 2(0.347 + 0.353)]$

$= 0.351$

so, $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$\therefore \quad y_2 = y_1 + k = 0.244 + 0.351 = 0.595$

Hence, y(0.2) = 0.244 and y(0.4) = 0.595 are the required solutions.

26. Applying Runge-Kutta fourth order method to find an approximate value of y when x = 0.3 given that: $y' = 2.5y + e^{0.3x}$ with an initial y(0) = 1, taking h = 0.3 [2020/Fall]

Scanned with CamScanner

Solution:

Given that;

$y' = 2.5y + e^{0.3x}$

$y(0) = 1 \quad \rightarrow x_0 = 0, \quad y_0 = 1$

$h = 0.3$

Now, using Runge-Kutta fourth other method

$k_1 = hf(x_0, y_0)$

$= 0.3f(0, 1)$

$= 0.3\,[2.5(1) + e^{0.3\times0}]$

$= 1.05$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.3f(0.15, 1.525)$

$= 0.3[2.5(1.525) + e^{0.3(0.15)}]$

$= 1.457$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.3f(0.15, 1.728)$

$= 1.609$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.3f(0.3, 2.609)$

$= 2.285$

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$

$= \frac{1}{6}[1.05 + 2.285 + 2(1.457 + 1.609)]$

$= 1.577$

Now,

$x_1 = x_0 + h = 0 + 0.3 + 0.3$

$\therefore \quad y_1 = y(0.3) = y_0 + k = 1 + 1.577 = 2.577$

27. Solve the Boundary value problem (BVP) using shooting method by dividing into four sub-interval employing Euler's method.

$$y'' + 2y' - y = x$$

Subjective to boundary condition y(1) = 2 and y(2) = 4. [2020/Fall]

Solution:

Given that;

$y' + 2y' - y = x$ ..... (1)

Let $y' = z$

Then, $y'' = z'$

So equation (1) becomes,

$x' + 2z - y = x$

or, $z' = x + y - 2z = f_2(x, y, z)$

and, $y' = z = f_1(x, y, z)$

Subject to

$y(1) = 2 \quad \rightarrow x_0 = 1, \quad y_0 = 2$

Assuming

$y'(1) = 4 \quad \rightarrow z_0 = 4$

And having four subintervals, h = 0.25

Now, using shooting method by employing Euler's method

At, $i = 0, x_0 = 1, y_0 = 2, z_0 = 4, h = 0.25$

$y_1 = y_0 + hf_1(x_0, y_0, z_0)$
$= 2 + 0.25f_1(1, 2, 4)$
$= 2 + 0.25 \times 4 = 3$

$z_1 = z_0 + hf_2(x_0, y_0, z_0)$
$= 4 + 0.25f_2(1, 2, 4)$
$= 4 + 0.25(1 + 2 - 2 \times 4)$
$= 2.75$

At, $i = 1, x_1 = x_0 + h = 1.25, y_1 = 3, z_1 = 1.25, h = 0.25$

$y_2 = y_1 + hf_1(x_1, y_1, z_1)$
$= 3 + 0.25f_1(1.25, 3, 2.75)$
$= 3 + 0.25(2.75)$
$= 3.687$

$z_2 = z_1 + hf_2(x_1, y_1, z_1)$
$= 2.75 + 0.25f_2(1.25, 3, 2.75)$
$= 2.75 + 0.25(1.25 + 3 - 2(2.75)$
$= 2.437$

At, $i = 2, x_2 = 1.5, y_2 = 3.687, z_2 = 2.43, h = 0.25$

$y_3 = y_2 + hf_1(x_2, y_2, z_2)$
$= 3.687 + 0.25f_1(1.5, 3.687, 2.437)$
$= 4.296$

$z_3 = z_2 + hf_2(1.5, 3.687, 2.437)$
$= 2.515$

At, $i = 3, x_3 = 1.75, y_3 = 4.296, z_3 = 2.515, h = 0.25$

$y_4 = y_3 + hf_1(x_3, y_3, z_3)$
$= 4.296 + 0.25f_1(1.75, 4.296, 2.515)$
$= 4.924$

$z_4 = z_3 + hf_2(x_3, y_3, z_3)$
$= 2.769$

Here, given $y(2) = 4$

and we obtain $y(2) = y_4 = 4.924$ which is greater than 4.

So, we choose $y'(0) = 1 = z_0$ and carry out the process

Scanned with CamScanner

At, $i = 0, x_0 = 1, y_0 = 2, z_0 = 1, h = 0.25$

$y_1 = y_0 + hf_1(x_0, y_0, z_0)$

$= 2 + 0.25f_1(1, 2, 1)$

$= 2.25$

$z_1 = z_0 + hf_2(1, 2, 1)$

$= 1.25$

At, $i = 1, x_1 = 1.25, y_1 = 2.25, z_1 = 1.25, h = 0.25$

$y_2 = y_1 + hf_1(x_1, y_1, z_1)$

$= 2.562$

$z_2 = z_1 + hf_2(x_1, y_1, z_1)$

$= 1.5$

At, $i = 2, x_2 = 1.5, y_2 = 2.562, z_2 = 1.5, h = 0.25$

$y_3 = y_2 + hf_1(x_2, y_2, z_2)$

$= 2.937$

$z_3 = z_2 + hf_2(x_2, y_2, z_2)$

$= 1.765$

At, $i = 3, x_3 = 1.75, y_3 = 2.937, z_3 = 1.765, h = 0.25$

$y_4 = y_3 + hf_1(x_3, y_3, z_3)$

$= 3.378$

$z_4 = z_3 + hf_2(x_3, y_3, z_3)$

$= 2.054$

Here, we obtain,

$y_4 = y(2) = 3.378$ at $y'(0) = 1$

Also, we have,

$y_4 = y(2) = 4.924$ at $y'(0) = 4$

So for better approximation

$P_1 = y'(0) = 4$, $Q_1 = y(2) = 4.924$

$P_2 = y'(0) = 1$, $Q_2 = y(2) = 3.378$

Then to obtain $y(2) = 4 = Q$

$$P = P_1 + \frac{P_2 - P_1}{Q_2 - Q_1}(Q - Q_1)$$

$$= 4 + \frac{1 - 4}{3.378 - 4.924}(4 - 4.924)$$

$= 2.206$

So, now using $y'(0) = 2.206 = z_0$ and continuing the process.

At, $i = 0, x_0 = 1, y_0 = 2, z_0 = 2.206, h = 0.25$

$y_1 = y_0 + hf_1(x_0, y_0, z_0)$

$= 2.551$

$z_1 = z_0 + hf_2(x_0, y_0, z_0)$

$= 1.853$

Scanned with CamScanner

At, $i = 1, x_1 = 1.25, y_1 = 2.551, z_1 = 1.853, h = 0.25$

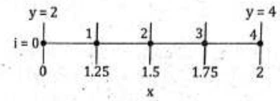
$y_2 = y_1 + hf_1(x_1, y_1, z_1)$
$= 3.014$
$z_2 = z_1 + hf_2(x_1, y_1, z_1)$
$= 1.876$

At, $i = 2, x_2 = 1.5, y_2 = 3.014, z_2 = 1.876, h = 0.25$

$y_3 = y_2 + hf_1(x_2, y_2, z_2)$
$= 3.483$
$z_3 = z_2 + hf_2(x_2, y_2, z_2)$
$= 2.066$

At, $i = 3, x_3 = 1.75, y_3 = 3.483, z_3 = 2.066, h = 0.25$

$y_4 = y_3 + hf_1(x_3, y_3, z_3)$
$= 3.995$
$z_4 = z_3 + hf_2(x_3, y_3, z_3)$
$= 2.341$



Here, we obtain $y_4 = y(2) = 3.995$ which is close to the exact value of $y(2) = 4$. Hence, the solution at $x = 2$ is $y = 3.995$.

28. Write short notes on: Finite differences. **[2020/Fall]**
Solution: See the topic 5.10 'B'.

29. Write short notes on: Picard's iterative formula. **[2020/Fall]**
Solution: See the topic 5.2.

30. Write short notes on: Solution of 2nd order differential equation. **[2016/Fall]**
Solution: See the topic 5.9.

31. Write short notes on: Boundary value problem. **[2017/Spring]**
Solution: See the topic 5.10.

A boundary value problem is a system of ordinary differential equations with solution and derivative values specified at more than one point. Most commonly, the solution and derivatives are specified at just two points (the boundaries) defining a two point boundary value problem. In the field of differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions. Boundary value problem arise in several branches of physics as any physical differential equation will have them.

Scanned with CamScanner

Boundary value problems are similar to initial value problems. A boundary value problem has conditions specified at the extremes ("boundaries") of the independent variable in the equation whereas on initial value problem has all of the conditions specified at the same value of the independent variable (and that value is at the lower boundary of the domain, thus the term "initial value). A boundary value is a data value that corresponds to a minimum or maximum input, internal or output value specified for a system or component.

32. Write short notes on: algorithm for second order Runge-Kutta (RK-2) method. **[2020/Fall]**

Solution:

1. Define function f(x, y)
2. Get values of x_0, y_0, h, x_n
   where, x_0 is starting value of x *i.e.*, x_0, x_n is the value of x for which y is to be determined.
3. If $x = x_n$ then go to step 7
   else
   $k_1 = h \times f(x, y)$
   $k_2 = h \times f(x + h, y + k_1)$
4. Compute $k = \left(\frac{k_1 + k_2}{2}\right)$ and,
   $x = x + h$
   $y = y + k$
5. Display x and y
6. Go to step 3
7. Stop.

33. Write short notes on: Taylor series for solving ordinary differential equations. **[2015/Spring]**

Solution: See the topic 5.3.

34. Write short notes on: Algorithm for Euler methods. **[2018/Spring]**

1. Define function df(x, y) *i.e.*, dy/dx
2. Get values of x_0, y_0, h, x
   where, x_0 is x_{n+0}
   x_1 is x_{n+1}
3. Assign $x_1 = x_0$ and $y_1 = y_0$
4. If $x_1 > x$, then go to step 7
   else
   Compute $y_1 += h \times df(x_1, y_1)$
   and, $x_1 += h$ *i.e.*, $x_1 = x_1 + h$
5. Display x_1 and y_1
6. Go to step 4.
7. Stop.

imp

Scanned with CamScanner

ADDITIONAL QUESTION SOLUTION

1. Solve $y' = \frac{y}{x^2 + y^2}$, $y(0) = 1$ using RK-2 method in the range of 0, 0.5, 1.

Solution:

Given that;

$$y' = \frac{y}{x^2 + y^2} = f(x, y)$$

Subject to

$y(0) = 1 \quad \rightarrow x_0 = 0$ and $y_0 = 1$

in the range of 0, 0.5, 1, so taking h = 0.5

Now, using RK-2 method

$$k_1 = hf(x_0, y_0)$$
$$= 0.5 \times f(0, 1)$$
$$= 0.5 \times \left(\frac{1}{0^2 + 1^2}\right)$$
$$= 0.5$$
$$k_2 = hf(x_0 + h, y_0 + k_1)$$
$$= 0.5 \times f(0.5, 1.5)$$
$$= 0.5 \times \left(\frac{1}{0.5^2 + 1.5^2}\right)$$
$$= 0.3$$

Then,

$$k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.5 + 0.3) = 0.4$$

so, $x_1 = x_0 + h = 0 + 0.5 = 0.5$

$y_1 = y_0 + k = 1 + 0.3 = 1.3$

Again,

$$k_1 = hf(x_1, y_1)$$
$$= 0.5 \times f(0.5, 1.3)$$
$$= 0.5 \times \left(\frac{1.3}{0.5^2 + 1.3^2}\right)$$
$$= 0.3351$$
$$k_2 = hf(x_1 + h, y_1 + k_1)$$
$$= 0.5 \times f(1, 1.6351)$$
$$= 0.5 \times \left(\frac{1.6351}{1^2 + 1.6351^2}\right)$$
$$= 0.2226$$

Scanned with CamScanner

Then,

$$k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.3351 + 0.2226) = 0.2788$$

so, $x_2 = x_1 + h = 0.5 + 0.5 = 1$

$y_2 = y_1 + k = 1.3 + 0.2788 = 1.5788$

2. Solve the BVP: $y'' + 3y' = y + x^2$, $y(0) = 2$, $y(2) = 5$ at $x = 0.5, 1, 1.5$ using finite difference method.

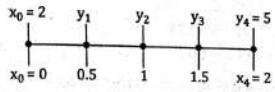
Solution:

Given that;

$$y'' + 3y' = y + x^2 \quad \text{..... (1)}$$

$$y(0) = 2 \text{ and } y(2) = 5$$

and, $h = 0.5$



Now, from finite difference approximation, we have,

$$\frac{dy}{dx} = y' = \frac{1}{2h}[y_{i+1} - y_{i-1}]$$

$$\frac{d^2y}{dx^2} = y'' = \frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}]$$

Now using the approximated value in equation (1),

$$\frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] + \frac{3}{2h}[y_{i+1} - y_{i-1}] = y_i + x_i^2$$

Put i = 1, at h = 0.5

$$\frac{1}{0.5^2}(y_2 - 2y_1 + y_0) + \frac{3}{2 \times 0.5}(y_2 - y_0) = y_1 + x_1^2$$

or, $4y_2 - 8y_1 + 4y_0 + 3y_2 - 3y_0 = y_1 + x_1^2$

Substituting the values of y_0 and x_1

$$4y_2 - 8y_1 + 4(2) + 3y_2 - 3(2) = y_1 + (0.5)^2$$

or, $7y_2 - 9y_1 = -1.75$ ..... (A)

Again,

Put i = 2,

$$4(y_3 - 2y_2 + y_1) + 3(y_3 - y_1) = y_2 + x_2^2$$

or, $7y_3 - 9y_2 + 4y_1 - 3y_1 = x_2^2$

Substituting the values

$$y_1 + 7y_3 - 9y_2 = 1^2$$

or, $y_1 - 9y_2 + 7y_3 = 1$ ..... (B)

Scanned with CamScanner

Again,

Put i = 3,

$$4(y_4 - 2y_3 + y_2) + 3(y_4 - y_2) = y_3 + x_3^2$$

or, $4y_4 - 8y_3 + 4y_2 - 3y_4 - 3y_2 = y_3 + x_3^2$

Substituting the values

or, $4(5) - 8y_3 - y_3 + 4y_2 - 3y_2 + 3(5) = (1.5)^2$

or, $y_2 - 9y_3 = -32.75$ ---- (C)

Now solving the equations (A), (B) and (C), we get,

$y_1 = 2.7716$

$y_2 = 3.3134$

$y_3 = 4.0070$

Hence, the required solutions are;

$x_1 = 0.5$, $y_1 = 2.7716$

$x_2 = 1$, $y_2 = 3.3134$

$x_3 = 1.5$, $y_3 = 4.0070$

3. Solve the following boundary value problem using the finite difference method by dividing the interval into four sub-intervals.

$$y'' = e^x + 2y' - y;\ y(0) = 1.5,\ y(2) = 2.5$$

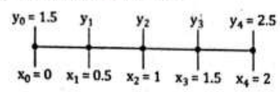
Solution:

Given that;

$y'' = e^x + 2y' - y$ ..... (1)

$y(0) = 1.5,\ y(2) = 2.5$

Dividing the interval into four sub-intervals



Here, h = 0.5

Now, for finite difference approximation, we have

$$\frac{dy}{dx} = y' = \frac{1}{2h}[y_{i+1} - y_{i-1}]$$

$$\frac{d^2y}{dx^2} = y'' = \frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}]$$

Now using the approximated value in equation (1),

$$\frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] = e^{x_i} + \frac{2}{2h}[y_{i+1} - y_{i-1}] - y_i$$

Put i = 1, at h = 0.5

$$\frac{1}{0.5^2}(y_2 - 2y_1 + y_0) = e^{x_1} + \frac{2}{2 \times 0.5}(y_2 - y_0) - y_1$$

or, $4y_2 - 8y_1 + 4y_0 = e^{x_1} + 2y_2 - 2y_0 - y_1$

Scanned with CamScanner

Substituting the values

$2y_2 - 7y_1 = e^{0.5} - 2(1.5) - 4(1.5)$

or, $2y_2 - 7y_1 = -7.3512$ ..... (A)

Put $i = 2$,

$4(y_3 - 2y_2 + y_1) = e^{x_2} + 2(y_3 - y_1) - y_2$

or, $4y_3 - 8y_2 + 4y_1 = e^{x_2} + 2y_3 - 2y_1 - y_2$

Substituting the values

$2y_3 - 7y_2 + 6y_1 = e^1$

or, $6y_1 - 7y_2 + 2y_3 = 2.7183$ ..... (B)

Put $i = 3$,

$4(y_4 - 2y_3 + y_2) = e^{x_3} + 2(y_4 - y_2) - y_3$

or, $4y_4 - 8y_3 + 4y_2 = e^{x_3} + 2y_4 - 2y_2 - y_3$

or, $2y_4 - 7y_3 + 6y_2 = e^{x_3}$

Substituting the values

or, $6y_2 - 7y_3 = e^{1.5} - 2(2.5)$

or, $6y_2 - 7y_3 = -0.5183$ ..... (C)

Now solving the equations (A), (B) and (C), we get,

$y_1 = 1.3487$

$y_2 = 1.0447$

$y_3 = 0.9695$

4. Solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ using RK-4 method, for y(0.4)

Given, y(0) = 1, h = 0.2

Solution:

We have,

$$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2} = f(x, y)$$

Subject to

$y(0) = 1 \quad \rightarrow x_0 = 0, \quad y_0 = 1$

At h = 0.2

Now, using RK-4 method

$$k_1 = hf(x_0, y_0)$$
$$= 0.2f(0, 1)$$
$$= 0.2 \times \left(\frac{1^2 - 0^2}{1^2 + 0^2}\right)$$
$$= 0.2$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.2f(0.1, 1.1)$$

Scanned with CamScanner

$$= 0.2 \times \left(\frac{1.1^2 - 0.1^2}{1.1^2 + 0.1^2}\right)$$

$= 0.1967$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$

$= 0.2f(0.1, 1.0983)$

$= 0.1967$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.2f(0.2, 1.1967)$

$= 0.1891$

Then,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0.2 + 0.1891 + 2(0.1967 + 0.1967)]$$

$= 0.1959$

so, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$\therefore$ $y_1 = y_0 + k = 1 + 0.1959 = 1.196$

Again,

$k_1 = hf(x_1, y_1)$

$= 0.2f(0.2, 1.196)$

$= 0.1891$

$$k_2 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right)$$

$= 0.2f(0.3, 1.2906)$

$= 0.1795$

$$k_3 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right)$$

$= 0.2f(0.3, 1.2858)$

$= 0.1793$

$k_4 = hf(x_1 + h, y_1 + k_3)$

$= 0.2f(0.4, 1.3753)$

$= 0.1688$

Then,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0.1891 + 0.1688 + 2(0.1795 + 0.1793)]$$

$= 0.1792$

so, $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$\therefore$ $y_2 = y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752$

Scanned with CamScanner

5. **Solve the following simultaneous differential equations using RK second order method at x = 0.1 and 0.2**

$$\frac{dy}{dx} = xz + 1; \frac{dz}{dx} = -xy \text{ with initial conditions } y(0) = 0, z(0) = 1$$

Solution:

Given that;

$$\frac{dy}{dx} = y' = 1 + xz = f_1(x, y, z)$$

and, $$\frac{dz}{dx} = z' = -xy = f_2(x, y, z)$$

Subject to

$y(0) = 0 \quad \rightarrow x_0 = 0, \quad y_0 = 0$

$z(0) = 1 \quad \rightarrow z_0 = 1$

At h = 0.1

Now, using Runge-Kutta method of second order

$k_1 = hf_1(x_0, y_0, z_0)$

$= 0.1f_1(0, 0, 1)$

$= 0.1 \times [1 + 0 \times (1)]$

$= 0.1$

$l_1 = 0.1f_2(x_0, y_0, z_0)$

$= 0.1f_2(0, 0, 1)$

$= 0.1 \times (-0 \times 0)$

$= 0$

$k_2 = hf_1(x_0 + h, y_0 + k_1, z_0 + l_1)$

$= 0.1f_1(0.1, 0.1, 1)$

$= 0.1 \times (1 + 0.1 \times 1)$

$= 0.11$

$l_2 = hf_2(0.1, 0.1, 1)$

$= 0.1 \times (-0.1 \times 0.1)$

$= -0.001$

Then, $k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.1 + 0.11) = 0.105$

$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}[0 + (-0.001)] = -0.0005$

So, $x_1 = x_0 + h = 0 + 0.1 = 0.1$

$y_1 = y_0 + k = 0 + 0.105 = 0.105$

$z_1 = z_0 + l = 1 - 0.0005 = 0.9995$

Again,

$k_1 = hf_1(x_1, y_1, z_1)$

$= 0.1f_1(0.1, 0.105, 0.9995)$

$= 0.1 \times (1 + 0.1 \times 0.9995)$

$= 0.11$

Scanned with CamScanner

$l_1 = hf_2(0.1, 0.105, 0.9995)$
$= -0.0011$
$k_2 = hf_1(x_1 + h, y_1 + k_1, z_1 + l_1)$
$= 0.1f_1(0.2, 0.215, 0.9984)$
$= 0.12$
$l_2 = hf_2(0.2, 0.215, 0.9984)$
$= -0.0043$

Then,

$$k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.11 + 0.12) = 0.115$$

$$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(-0.0011 - 0.0043) = -0.0027$$

Hence,

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$
$\therefore$ $y_2 = y_1 + k = 0.105 + 0.115 = 0.22$
$\therefore$ $z_2 = z_1 + l = 0.9995 - 0.0027 = 0.9968$

6. Solve $\frac{dy}{dx} = \log(x + y)$, $y(0) = 2$ for $x = 0.8$ taking $h = 0.1$ using Euler's method.

Solution:

We have,

$$\frac{dy}{dx} = \log(x + y)$$

Subject to

$y(0) = 2 \quad \rightarrow x_0 = 0, \quad y_0 = 2$

Taking $h = 0.1$

Now, using Euler's method in tabular form

S.N.	x	y	$\frac{dy}{dx} = \log(x + y)$	$y_{new} = y_{old} + h\frac{dy}{dx}$
1	0	2	log(0 + 2) = 0.30102	2 + 0.1(0.30102) = 2.03010
2	0.1	2.03010	0.32840	2.06294
3	0.2	2.06294	0.35467	2.09840
4	0.3	2.09840	0.37992	2.13639
5	0.4	2.13639	0.40421	2.17681
6	0.5	2.17681	0.42761	2.21957
7	0.6	2.21957	0.45018	2.26458
8	0.7	2.26458	0.47196	2.31177
9	0.8	2.31177		

Hence the required approximate value is 2.31177 for x = 0.8.

Scanned with CamScanner

7. **Solve the following by Euler's modified method:**

$\frac{dy}{dx} = \log(x + y)$, $y(0) = 2$ at $x = 1.2$ and 1.4 with $h = 0.2$

Solution:

Given that;

$$\frac{dy}{dx} = \log(x + y)$$

Subject to

$y(0) = 2 \quad \rightarrow x_0 = 0, \quad y_0 = 2$

At $h = 0.2$

Now, solving in tabular form

S.N.	x	$\frac{dy}{dx} = \log(x + y)$	Mean slope	$y_{new} = y_{old} + h(\text{mean slope})$
1	0	log(0+2)=0.301	–	2+0.2(0.301)=2.0602
2	0.2	log(0.2+2.0602)	$\frac{1}{2}$(0.301+0.3541)	2+0.2(0.3276)=2.0655
3	0.2	log(0.2+2.0655)	$\frac{1}{2}$(0.301+0.3552)	2+0.2(0.3281)=2.0656

Here, last two values are equal at $y_1 = 2.0656$.

S.N.	x	$\frac{dy}{dx} = \log(x + y)$	Mean slope	$y_{new} = y_{old} + h(\text{mean slope})$
4	0.2	0.3552	–	2.0656+0.2(0.3552)=2.1366
5	0.4	log(0.4+2.1366)	$\frac{1}{2}$(0.3552+0.4042)	2.0656+0.2(0.3797)=2.1415
6	0.4	log(0.4+2.1415)	$\frac{1}{2}$(0.3552+0.4051)	2.0656+0.2(0.3801)=2.1416

Here, last two values are equal at $y_2 = 2.1416$.

S.N.	x	$\frac{dy}{dx} = \log(x + y)$	Mean slope	$y_{new} = y_{old} + h(\text{mean slope})$
7	0.4	0.4051	–	2.1416+0.2(0.4051)=2.2226
8	0.6	log(0.6+2.2226)	$\frac{1}{2}$(0.4051+0.4506)	2.1416+0.2(0.4279)=2.2272
9	0.6	log(0.6+2.2272)	$\frac{1}{2}$(0.4051+0.4514)	2.1416+0.2(0.4282)=2.2272

Here, last two values are equal at $y_3 = 2.2272$.

S.N.	x	$\frac{dy}{dx} = \log(x + y)$	Mean slope	$y_{new} = y_{old} + h(\text{mean slope})$
10	0.6	0.4514	–	2.2272+0.2(0.4514)=2.3175
11	0.8	log(0.8+2.3175)	$\frac{1}{2}$(0.4514+0.4938)	2.2272+0.2(0.4726)=2.3217
12	0.8	log(0.8+2.3217)	$\frac{1}{2}$(0.4514+0.4943)	2.2272+0.2(0.4727)=2.3217

Scanned with CamScanner

Here, last two values are equal at $y_4 = 2.3217$.

S.N.	x	$\frac{dy}{dx} = \log(x+y)$	Mean slope	$y_{new} = y_{old} + h(\text{mean slope})$
13	0.8	0.4943	–	2.3217+0.2(0.4943)=2.4206
14	1	log(1+2.4206)	$\frac{1}{2}$(0.4943+0.5341)	2.3217+0.2(0.5142)=2.4245
15	1	log(1+2.4245)	$\frac{1}{2}$(0.4943+0.5346)	2.3217+0.2(0.5144)=2.4245

Here, last two values are equal at $y_5 = 2.4245$.

S.N.	x	$\frac{dy}{dx} = \log(x+y)$	Mean slope	$y_{new} = y_{old} + h(\text{mean slope})$
16	1	0.5346	–	2.4245+0.2(0.5346)=2.5314
17	1.2	log(1.2+2.5314)	$\frac{1}{2}$(0.5346+0.5719)	2.4245+0.2(0.5532)=2.5351
18	1.2	log(1.2+2.5351)	$\frac{1}{2}$(0.5346+0.5723)	2.4245+0.2(0.5534)=2.5351

Here, last two values are equal at $y_6 = 2.5351$.

S.N.	x	$\frac{dy}{dx} = \log(x+y)$	Mean slope	$y_{new} = y_{old} + h(\text{mean slope})$
19	1.2	0.5723	–	2.5351+0.2(0.5723)=2.6496
20	1.4	log(1.4+2.6496)	$\frac{1}{2}$(0.5723+0.6074)	2.5351+0.2(0.5898)=2.6531
21	1.4	log(1.4+2.6531)	$\frac{1}{2}$(0.5723+0.6078)	2.5351+0.2(0.5900)=2.6531

Here, last two values are equal at $y_7 = 2.6531$.

Hence, $y(1.2) = 2.5351$ and $y(1.4) = 2.6531$ are the required approximated values.

Scanned with CamScanner

Chapter

6

SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

♠♠♠

6.1 INTRODUCTION

Partial differential equations arise in the study of many branches of applied mathematics. For example; in fluid dynamics, heat transfer, boundary layer flow, elasticity, quantum mechanics and electro-magnetic theory. Only a few of these equations can be solved by analytical methods which are also complicated by requiring use of advanced mathematical techniques. In most of the cases, it is easier to develop approximate solutions by numerical methods. Of all the numerical methods available for the solution of partial differential equations, the method of finite differences is most commonly used. In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite difference equations. Then the given equation is changed to a system of linear equations which are solved by iterative procedures. This process is slow but produces good results in many boundary value problems. An added advantage of this method is that the computation can be carried by electronic computers. To accelerate the solution, sometimes the method of relaxation proves quite effective.

Scanned with CamScanner

6.2 CLASSIFICATION OF SECOND ORDER EQUATIONS

The general linear partial differential equation of the second order in two independent variables is of the form.

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + B(x, y)\frac{\partial^2 u}{\partial x \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} + \left(x, y, u\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad \text{..... (1)}$$

Such a partial differential equation is said to be

a) Elliptic if $B^2 - 4AC < 0$

b) Parabolic if $B^2 - 4AC = 0$

c) Hyperbolic if $B^2 - 4AC > 0$

A partial equation is classified according to the region in which it is desired to be solved. For instance, the partial differential equation $f_{xx} + f_{yy} = 0$ is elliptic if $y > 0$, parabolic if $y = 0$ and hyperbola if $y < 0$.

A. Finite Difference Approximations to Partial Derivatives

Consider a rectangular region R in the x, y plane. Divide this region into a rectangular network of sides $\Delta x = h$ and $\Delta y = k$ as shown in figure 6.1. The points of intersection of the dividing lines are called mesh points, nodal points or grid points.

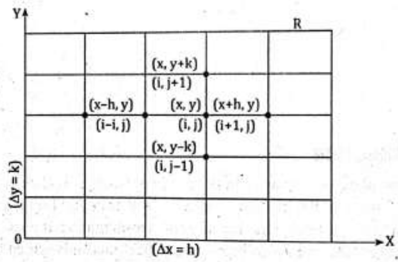


Figure 6.1

Then we have the finite difference approximations for the partial derivatives in x-direction.

$$\frac{\partial u}{\partial x} = \frac{u(x+h, y) - u(x, y)}{h} + O(h)$$

$$= \frac{u(x, y) - u(x-h, y)}{h} + O(h)$$

$$= \frac{u(x+h, y) - u(x-h, y)}{2h} + O(h^2)$$

and, $$\frac{\partial^2 u}{\partial x^2} = \frac{(x-h, y) - 2u(x, y) + u(x+h, y)}{h^2} + O(h^2)$$

Scanned with CamScanner

Writing u(x, y) = u(ih, jk) as simply $u_{i,j}$ the above approximations become,

$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h} + O(h) \quad \text{..... (1)}$$

$$= \frac{u_{i,j} - u_{i-1,j}}{h} + O(h) \quad \text{..... (2)}$$

$$= \frac{u_{i+1} - u_{i-1,j}}{2h} + O(h^2) \quad \text{..... (3)}$$

$$u_{xx} = \frac{u_{i-1} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(h^2) \quad \text{..... (4)}$$

Similarly, we have approximations for the derivatives with respect to y,

$$u_y = \frac{u_{i,j+1} - u_{i,j}}{k} + O(k) \quad \text{..... (5)}$$

$$= \frac{u_{i,j} - u_{i,j-1}}{k} + O(k) \quad \text{..... (6)}$$

$$= \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2) \quad \text{..... (7)}$$

$$u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + O(k^2) \quad \text{..... (8)}$$

Replacing the derivatives in any partial differential equation by their corresponding difference approximations (1) to (8), we obtain the finite-difference analogues of the given equation.

B. Elliptic Equations

The Laplace equation,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and the Poisson's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

are examples of elliptic partial differential equations. The Laplace equation arises in steady-state flow and potential problem. Poisson's equation arises in fluid mechanics, electricity and magnetism and torsion problem.

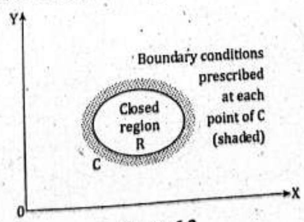


Figure 6.2

The solution of these equations is a function u(x, y) which is satisfied at every point of region R subject to certain boundary conditions specified on the closed curve.

In general, problem concerning steady viscous flow, equilibrium stress in elastic structures etc lead to elliptic type of equations.

C. Solutions of Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{..... (1)}$$

Consider a rectangular region R for which u(x, y) is known at the boundary. Divide this region into a network of square mesh of side h as shown in figure 6.3. (Assuming that an exact sub-division of R is possible). Replacing the derivatives in (1) by their difference approximations, we have,

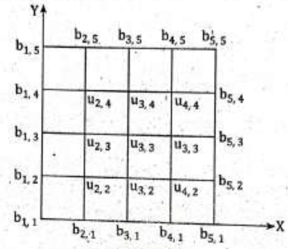


Figure 6.3

$$\frac{1}{h^2}[u_{i-1} - 2u_{i+1,j}] + \frac{1}{h^2}[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}] = 0$$

or, $$u_{i,j} = \frac{1}{4}[u_{i-1} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}] \qquad \text{..... (2)}$$

This shows that the value of u at any interior mesh point is the average of its values at four neighboring points to the left, right, above and below. Equation (2) is called the standard 5-point formula which is exhibited in figure 6.4.

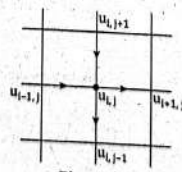


Figure 6.4

Sometimes a formula similar to equation (2) is used which is given by,

$$u_{i,j} = \frac{1}{4}(u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}) \qquad \text{..... (3)}$$

Scanned with CamScanner

This shows that the value of $u_{i,j}$ is the average of its values at the four neighboring diagonal mesh points. Equation (3) is called the diagonal five-point formula which is represented in figure 6.5. Although equation (3) is less accurate than equation (2), yet it serves as a reasonably good approximation for obtaining the starting values at the mesh points.

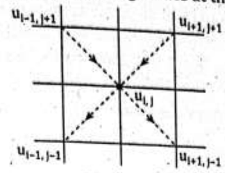


Figure 6.5

Now, to find the initial values of u at the interior mesh points, we first use the diagonal five-point formula (3) and compute $u_{3,3}$, $u_{2,4}$, $u_{4,4}$, $u_{4,2}$ and $u_{2,2}$ in this order. Thus, we get,

$$u_{3,3} = \frac{1}{4}(b_{1,5} + b_{5,1} + b_{5,5} + b_{1,1})$$

$$u_{2,4} = \frac{1}{4}(b_{1,5} + u_{3,3} + b_{3,5} + b_{1,3})$$

$$u_{4,4} = \frac{1}{4}(b_{3,5} + b_{5,3} + b_{3,5} + u_{3,3})$$

$$u_{4,2} = \frac{1}{4}(u_{3,3} + b_{5,1} + b_{3,1} + b_{5,3})$$

$$u_{2,2} = \frac{1}{4}(b_{1,3} + b_{3,1} + u_{3,3} + b_{1,1})$$

The values at the remaining interior points *i.e.*, $u_{2,3}$, $u_{3,4}$, $u_{4,3}$ and $u_{3,2}$ are computed by the standard five-point formula (2). Thus, we get,

$$u_{2,3} = \frac{1}{4}(b_{1,3} + u_{3,3} + u_{2,4} + u_{2,2})$$

$$u_{3,4} = \frac{1}{4}(u_{2,4} + u_{4,4} + b_{3,5} + u_{3,3})$$

$$u_{4,3} = \frac{1}{4}(u_{3,3} + b_{5,3} + u_{4,4} + u_{4,2})$$

$$u_{3,2} = \frac{1}{4}(u_{2,2} + u_{4,2} + u_{3,3} + u_{3,1})$$

Having found all the nine values of $u_{i,j}$ once, their accuracy is improved by either of the following iterative methods. In each case, the method is repeated until the difference between two consecutive iterates becomes negligible.

i) Jacobi's Method

Denoting th n^{th} iterative value of $u_{i,j}$ by $u^{n}_{i,j}$, the iterative formula to solve (2) is,

$$u^{(n+1)}_{i,j} = \frac{1}{4}\left[u^{(n)}_{i-1,j} + u^{(n)}_{i+1,j} + u^{(n)}_{i,j+1} + u^{(n)}_{i,j-1}\right] \quad \text{..... (4)}$$

It gives improved values of $u_{i,j}$ at the interior mesh points and is called the point of Jacobi's formula.

Scanned with CamScanner

ii) Gauss-Seidel Method

In this method, the iteration formula is,

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n+1)} + u_{i,j-1}^{(n)}\right] \quad \text{..... (5)}$$

It utilizes the latest derivative value available and scans the mesh points symmetrically from left to right along successive rows.

The accuracy of calculations depends on the mesh seize *i.e.*, smaller the h, the better the accuracy. But if h is too small, it may increase rounding off errors and also increases the labor of computation.

D. Solution of Poisson' Equation

Here, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ ..... (1)

Its method of solution is similar to that of the Laplace equation. Here the standard five-point formula for (1) takes the form,

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh) \quad \text{..... (2)}$$

By applying (2) at each interior mesh points, we arrive at linear equations in the nodal values $u_{i,j}$. These equations can be solved by the Gauss-seidal method.

E. Parabolic Equations

The one-dimensional heat conduction equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is a well known example of parabolic partial differential equations. The solution of this equation is a temperature function u(x, t) which is defined for values of x from 0 to 1 and for values of time t from 0 to ∞. The solution is not defined in a closed domain but advances in an open-ended region from initial values, satisfying the prescribed boundary conditions.

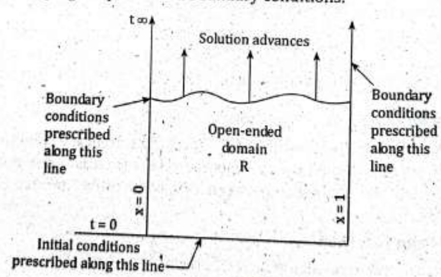


Figure 6.6

In general, the study of pressure waves in a fluid, propagation of heat and unsteady state problems lead to parabolic type of equations.

Solution of One Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{.... (1)}$$

where, $c^2 = \frac{k}{s\rho}$ is the diffusivity of the substance (cm²/sec)

Schmidt Method

Consider a rectangular mesh in the x-t plane with spacing h along x-direction and k along time t direction. Denoting a mesh point $(x, t) = (ih, jk)$ as simply i, j

We have,

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k}$$

and, $$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

Replacing these in equation (1), we get,

$$u_{i,j+1} - u_{i,j} = \frac{kc^2}{h^2}\left[u_{i,j} - 2\,u_{i,j} + u_{i+1,j}\right]$$

or, $$u_{i,j+1} = \alpha\, u_{i-1,j} + (1 - 2\alpha)\, u_{i,j} + \alpha\, u_{i+1,j} \qquad \text{.... (2)}$$

where, $\alpha = \frac{kc^2}{h^2}$ is the mesh ratio parameter

This formula enables us to determine the value of u at the $(i, j+1)^{th}$ mesh point in terms of the known function values at the points x_{i-1}, x_i and x_{i+1} at the instant t_j. It is a relation between the function values at the two time levels j + 1 and j and is called a two level formula. In schematic form equation (2) is shown in figure 6.7.

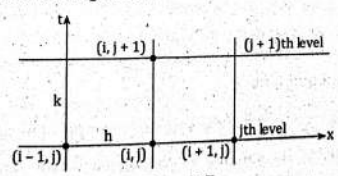


Figure 6.7

Hence, equation (2) is called the Schmidt explicit formula which is valid only for $0 < \alpha \le 12$.

Solution of Two Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \text{.... (1)}$$

The methods employed for the solution of one dimensional heat equation can be readily extended to the solution of equation (1).

Scanned with CamScanner

Consider the square region $0 \leq x \leq y \leq a$ and assume that u is known at all points within and on the boundary of this square.

If h is the step-size, then a mesh point $(x, y, t) = (ih, jh, nl)$ may be denoted as simply (i, j, n). Replacing the derivatives in (1) by their finite difference approximations, we get,

$$\frac{u_{i,j,n+1} - u_{i,j,n}}{l} = \frac{c^2}{h^2}[(u_{i-1,j,n} - 2u_{i,j,n} + u_{i+1,j,n} + (u_{i,j-1,n} - 2u_{i,j,n} + u_{i,j+1,n})]$$

i.e., $$u_{i,j,n+1} = u_{i,j,n} + \alpha(u_{i-1,j,n} + u_{i+1,j,n} + u_{i,j-1,n} - 4u_{i,j,n}) \quad \text{..... (2)}$$

where, $\alpha = \dfrac{lc^2}{h^2}$

This equation needs the five points available on the n^{th} plane.

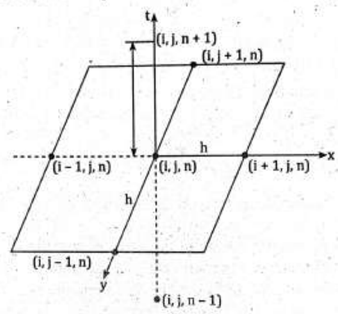


Figure 6.8

The computation process consists of point-by-point evaluation in the $(n + 1)^{th}$ plane using the points on the n^{th} plane. It is followed by plane by plane evaluation. This method is known as alternating direction explicit method.

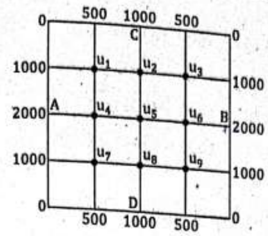
F. Hyperbolic Equations

The wave equation $\dfrac{\partial^2 u}{\partial x^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ is the simplest example of hyperbolic partial differential equations. Its solution is the displacement function $u(x, t)$ defined for values of x from 0 to 1 and for t from 0 to ∞, satisfying the initial and boundary conditions. In the case of hyperbolic equations, however, we have two initial conditions and two boundary conditions.

Such equations arise from connective type of problems in vibrations, wave mechanics and gas dynamics.

Example 6.1

Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in figure.



Solution:

Let, u_1, u_2, u_3 up to u_9 be the values of u at the interior mesh points. Since the boundary values of u are symmetrical about AB,

$\therefore \quad u_7 = u_1,\ u_8 = u_2,\ u_9 = u_3$

Also the values of u being symmetrical about CD

$u_3 = u_1,\ u_6 = u_4,\ u_9 = u_7$

Thus it is sufficient to find the values of u_1, u_2, u_4 and u_5

Now, we find their initial values in the following order

$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000)$[using standard 5 point formula]

$= 1500$

$u_1 = \frac{1}{4}(0 + 1500 + 1000 + 2000)$ [using diagonal 5 point formula]

$= 1125$

$u_2 = \frac{1}{4}(1125 + 1125 + 1000 + 1500)$ [using standard 5 point formula]

≈ 1188

$u_4 = \frac{1}{4}(2000 + 1125 + 1500 + 1125)$[using standard 5 point formula]

≈ 1438

Now, we carryout the iteration process using the standard formulae:

$$u_1^{n+1} = \frac{1}{4}[1000 + u_2^n + 500 + u_4^n]$$

$$u_2^{n+1} = \frac{1}{4}[u_1^{n+1} + u_1^n + 1000 + u_5^n]$$

$$u_4^{n+1} = \frac{1}{4}[u_1^{n+1} + u_5^n + 2000 + u_1^n]$$

$$u_5^{n+1} = \frac{1}{4}[u_4^{n+1} + u_2^{n+1} + u_4^n + u_2^n]$$

First iteration, put n = 0

$$u_1^1 = \frac{1}{4}(1000 + 1188 + 500 + 1438) \approx 1032$$

$$u_2^1 = \frac{1}{4}(1032 + 1125 + 1000 + 1500) = 1164$$

$$u_4^1 = \frac{1}{4}(2000 + 1500 + 1032 + 1125) = 1414$$

$$u_5^1 = \frac{1}{4}(1414 + 1438 + 1164 + 1188) = 1301$$

Scanned with CamScanner

Second iteration, put n = 1

$$u_1^2 = \frac{1}{4}(1000 + 1164 + 500 + 1414) = 1020$$

$$u_2^2 = \frac{1}{4}(1020 + 1032 + 1000 + 1301) = 1088$$

$$u_4^2 = \frac{1}{4}(2000 + 1301 + 1020 + 1032) = 1338$$

$$u_5^2 = \frac{1}{4}(1338 + 1414 + 1088 + 1164) = 1251$$

Third iteration, put n = 2

$$u_1^3 = \frac{1}{4}(1000 + 1088 + 500 + 1338) = 982$$

$$u_2^3 = \frac{1}{4}(982 + 1020 + 1000 + 1251) = 1063$$

$$u_4^3 = \frac{1}{4}(2000 + 1251 + 982 + 1020) = 1313$$

$$u_5^3 = \frac{1}{4}(1313 + 1338 + 1063 + 1088) = 1201$$

Fourth iteration, put n = 3

$$u_1^4 = \frac{1}{4}(1000 + 1063 + 500 + 1313) = 969$$

$$u_2^4 = \frac{1}{4}(969 + 982 + 1000 + 1201) = 1038$$

$$u_4^4 = \frac{1}{4}(2000 + 1201 + 969 + 982) = 1288$$

$$u_5^4 = \frac{1}{4}(1288 + 1313 + 1038 + 1063) = 1176$$

Similarly,

$u_1^5 = 957$, $u_2^5 = 1026$, $u_4^5 = 1276$, $u_5^5 = 1157$

$u_1^6 = 951$, $u_2^6 = 1016$, $u_4^6 = 1266$, $u_5^6 = 1146$

$u_1^7 = 946$, $u_2^7 = 1011$, $u_4^7 = 1260$, $u_5^7 = 1138$

$u_1^8 = 943$, $u_2^8 = 1007$, $u_4^8 = 1257$, $u_5^8 = 1134$

$u_1^9 = 941$, $u_2^9 = 1005$, $u_4^9 = 1255$, $u_5^9 = 1131$

$u_1^{10} = 940$, $u_2^{10} = 1003$, $u_4^{10} = 1253$, $u_5^{10} = 1129$

$u_1^{11} = 939$, $u_2^{11} = 1002$, $u_4^{11} = 1252$, $u_5^{11} = 1128$

There is a negligible difference between the values obtained in the tenth and eleventh iterations.

Hence,

$u_1 = 939$, $u_2 = 1002$,

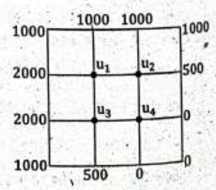
$u_4 = 1252$, $u_5 = 1128$

Example 6.2

Given the values of u(x, y) on the boundary of the square in the figure, Evaluate the function u(x, y) satisfying the Laplace equation $\nabla^2 u = 0$ at the pivotal points of this figure by

a) Jacobi's method

b) Gauss-Seidel method



Solution

To get the initial values of u_1, u_2, u_3, u_4, we assume that $u_4 = 0$. Then,

$u_1 = \frac{1}{4}(1000 + 0 + 1000 + 2000) = 1000$ [Diag. formula]

$u_2 = \frac{1}{4}(1000 + 500 + 1000 + 0) = 625$ [Standard formula]

$u_3 = \frac{1}{4}(2000 + 0 + 1000 + 500) = 875$ [Standard formula]

$u_4 = \frac{1}{4}(875 + 0 + 625 + 0) = 375$ [Standard formula]

a) We carry out the successive iterations, using Jacobi's formulae;

Itn.	$u_1 = \frac{1}{4}(3000 + u_2 + u_3)$	$u_2 = \frac{1}{4}(u_1 + 1500 + u_4)$	$u_3 = \frac{1}{4}(2500 + u_1 + u_4)$	$u_4 = \frac{1}{4}(u_2 + u_3)$
1	$\frac{1}{4}(3000+625+875) = 1125$	$\frac{1}{4}(1000+1500+375) = 719$	$\frac{1}{4}(2500+1000+375) = 969$	375
2	1172	750	1000	422
3	1188	774	1024	438
4	1200	782	1032	450
5	1204	788	1038	454
6	1206.5	790	1040	456.5
7	1208	791	1041	458
8	1208	791.5	1041.5	458

There is no significant difference between 7th and 8th iteration values.

Hence, $u_1 = 1208, u_2 = 792, u_3 = 1042, u_4 = 458$

b) We carry out the successive iterations, using Gauss-Seidel formulae

Itn.	$u_1 = \frac{1}{4}(3000 + u_2 + u_3)$	$u_2 = \frac{1}{4}(u_1 + 1500 + u_4)$	$u_3 = \frac{1}{4}(2500 + u_1 + u_4)$	$u_4 = \frac{1}{4}(u_2 + u_3)$
1	$\frac{1}{4}(3000 + 625 + 875) = 1125$	$\frac{1}{4}(1125 + 1500 + 375) = 750$	$\frac{1}{4}(2500 + 375 + 1125) = 1000$	$\frac{1}{4}(1000 + 750) = 438$
2	1188	782	1032	454
3	1204	789	1040	458
4	1207	791	1041	458
5	1208	791.5	1041.5	458.25

There is no significant difference between last two iterations

Hence, $u_1 = 1208, u_2 = 792, u_3 = 1042$ and $u_4 = 458$

Example 6.3

Solve the Poisson equation $u_{xx} + u_{yy} = -81\,xy$, $0 < x < 1$, $0 < y < 1$ given that $u(0, y) = 0$, $u(x, 0) = 0$, $u(1, y) = 100$, $u(x, 1) = 100$ and $h = \frac{1}{3}$.

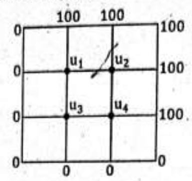
Scanned with CamScanner

Solution:

Given that;

$$u_{xx} + u_{yy} = -81\,xy$$

From the given boundary, the figure can be illustrated as,



Here $h = \frac{1}{3}$

The standard five point formula for the given equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh)$$

$$= h^2\,[-81\,(ih \cdot jh)]$$

$$= h^4\,(-81)ij$$

$$= -ij \qquad \text{..... (1)}$$

For u_1 ($i = 1, j = 2$)

$$0 + u_2 + u_3 + 100 - 4u_1 = -2$$

i.e., $-4u_1 + u_2 + u_3 = -102$ ..... (2)

For u_2 ($i = 2, j = 2$)

$$u_1 + 100 + u_4 + 100 - 4u_2 = -4$$

i.e., $u_1 - 4u_2 + u_4 = -204$ ..... (3)

For u_3 ($i = 1, j = 1$)

$$0 + u_4 + 0 + u_1 - 4u_3 = -1$$

i.e., $u_1 - 4u_3 + u_4 = -1$ ..... (4)

For u_4 ($i = 2, j = 1$)

$$u_3 + 100 + u_2 - 4u_4 = -2$$

i.e., $u_2 + u_3 - 4u_4 = -102$ ..... (5)

Subtracting (5) from (2),

$$-4u_1 + 4u_4 = 0$$

i.e., $u_1 = u_4$

Then (3) becomes

$$2u_1 - 4u_2 = -240 \qquad \text{..... (6)}$$

and, (4) becomes

$$2u_1 - 4u_3 = -1 \qquad \text{..... (7)}$$

Now, 4 × equation (2) + equation (6) gives,

$$-14u_1 + 4u_3 = -612 \qquad \text{..... (8)}$$

and, (7) + (8) gives

$-12\, u_1 = -613$

Thus,

$$u_1 = \frac{613}{12} = 51.0833 = u_4$$

From (6), $u_2 = \frac{1}{2}(u_1 + 102) = 76.5477$

From (7), $u_3 = \frac{1}{2}\left(u_1 + \frac{1}{2}\right) = 25.7916$

Example 6.4

Solve the boundary value problem $u_t = u_{xx}$ under the conditions $u(0, t) = u(1, t) = 0$ and $u(x, 0) = \sin px$, $0 \le x \le$ using the Schmidt method (take $h = 0.2$ and $\alpha = \frac{1}{2}$).

Solution:

Since $h = 0.2$ and $\alpha = \frac{1}{2}$

$\therefore \quad \alpha = \frac{k}{h^2}$ gives $k = 0.02$

Since $\alpha = \frac{1}{2}$, we use the Bendre-Schmidt relation

$$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j}) \quad \text{..... (1)}$$

we have, $u(0, 0) = 0$, $u(0.2, 0) \sin\frac{\pi}{5} = 0.5875$

$$u(0.4, 0) = \sin\frac{2\pi}{5} = 0.9511,\ u(0.6, 0) = \sin\frac{3\pi}{5} = 0.951$$

$$u(0.8, 0) = 0.5875,\ u(1, 0) = \sin\pi = 0$$

The values of u at the mesh points can be obtained by using the recurrence relation (1) as shown in the table below.

x →		0	0.2	0.4	0.6	0.8	1.0
t ↓ 0	j \ i	0	1	2	3	4	5
	0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	1	0	0.4756	0.7695	0.7695	0,4756	0
0.04	2	0	0.3848	0.6225	0.6225	0.3848	0
0.06	3	0	0.3113	0.5036	0,5036	0.3113	0
0.08	4	0	0.2518	0.4074	0,4074	0,2518	0
0.1	5	0	0.2037	0.3296	0.3296	0.2037	0

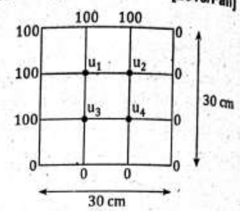
Scanned with CamScanner

BOARD EXAMINATION SOLVED QUESTIONS

1. The steady state two dimensional heat flow in a metal plate of size 30 × 30 cm is defined by $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$. Two adjacent sides are placed at 100° C and other side at 0° C. Find the temperature at inner points, assuming the grid size of 10 × 10 cm. **[2013/Fall]**

Solution:

The metal plate can be drawn as,



Given that;

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Let the inner points be defined as u_1, u_2, u_3 and u_4. Now using standard five point formula. We have,

$$u_1 = \frac{1}{4}(100 + 100 + u_2 + u_3)$$

$$= \frac{1}{4}(200 + u_2 + u_3)$$

$$u_2 = \frac{1}{4}(0 + 100 + u_1 + u_4) = \frac{1}{4}(100 + u_1 + u_4)$$

$$u_3 = \frac{1}{4}(0 + 100 + u_1 + u_4) = \frac{1}{4}(100 + u_1 + u_4)$$

$$u_4 = \frac{1}{4}(0 + 0 + u_2 + u_3) = \frac{1}{4}(u_2 + u_3)$$

To obtain the values let initial values of

$u_1 = 0, u_2 = 0, u_3 = 0$ and $u_4 = 0$ then

Using gauss Siedal method of iteration in tabular form

Itn.	$u_1 = \frac{1}{4}(200 + u_2 + u_3)$	$u_2 = \frac{1}{4}(100 + u_1 + u_4)$	$u_3 = \frac{1}{4}(100 + u_1 + u_4)$	$u_4 = \frac{1}{4}(u_2 + u_3)$
1	$\frac{1}{4}(200 + 0 + 0)$ = 50	$\frac{1}{4}(100 + 50 + 0)$ = 37.5	$\frac{1}{4}(100 + 50 + 0)$ = 37.5	$\frac{1}{4}(37.5 + 37.5)$ = 18.75
2	68.75	46.875	46.875	23.437
3	73.437	49.218	49.218	24.409
4	74.609	49.804	49.804	24.902
5	74.902	49.951	49.951	24.975
6	74.975	49.987	49.987	24.993
7	74.993	49.996	49.996	24.998
8	74.998	49.999	49.999	24.9995
9	74.9995	49.9997	49.9997	24.9998

Scanned with CamScanner

Here the values of u_1, u_2, u_3 and u_4 are correct up to 3 decimal places

Hence, the required temperature at inner points are;

$u_1 = 74.9995 \approx 75°C$

$u_2 = 49.9997 \approx 50°C$

$u_3 = 49.9997 \approx 50°C$

and, $u_4 = 24.9998 \approx 25°C$

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = u_1$, $B = u_2$, $C = u_3$, $D = u_4$

Step 1: Set the following in calculator;

$A = \frac{200 + B + C}{4} : B = \frac{100 + A + D}{4} : C = \frac{100 + A + D}{4} : D = \frac{B + C}{4}$

Step 2: Press CALC then,

enter the value of B? then press =

enter the value of C? then press =

enter the value of D? then press =

Step 3: Now press = only, again and again to get the values for the respective row for each column.

Step 4: The values are updated automatically so continue pressing = till the required number of iterations.

2. **Solve the Poisson equation $\nabla^2 f = 2x^2 y^2$ over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with f = 0 on the boundary and h = 1.**

[2013/Spring, 2014/Spring, 2018/Spring]

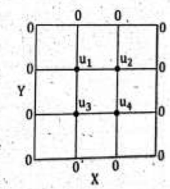
Solution:

Given that;

$\nabla^2 f = 2x^2 y^2$ ..... (1)

Also the square domain of $0 \le x \le 3$ and $0 \le y \le 3$ with f = 0 on the boundary.

It is illustrated in figure as,



Here, let u_1, u_2, u_3 and u_4 be the initial nodes of Poisson equation and replacing $\nabla^2 f$ by difference equation with x = ih, y = jk where, (h = k = 1)

Then, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = (2i^2 j^2)(1)^2$

For node u_1, put i = 1, j = 2

$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 2(1)^2(2)^2$

or, $0 + u_2 + u_3 + 0 - 4u_1 = 8$

or, $u_2 + u_3 - 4u_1 = 8$

or, $u_1 = \frac{1}{4}(u_2 + u_3 - 8)$

Likewise,

For node u_2, put i = 2, j = 2

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 2(2)^2(2)^2$$

or, $u_1 + 0 + u_4 + 0 - 4u_2 = 32$

$$u_2 = \frac{1}{4}(u_1 + u_4 - 32) \quad \text{.... (2)}$$

For node, u_4 put i = 2, j = 1

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 2(2)^2(1)^2$$

or, $u_3 + 0 + 0 + u_2 - 4u_4 = 8$

or, $u_3 + u_2 - 4u_4 = 8$

or, $u_4 = \frac{1}{4}(u_3 + u_2 - 8)$

or, $u_4 = u_1$

Equation (2) becomes

$$u_2 = \frac{1}{4}(2u_1 - 32) = \frac{1}{2}(u_1 - 16)$$

For node u_3, put i = 1, j = 1

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 2(1)^2(1)^2$$

or, $0 + u_4 + 0 + u_1 - 4u_3 = 2$

or, $u_3 = \frac{1}{4}(u_1 + u_4 - 2)$

or, $u_3 = \frac{1}{4}(2u_1 - 2)$

or, $u_3 = \frac{1}{2}(u_1 - 1)$

Now, let initial guess for u_1, u_2, u_3 and u_4 be 0. Then using Gauss Seidel method of iteration in tabular form.

Iteration	$u_1 = \frac{1}{4}(u_2 + u_3 - 8) = u_4$	$u_2 = \frac{1}{2}(u_1 - 16)$	$u_3 = \frac{1}{2}(u_1 - 1)$
1	$\frac{1}{4}(0 + 0 - 8) = -2$	$\frac{1}{2}(-2 - 16) = -9$	$\frac{1}{2}(-2 - 1) = -1.5$
2	−4.625	−10.312	−2.812
3	−5.281	−10.640	−3.140
4	−5.445	−10.722	−3.222
5	−5.486	−10.743	−3.243
6	−5.496	−10.748	−3.248
7	−5.499	−10.749	−3.249
8	−5.499	−10.749	−3.249

Hence the required values of nodes are

$$u_1 = u_4 = -5.499 \approx -5.5$$
$$u_2 = -10.749 \approx -10.75$$
$$u_3 = -3.249 \approx -3.25$$

Scanned with CamScanner

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = u_1 = u_4$, $B = u_2$, $C = u_3$
Set the following in calculator;

$$A = \frac{B + C - 8}{4} : B = \frac{A - 16}{2} : C = \frac{A - 1}{2}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

3. Torsion on a square bar of size 9 cm × 9 cm subject to twisting is governed by $\nabla^2 u = -4$ with Dirichlet boundary condition of $u(x, y) = 0$ and h = 1. Calculate the steady state temperatures at interior points. Assume a grid size of 3 cm × 3 cm. Iterate until the minimum difference at any point is correct to two decimal places by applying Gauss point is correct to two decimal places by applying gauss Seidel method. **[2014/Fall]**

Solution:

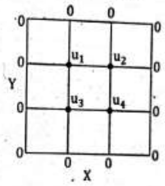
Given that;

$$\nabla^2 u = -4$$

with $u(x, y) = 0$ and $h = 1$ ..... (1)

Torsion on a square bar of size 9 cm × 9 cm with grid size of 3 cm × 3 cm

It is illustrated in figure as:



Let u_1, u_2, u_3 and u_4 be the internal points of Poisson equation and replacing $\nabla^2 u$ by difference equation with $x = ih$, $y = jk$ where $(h = k = 1)$

Then, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -4\,(1)^2$

For node u_1, put $i = 1$, $j = 2$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4\,u_{1,2} = -4$$

or, $0 + u_2 + u_3 + 0 - 4u_1 = -4$

or, $u_2 + u_3 - 4u_1 = -4$

or, $u_1 = \frac{1}{4}(u_2 + u_3 + 4)$

For node u_4, put $i = 2$, $j = 1$

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4\,u_{2,1} = -4$$

or, $u_3 + 0 + 0 + u_2 - 4\,u_4 = -4$

or, $u_3 + u_2 - 4u_4 = -4$

or, $u_4 = \frac{1}{4}(u_3 + u_2 + 4)$

or, $u_4 = u_1$

For node u_2, put $i = 2$, $j = 2$

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = -4$$

or, $u_1 + 0 + u_4 + 0 - 4u_2 = -4$

or, $u_2 = \frac{1}{4}(u_1 + u_4 + 4)$

or, $u_2 = \frac{1}{2}(u_1 + 2)$ [$\because u_1 = u_4$]

For node u_3, put $i = 1$ $j = 1$

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = -4$$

or, $0 + u_4 + 0 + u_1 - 4u_3 = -4$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 4)$

or, $u_3 = \frac{1}{2}(u_1 + 2)$

or, $u_3 = u_2$

Now, let the initial guess for u_1, u_2, u_3 and u_4 be 0.

Then using Gauss Seidel method of iteration in tabular form,

Iteration	$u_1 = u_4 = \frac{1}{2}(u_2 + 2)$	$u_2 = u_3 = \frac{1}{2}(u_1 + 2)$
1	$\frac{1}{2}(0 + 2) = 1$	1.5
2	1.75	1.875
3	1.9375	1.9688
4	1.9844	1.9922
5	1.9961	1.9980

Here, the obtained values are correct up to two decimal places.

Hence the required steady state temperatures at interior points are

$$u_1 = u_2 = u_3 = u_4 = 1.99 \approx 2.$$

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = u_1 = u_4$, $B = u_2 = u_3$

Set the following in calculator;

$$A = \frac{B+2}{2} : B = \frac{A+2}{2}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

trick

4. Solve the Poisson equation $\nabla^2 f = (2 + x^2 y)$, over the square domain of $0 \le x \le 3$ and $0 \le y \le 3$ with $f = 0$ on the boundary and $h = 1$.

[2015/Fall]

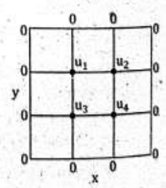
Solution:

Given that;

$$\nabla^2 f = 2 + x^2 y$$

Over the square domain of $0 \le x \le 3$ and $0 \le y \le 3$ with $f = 0$ on the boundary.

It is illustrated on the figure as;



Let u_1, u_2, u_3 and u_4 be the interior points and using Poisson formula with $x = ih$, $y = jk$ where ($h = k = 1$)

Scanned with CamScanner

$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} - 4u_{i,j} = (2 + i^2j)(1)^2$

Now, for interior point u_1, put $i = 1, j = 2$

$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 2 + (1)^2 2$

or, $0 + u_2 + u_3 + 0 - 4u_1 = 4$

or, $u_1 = \frac{1}{4}(u_2 + u_3 - 4)$

For interior point u_4, put $i = 2, j = 1$

$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 2 + (2)^2 \cdot 1$

or, $u_3 + 0 + 0 + u_2 - 4u_4 = 6$

or, $u_4 = \frac{1}{4}(u_2 + u_3 - 6)$

For interior point u_2, put $i = 2, j = 2$

$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 2 + (2)^2 \cdot 2$

or, $u_1 + 0 + u_4 + 0 - 4u_2 = 10$

or, $u_2 = \frac{1}{4}(u_1 + u_4 - 10)$

For interior point u_3, put $i = 1, j = 1$

$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 2 + (1)^2 \cdot 1$

or, $0 + u_4 + 0 + u_1 - 4u_3 = 3$

or, $u_3 = \frac{1}{4}(u_1 + u_4 - 3)$

Now, let the initial guess for u_1, u_2, u_3 and u_4 be 0.

Then using Gauss Seidel method of iteration in tabular form,

Iteration	$u_1 = \frac{1}{4}(u_2 + u_3 - 4)$	$u_2 = \frac{1}{4}(u_1 + u_4 - 10)$	$u_3 = \frac{1}{4}(u_1 + u_4 - 3)$	$u_4 = \frac{1}{4}(u_2 + u_3 - 6)$
1	$\frac{1}{4}(0 - 4) = -1$	$\frac{1}{4}(-1 + 0 - 10) = -2.75$	$\frac{1}{4}(-1 + 0 - 3) = -1$	$\frac{1}{4}(-2.75 - 1 - 6) = -2.437$
2	−1.9375	−3.5936	−1.8436	−2.8593
3	−2.3593	−3.8047	−2.0547	−2.9648
4	−2.4649	−3.8574	−2.1074	−2.9912
5	−2.4912	−3.8706	−2.1206	−2.9978
6	−2.4978	−3.8739	−2.1239	−2.9995
7	−2.4995	−3.8747	−2.1248	−2.9999
8	−2.4999	−3.8749	−2.1250	−3.0000
9	−2.5000	−3.8750	−2.1250	−3.0000
10	−2.5000	−3.8750	−2.1250	−3.0000

Here, the obtained values are values are correct up to 4 decimal places.

Hence the required interior points are,

$u_1 = -2.5$

$u_2 = -3.875$

Scanned with CamScanner

$u_3 = -2.125$

and, $u_4 = -3$

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = u_1$, $B = u_2$, $C = u_3$, $D = u_4$

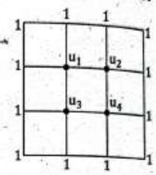
Set the following in calculator;

$$A = \frac{B + C - 4}{4} : B = \frac{A + D - 10}{4} : C = \frac{A + D - 3}{4} : D = \frac{B + C - 6}{4}$$

Now press CALC and enter the initial value of B, C and D and continue pressing = only for the required number of iterations.

5. Solve the Poisson equation $\nabla^2 f = 2x^2 + y$, over the square domain $1 \le x \le 3$, $1 \le, \le 3$ with $f = 1$ on the boundary. Take $h = k = 1$. [2015/Spring]

Solution:



Given that;

$$\nabla^2 f = 2x^2 + y$$

Over the square domain $1 \le x \le 3$, $1 \le y \le 3$

With $f = 1$ on the boundary.

It is illustrated in figure as:

Let u_1, u_2, u_3 and u_4 be the interior points and using Poisson formula with $x = ih$, $y = jk$ where, $(h = k = 1)$

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = (2i^2 + j)(1)^2$$

Now for interior point u_1, put $i = 1, j = 2$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 2(1)^2 + 2$$

or, $1 + u_2 + u_3 + 1 - 4u_1 = 4$

or, $u_2 + u_3 - 4u_1 = 2$

or, $u_1 = \frac{1}{2}(u_2 + u_3 - 2)$

For interior point u_2, put $i = 2, j = 2$

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 2(2)^2 + 2$$

or, $u_1 + 1 + u_4 + 1 - 4u_2 = 10$

or, $u_2 = \frac{1}{4}(u_1 + u_4 - 8)$

For interior point u_3, put $i = 1, j = 1$

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 2(1)^2 + 1$$

or, $1 + u_4 + 1 + u_1 - 4u_3 = 3$

or, $u_3 = \frac{1}{4}(u_1 + u_4 - 1)$

For interior point u_4, put $i = 2, j = 1$

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 2(2)^2 + 1$$

or, $u_3 + 1 + 1 + u_2 - 4u_4 = 9$

or, $u_4 = \frac{1}{4}(u_2 + u_3 - 7)$

Now let the initial guess for u_1, u_2, u_3 and u_4 be 0.

Scanned with CamScanner

Then using Gauss Seidel method of iteration in tabular form

Itn.	$u_1 = \frac{1}{4}(u_2+u_3-2)$	$u_2 = \frac{1}{4}(u_1+u_4-8)$	$u_3 = \frac{1}{4}(u_1+u_4-1)$	$u_4 = \frac{1}{4}(u_2+u_3-7)$
	-0.5	-2.1250	-0.3750	-2.3750
1	-1.1250	-2.8750	-1.1250	-2.7500
2	-1.5000	-3.0625	-1.3125	-2.8438
3	-1.5938	-3.1094	-1.3594	-2.8672
4	-1.6172	-3.1211	-1.3711	-2.8731
5	-1.6231	-3.1240	-1.3741	-2.8745
6	-1.6245	-3.1248	-1.3748	-2.849
7	-1.6249	-3.1250	-1.3750	-2.8750
8	-1.6250	-3.1250	-1.3750	-2.8750

Here, the obtained values are correct up to 4 decimal places

Hence the required interior points are

$u_1 = -1.6250$

$u_2 = -3.1250$

$u_3 = -1.3750$

and, $u_4 = -2.8750$

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = u_1$, $B = u_2$, $C = u_3$, $D = u_4$

Set the following in calculator;

$$A = \frac{B+C-2}{4} : B = \frac{A+D-8}{4} : C = \frac{A+D-1}{4} : D = \frac{B+C-7}{4}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

6. Given the Poisson's equation. $\Delta^2 f = -10(x^2 + y^2 + 10)$ over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with Dirichlet boundary condition of $f(x, y) = 0$ and $h = 1$. Calculate the steady state temperatures at the interior nodes by using Gauss Seidel method. [2016/Fall, 2018/Fall]

Solution:

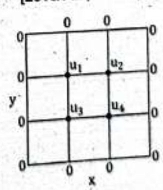
Given that;

$\Delta^2 f = -10(x^2 + y^2 + 10)$

Over the square domain $0 \le x \le 3$ and $0 \le y \le 3$

With Dirichlet boundary condition of $f(x, y) = 0$

It is illustrated in figure as:



Let, u_1, u_2, u_3, u_4 be the interior nodes and using Poisson formula with $x = ih$, $y = jk$ where $(h = k = 1)$

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10(i^2 + j^2 + 10)\cdot(1)^2$$

Scanned with CamScanner

square domain ??

Now, for interior node u_1, put i = 1, j = 2

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = -10((1)^2 + (2)^2 + 10)$$

or, $0 + u_2 + u_3 + 0 - 4u_1 = -150$

or, $u_1 = \frac{1}{4}(u_2 + u_3 + 150)$

For interior node u_2, put i = 2, j = 2

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = -10((2)^2 + (2)^2 + 10)$$

or, $u_1 + 0 + u_4 + 0 - 4u_2 = -180$

or, $u_1 = \frac{1}{4}(u_1 + u_4 + 180)$

For interior node u_3, put i = 1, j = 1

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = -10[(1)^2 + (1)^2 + 10]$$

or, $0 + u_4 + 0 + u_1 - 4u_3 = -120$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 120)$

For interior node u_4, put i = 2, j = 1

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = -10[(2)^2 + (1)^2 + 10]$$

or, $u_3 + 0 + 0 + u_2 - 4u_4 = -150$

or, $u_4 = \frac{1}{4}(u_2 + u_3 + 150)$

Here, $u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 150)$

so, $u_2 = \frac{1}{2}(u_1 + 90)$ and $u_3 = \frac{1}{2}(u_1 + 60)$

Now, let initial Guess for u_1, u_2, u_3 and u_4 be 0.

Now, solving the equations by the Gauss Seidel method,

Iteration	$u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 150)$	$u_2 = \frac{1}{2}(u_1 + 90)$	$u_3 = \frac{1}{2}(u_1 + 60)$
1	37.5	63.75	48.75
2	65.625	77.8125	62.8125
3	72.6563	81.3281	66.3282
4	74.4141	82.2070	67.2071
5	74.8535	82.4268	67.4268
6	74.9634	82.4817	67.4817
7	74.9909	82.4954	67.4955
8	74.9977	82.4989	67.4989
9	74.9994	82.4997	67.4997
10	74.9999	82.4999	67.4999

Hence the required steady state temperatures at the interior nodes are

$u_1 = u_4 = 75$

$u_2 = 82.5$

and, $u_3 = 67.5$

Scanned with CamScanner

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = u_1 = u_4$, $B = u_2$, $C = u_3$
Set the following in calculator;

$$A = \frac{B + C + 150}{4} : B = \frac{A + 90}{2} : C = \frac{A + 60}{2}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

7. **Solve the parabolic equation $2f_{xx}(x, t) = f_t(x, t)$, $0 \le t \le 1.5$ and given initial condition $f(x, 0) = 50(4 - x)$, $0 \le x \le 4$ with boundary condition $f(0, t) = 0 = f(4, t)$ $0 \le t \le 1.5$.** **[2017/Fall]**

imp

Solution:
Given that;

$$f_t(x, t) = 2f_{xx}(x, t)$$

We have the parabolic equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

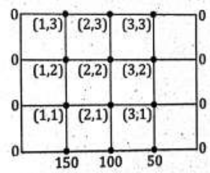
where, c^2 is the diffusivity of the substance

$$c^2 = 2$$

Let, $h = 1 \rightarrow$ Spacing along x-direction, $0 \le x \le 4$
Let, $k = 0.5 \rightarrow$ Spacing along time, t-direction, $0 \le t \le 1.5$
Now, solving the parabolic equation using Schmidt method.
We have,

$$\alpha = \frac{kc^2}{h^2} = \frac{0.5 \times 2}{1^2} = 1$$

Here, α lies between $0 < \alpha \le 12$ which satisfies the condition
The figure is illustrated as shown for $f(0, t) = 0 = f(4, t)$



Here, boundary values for

$$u_{1,0} = 50(4 - x) = 50(4 - 1) = 150$$
$$u_{2,0} = 50(4 - 2) = 100$$
$$u_{3,0} = 50(4 - 3) = 50$$

From Schmidt's formula, we have,

$$u_{i,j+1} = \alpha\, u_{i-1,j} + (1 - 2\alpha)\, u_{i,j} + \alpha\, u_{i+1,j}$$

Scanned with CamScanner

Substituting the value of $\alpha = 1$

$$u_{i,j+1} = u_{i-1,j} - u_{i,j} + u_{i+1,j}$$

Now, for i = 1, 2, 3 and j = 0

$$u_{1,1} = [u_{0,0} - u_{1,0} + u_{2,0}] = 0 - 150 + 100 = -50$$
$$u_{2,1} = [u_{1,0} - u_{2,0} + u_{3,0}] = 150 - 100 + 50 = 100$$
$$u_{3,1} = [u_{2,0} - u_{3,0} + u_{4,0}] = 100 - 50 + 0 = 50$$

For i = 1, 2, 3 and j = 1

$$u_{1,2} = [u_{0,1} - u_{1,1} + u_{2,1}] = 0 + 50 + 100 = 150$$
$$u_{2,2} = [u_{1,1} - u_{2,1} + u_{3,1}] = -50 - 100 + 50 = -100$$
$$u_{3,2} = [u_{2,1} - u_{3,1} + u_{4,1}] = 100 - 50 + 0 = 50$$

For i = 1, 2, 3 and j = 2

$$u_{1,3} = [u_{0,2} - u_{1,2} + u_{2,2}] = 0 - 150 + (-100) = -250$$
$$u_{2,3} = [u_{1,2} - u_{2,2} + u_{3,2}] = 150 + 100 + 50 = 300$$
$$u_{3,3} = [u_{2,2} - u_{3,2} + u_{4,2}] = -100 - 50 + 0 = -150$$

8. Given the Poisson's equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square domain such that $0 \le x \le 3$ and $0 \le y \le 3$ with Dirichlet boundary condition of u(x, y) = 0. Calculate the steady state temperature at interior points by suing successive over relaxation method up to 5th iteration. Assume h = k = 1. [2017/Spring]

Solution:

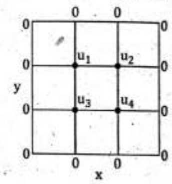
Given that;

$$\nabla^2 u = -10(x^2 + y^2 + 10)$$

Over the square domain; $0 \le x \le 3$ and $0 \le y \le 3$

With Dirichlet boundary condition of u(x, y) = 0

It is illustrated in figure as:



Let u_1, u_2, u_3 and u_4 be the interior points and using Poisson formula with x = ih, y = jk where (h = k = 1)

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10(i^2 + j^2 + 10) \cdot (1)^2$$

Now, for interior point u_1, put i = 1, j = 2

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = -10[(1)^2 + (2)^2 + 10]$$

or, $0 + u_2 + u_3 + 0 - 4u_1 = -150$

or, $u_1 = \frac{1}{4}(u_2 + u_3 + 150)$

For interior node u_2, put i = 2, j = 2

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = -10[(2)^2 + (2)^2 + 10]$$

or, $u_1 + 0 + u_4 + 0 - 4u_2 = -180$

or, $u_2 = \frac{1}{4}(u_1 + u_4 + 180)$

For interior node u_3, put i = 1, j = 1

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = -10[(1)^2 + (1)^2 + 10]$$

$0 + u_4 + 0 + u_1 - 4u_3 = -120$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 120)$

or, For interior node u_4, put $i = 2, j = 1$

$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = -10\,[(2)^2 + (1)^2 + 10]$

or, $u_3 + 0 + 0 + u_2 - 4u_4 = -150$

or, $u_4 = \frac{1}{4}(u_2 + u_3 + 150)$

Here, $u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 150)$

so, $u_2 = \frac{1}{2}(u_1 + 90)$ and $u_3 = \frac{1}{2}(u_1 + 60)$

Now, using successive over relaxation method

We have,

$$x_i^{n+1} = (1 - w)\,x_i^n + w\,[\text{Gauss Seidel iteration}]$$

Here, w is relaxation parameter which value lies from $0 < w < 2$ for convergence reason.

Lets choose $w = 1.25$

$$x_i^{n+1} = -0.25x_i^n + 1.25\,[\text{Gauss Seidel iteration}]$$

Now, the equations are formed as

$$u_1^{n+1} = -0.25\,u_1^n + \frac{1.25}{4}(u_2^n + u_3^n + 150)$$

$$u_2^{n+1} = -0.25\,u_2^n + \frac{1.25}{4}(u_1^{n+1} + u_4^n + 180)$$

$$u_3^{n+1} = -0.25\,u_3^n + \frac{1.25}{4}(u_1^{n+1} + u_4^n + 120)$$

$$u_4^{n+1} = -0.25\,u_4^n + \frac{1.25}{4}(u_2^{n+1} + u_3^{n+1} + 150)$$

Here, $u_1^{n+1} = u_4^{n+1}$

Then, $u_1^{n+1} = -0.25\,u_1^n + 0.3125\,(u_2^n + u_3^n + 150) = u_4^{n+1}$

$u_2^{n+1} = -0.25\,u_2^n + 0.3125\,(u_1^{n+1} + u_4^n + 180)$

$u_3^{n+1} = -0.25\,u_3^n + 0.3125\,(u_1^{n+1} + u_4^n + 120)$

Let the initial guess for u_1, u_2, u_3 and u_4 be 0.

Now, 1st iteration,

For $n = 0$

$u_4^1 = u_1^{(1)} = -0.25\,u_1^0 + 0.3125\,(u_2^0 + u_3^0 + 150)$

$= -0.25 \times 0 + 0.3125\,(0 + 0 + 150)$

$= 46.875$

$u_2^1 = -0.25\,u_2^0 + 0.3125\,(u_1^1 + u_4^0 + 180)$

$= 0 + 0.3125\,(46.875 + 0 + 180)$

$= 70.8984$

$u_3^1 = -0.25\, u_3^0 + 0.3125\, (u_1^1 + u_4^0 + 120)$
$= 0 + 0.3125\, (46.875 + 0 + 120)$
$= 52.1484$

Likewise,

2nd iteration, $n = 1$

$u_1^2 = -0.25\, u_1^1 + 0.3125\, (u_2^1 + u_3^1 + 150)$
$= 73.6084$

$u_2^2 = -0.25\, u_2^1 + 0.3125\, (u_1^2 + u_4^1 + 180)$
$= 76.1765$

$u_3^2 = -0.25\, u_3^1 + 0.3125\, (u_1^2 + u_4^1 + 120)$
$= 62.1140$

$u_4^2 = u_1^2 = 73.6084$

3rd iteration, $n = 2$

$u_1^3 = -0.25\, u_1^2 + 0.3125\, (u_2^2 + u_3^2 + 150)$
$= 71.6887$

$u_2^3 = -0.25\, u_2^2 + 0.3125\, (u_1^3 + u_4^2 ++ 180)$
$= 82.6112$

$u_3^3 = -0.25\, u_3^2 + 0.3125\, (u_1^3 + u_4^2 + 120)$
$= 67.3768$

$u_4^3 = u_1^3 = 71.6887$

4th iteration

$n = 3$ then

$u_1^4 = -0.25\, u_1^3 + 0.3125\, (u_2^3 + u_3^3 + 150)$
$= 75.8241$

$u_2^4 = -0.25\, u_2^3 + 0.3125\, (u_1^4 + u_4^3 + 180)$
$= 81.6950$

$u_3^4 = -0.25\, u_3^3 + 0.3125\, (u_1^4 + u_4^3 + 120)$
$= 66.7536$

$u_4^4 = u_1^4 = 75.8241$

5th iteration, $n = 4$

$u_1^5 = -0.25\, u_1^4 + 0.3125\, (u_2^4 + u_3^4 + 150)$
$= 74.3092$

$u_2^5 = -0.25\, u_2^4 + 0.3125\, (u_1^5 + u_4^4 + 180)$
$= 82.7429$

$u_3^5 = -0.25\, u_3^4 + 0.3125\, (u_1^5 + u_4^4 + 120)$
$= 67.7283$

$u_4^5 = u_1^5 = 74.3092$

Scanned with CamScanner

Hence the required steady state temperature at interior points are

$u_1 = u_4 = 74.3092$

$u_2 = 82.7429$

$u_3 = 67.7283$

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = u_1 = u_4$, $B = u_2$, $C = u_3$

Set the following in calculator;

$X = -0.25A + 0.3125(150 + B + C) : Y = -0.25B + 0.3125(180 + X + A) :$

$M = -0.25C + 0.3125(120 + X + A)$

Press CALC and enter the initial value of A, B and C and continue pressing = only for the required row for each column.

Update the values of A?, B? and C? when asked again.

9. **Given the Poisson's equation $\Delta^2 f = 4x^2 y^2$ over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with Dirichlet boundary condition of $f(x, y) = 100$ and $h = k = 1$. Calculate the steady state temperatures at the interior nodes by using Gauss Seidel method. Iterate until the successive values at any point is correct to two decimal places. [2019/Fall]**

Solution:

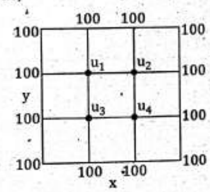
Given that;

$\Delta^2 f = 4x^2 y^2$

Over the square domain $0 \le x \le 3$ and $0 \le y \le 3$

With Dirichlet boundary condition of $f(x, y) = 100$

It is illustrated in figure as,



Let u_1, u_2, u_3 and u_4 be the interior nodes of Poisson's equation with $x = ih$, $y = jk$ where $h = k = 1$

Then, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = (4\, i^2 j^2)(1)^2$

For node u_1, put $i = 1, j = 2$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 4(1)^2 (2)^2$$

or, $$100 + u_2 + u_3 + 100 - 4u_1 = 16$$

or, $$u_1 = \frac{1}{4}(u_2 + u_3 + 184)$$

For node u_2, put $i = 2, j = 2$

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 4(2)^2 (2)^2$$

Scanned with CamScanner

or, $u_1 + 100 + u_4 + 100 - 4u_2 = 64$

or, $u_2 = \frac{1}{4}(u_1 + u_4 + 136)$

For node u_3; put $i = 1, j = 1$

$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 4(1)^2(1)^2$

or, $100 + u_4 + 100 + u_1 - 4u_3 = 4$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 196)$

For node u_4, put $i = 2, j = 1$

$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 4(2)^2(1)^2$

or, $u_3 + 100 + 100 + u_2 - 4u_4 = 16$

or, $u_4 = \frac{1}{4}(u_2 + u_3 + 184)$

Here, $u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 184)$ then,

$u_2 = \frac{1}{2}(u_1 + 68)$

$u_3 = \frac{1}{2}(u_1 + 98)$

Let the initial guess for u_1, u_2, u_3, u_4 be zero.

Now, using Gauss Seidel method in tabular form,

Iteration	$u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 184)$	$u_2 = \frac{1}{2}(u_1 + 68)$	$u_3 = \frac{1}{2}(u_1 + 98)$
1	46	57	72
2	78.25	73.125	88.125
3	86.3125	77.1563	92.1563
4	88.3281	78.1641	93.1641
5	88.8320	78.4160	934160
6	88.9580	78.4790	93.4790
7	88.9895	78.4948	93.4948
8	88.9974	78.4987	93.4987

Hence the required values of temperatures at interior nodes are,

$u_1 = u_4 = 88.9974$

$u_2 = 78.4987$

and, $u_3 = 93.4987$

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = u_1 = u_4$, $B = u_2$, $C = u_3$

Set the following in calculator;

$A = \frac{B + C + 184}{4}$: $B = \frac{A + 68}{2}$: $C = \frac{A + 98}{2}$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

Scanned with CamScanner

10. **Solve the Poisson's equation $u_{xx} + u_{yy} = 243(x^2 + y^2)$ over a square domain $0 \le x \le 1, 0 \le y \le 1$ with step size $h = \frac{1}{3}$ with $u = 100$ on the boundary.** **[2019/Spring]**

Solution:

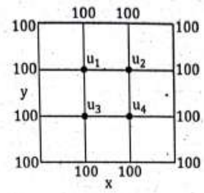
Given that;

$$u_{xx} + u_{yy} = 243(x^2 + y^2)$$

Over a square domain $0 \le x \le 1, 0 \le y \le 1$

With $u = 100$ on the boundary.

It is illustrated in the figure as,



Let u_1, u_2, u_3 and u_4 be the interior nodes of Poisson's equation and replacing $u_{xx} + u_{yy}$ by difference equation with $x = ih$, $y = jk$ where $h = k = \frac{1}{3}$.

Then, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jk)$

or, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = \frac{1}{9} \times 243\left(\frac{i^2}{9} + \frac{j^2}{9}\right)$

or, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 3(i^2 + j^2)$

Now, for node u_1, put $i = 1, j = 2$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 3(1^2 + 2^2)$$

or, $100 + u_2 + u_3 + 100 - 4u_1 = 15$

or, $u_1 = \frac{1}{4}(u_2 + u_3 + 185)$

For node u_2, put $i = 2, j = 2$

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 3[(2)^2 + (2)^2]$$

or, $u_1 + 100 + u_4 + 100 - 4u_2 = 24$

or, $u_2 = \frac{1}{4}(u_1 + u_4 + 176)$

For node u_3, put $i = 1, j = 1$

$$u_{0,1}\ u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 3[(1)^2 + (1)^2]$$

or, $100 + u_4 + 100 + u_1 - 4u_3 = 6$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 194)$

For node u_4, put $i = 2, j = 1$

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 3[(2)^2 + (1)^2]$$

or, $u_3 + 100 + 100 + u_2 - 4u_4 = 15$

or, $u_4 = \frac{1}{4}(u_2 + u_3 + 185)$

Here, $u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 185)$ and

$u_2 = \frac{1}{2}(u_1 + 88)$, $u_3 = \frac{1}{2}(u_1 + 97)$

Let the initial guess for u_1, u_2, u_3 and u_4 be zero.

Now, using Gauss Seidel method in tabular form,

Iteration	$u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 185)$	$u_2 = \frac{1}{2}(u_1 + 88)$	$u_3 = \frac{1}{2}(u_1 + 97)$
1	46.250	67.125	71.625
2	80.938	84.469	88.969
3	89.610	88.805	93.305
4	91.778	89.889	94.389
5	92.320	90.160	94.660
6	92.455	90.228	94.728
7	92.489	90.244	94.745
8	92.497	90.249	94.749
9	92.499	90.250	94.750
10	92.500	90.250	94.750

Hence the required values of interior points are,

$u_1 = u_4 = 92.5$, $u_2 = 90.25$ and $u_3 = 94.75$

NOTE:

Procedure to iterate in programmable calculator:

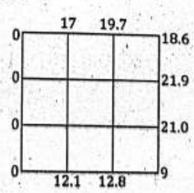
Let, $A = u_1 = u_4$, $B = u_2$, $C = u_3$

Set the following in calculator;

$$A = \frac{B + C + 185}{4} : B = \frac{A + 88}{2} : C = \frac{A + 97}{2}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

11. Solve the Poisson equation $\nabla^2 f = 4x^2 y + 3xy^2$, over the square domain $x \le 3$, $1 \le y \le 3$, with f on the boundary is given in figure below. Take $h = k = 1$. **[2020/Fall]**

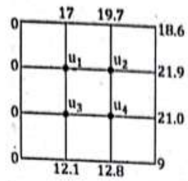


Solution:

Given that;

$$\nabla^2 f = 4x^2 y + 3x y^2$$

Over the square domain $x \leq 3$, $1 \leq y \leq 3$ with f on the boundary



Let u_1, u_2, u_3 and u_4 be the interior nodes of Poisson's equation and replacing $\nabla^2 f$ by difference equation with $x = ih$, $y = jk$ where, $(h = k = 1)$

Then,

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = (1)^2 (4i^2 j + 3ij^2)$$

or, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 4i^2 j + 3ij^2$

Now, for node, u_1, put $i = 1$, $j = 2$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 4(1)^2 (2) + 3(1)(2)^2$$

or, $0 + u_2 + u_3 + 17 - 4u_1 = 20$

or, $u_1 = \frac{1}{4}(u_2 + u_3 - 3)$

Now for node u_2, put $i = 2$, $j = 2$

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 4(2)^2 (2) + 3(2)(2)^2$$

or, $u_1 + 21.9 + u_4 + 19.7 - 4u_2 = 56$

or, $u_2 = \frac{1}{4}(u_1 + u_4 - 14.4)$

For node u_3, put $i = 1$, $j = 1$

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 4(1)^2 (1) + 3(1)(1)^2$$

or, $0 + u_4 + 12.1 + u_1 - 4u_3 = 7$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 5.7)$

For node u_4, put $i = 2$, $j = 1$

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 4(2)^2 (1) + 3(2)(1)^2$$

or, $u_3 + 21 + 12.8 + u_2 - 4u_4 = 22$

or, $u_4 = \frac{1}{4}(u_2 + u_3 + 11.8)$

Let the initial guess for u_1, u_2, u_3 and u_4 be zero.

Now, using Gauss Seidel method in tabular form,

Scanned with CamScanner

Itn.	$u_1 = \frac{1}{4}(u_2 + u_3 - 3)$	$u_2 = \frac{1}{4}(u_1 + u_4 - 14.4)$	$u_3 = \frac{1}{4}(u_1 + u_4 + 5.7)$	$u_4 = \frac{1}{4}(u_2 + u_3 + 11.8)$
1	−0.75	−3.788	1.238	2.312
2	−1.388	−3.369	1.656	2.522
3	−1.178	−3.264	1.761	2.574
4	−1.126	−3.238	1.787	2.587
5	−1.113	−3.231	1.794	2.591
6	−1.109	−3.230	1.796	2.592
7	−1.109	−3.229	1.796	2.592
8	−1.108	−3.229	1.796	2.592

Hence the required values of interior points are

$u_1 = -1.108$

$u_2 = -3.229$

$u_3 = 1.796$

and, $u_4 = 2.592$

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = u_1$, $B = u_2$, $C = u_3$, $D = u_4$

Set the following in calculator:

$$A = \frac{B + C - 3}{4} : B = \frac{A + D - 14.4}{4} : C = \frac{A + D + 5.7}{4} : D = \frac{B + C + 11.8}{4}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

12. Write short notes on: Laplacian equation. **[2013/Fall, 2013/Spring, 2016/Fall, 2016/Spring]**

Solution: See the topic 6.2 'C'.

13. Write short notes on; Hyperbolic equations. **[2015/Spring]**

Solution: See the topic 6.2. 'F'.

14. Write short notes on: Laplace method for partial differential. **[2017/Fall, 2018/Fall]**

Solution: See the topic 6.2 'C'.

15. Write short notes on: Parabolic equation. **[2017/Spring]**

Solution: See the topic 6.2.'E'.

16. Write short notes on Elliptical equations. **[2017/Spring]**

Solution: See the topic 6.2. 'B'.

Scanned with CamScanner

ADDITIONAL QUESTION SOLUTION

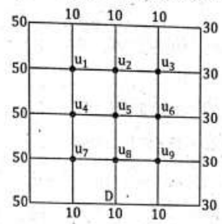
1. **Solve the elliptic equation $\nabla^2 u = 0$ in the square plate of size 8 cm × 8 cm if the boundary values are given 50 on one side of the plate and 30 on its opposite side. On the other sides the values are given 10. Assume the square grids of size 2 cm × 2 cm.**

Solution:

Given that;

Elliptic equation $\nabla^2 u = 0$.

From the given boundary values, the figure can be illustrated as,



Let the inner points be defined as u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8 and u_9 as shown

Then we find the first initial values as

$u_5 = \frac{1}{4}[50 + 10 + 30 + 10]$ (Using standard 5-point formula)

$= 25$

$u_1 = \frac{1}{4}[10 + 50 + 50 + 25]$ (Using diagonal 5-point formula)

$= 33.75$

Likewise,

$u_3 = \frac{1}{4}[10 + 30 + 30 + 25]$ (Using diagonal 5-point formula)

$= 23.75$

$u_2 = \frac{1}{4}[10 + u_1 + u_3 + u_5]$ (Using standard 5-point formula)

$= \frac{1}{4}[10 + 33.75 + 23.75 + 25]$

$= 23.125$

$u_7 = \frac{1}{4}[10 + 50 + 50 + 25]$ (Using diagonal 5-point formula)

$= 33.75$

$u_9 = \frac{1}{4}[10 + 30 + 30 + 25]$ (Using diagonal 5-point formula)

$= 23.75$

Scanned with CamScanner

$u_4 = \frac{1}{4}[50 + u_1 + u_5 + u_7]$ (Using standard 5-point formula)

$= \frac{1}{4}[50 + 33.75 + 25 + 33.75]$

$= 35.625$

$u_6 = \frac{1}{4}[30 + u_3 + u_5 + u_9]$ (Using standard 5-point formula)

$= \frac{1}{4}[30 + 23.75 + 25 + 23.75] = 25.625$

$u_8 = \frac{1}{4}[u_5 + u_7 + u_9 + 10]$ (Using standard 5-point formula)

$= \frac{1}{4}[25 + 33.75 + 23.75 + 10]$

$= 23.125$

Now, we can carry out Gauss Seidel iteration using standard 5-point formula.

Iteration 1, put n = 0 at

$u_1^{n+1} = \frac{1}{4}[10 + 50 + u_2^n + u_4^n] = \frac{1}{4}[60 + u_2^0 + u_4^0]$

$\therefore \quad u_1^1 = \frac{1}{4}[60 + 23.125 + 35.625] = 29.6875$

$u_2^{n+1} = \frac{1}{4}[10 + u_1^{n+1} + u_3^n + u_5^n] = \frac{1}{4}[10 + u_1^1 + u_3^0 + u_5^0]$

$\therefore \quad u_2^1 = \frac{1}{4}[10 + 29.6875 + 23.75 + 25] = 22.1094$

$u_3^{n+1} = \frac{1}{4}[10 + 30 + u_2^{n+1} + u_6^n] = \frac{1}{4}[40 + u_2^1 + u_6^0]$

$\therefore \quad u_3^1 = \frac{1}{4}[40 + 22.1094 + 25.625] = 21.9336$

$u_4^{n+1} = \frac{1}{4}[u_1^{n+1} + u_5^n + u_7^n + 50] = \frac{1}{4}[u_1^1 + u_5^0 + u_7^0 + 50]$

$\therefore \quad u_4^1 = \frac{1}{4}[29.6875 + 25 + 33.75 + 50] = 34.6094$

$u_5^{n+1} = \frac{1}{4}[u_2^{n+1} + u_4^{n+1} + u_6^n + u_8^n] = \frac{1}{4}[u_2^1 + u_4^1 + u_6^0 + u_8^0]$

$\therefore \quad u_5^1 = \frac{1}{4}[29.6875 + 34.6094 + 25.625 + 23.125] = 28.2617$

$u_6^{n+1} = \frac{1}{4}[u_3^{n+1} + u_5^{n+1} + u_9^n + 30] = \frac{1}{4}[u_3^1 + u_5^1 + u_9^0 + 30]$

$\therefore \quad u_6^1 = \frac{1}{4}[21.9336 + 28.2617 + 23.75 + 30] = 25.9863$

$u_7^{n+1} = \frac{1}{4}[u_4^{n+1} + 50 + 10 + u_8^n] = \frac{1}{4}[u_4^1 + 60 + u_8^0]$

$\therefore \quad u_7^1 = \frac{1}{4}[34.6094 + 60 + 23.125] = 29.4336$

$u_8^{n+1} = \frac{1}{4}[u_5^{n+1} + u_7^{n+1} + 10 + u_9^n] = \frac{1}{4}[u_5^1 + u_7^1 + 10 + u_9^0]$

$\therefore \quad u_8^1 = \frac{1}{4}[28.2617 + 29.4336 + 10 + 23.75] = 22.8613$

Scanned with CamScanner

$$u_9^{n+1} = \frac{1}{4}[u_6^{n+1} + u_8^{n+1} + 10 + 30] = \frac{1}{4}[u_6^1 + u_8^1 + 40]$$

$$\therefore \quad u_9^1 = \frac{1}{4}[25.9863 + 22.8613 + 40] = 22.2119$$

Likewise, put n = 1 and carry out the iterations

$u_1^2 = 29.1797$ $\qquad$ $u_2^2 = 22.3438$

$u_3^2 = 22.0825$ $\qquad$ $u_4^2 = 34.2188$

$u_5^2 = 26.3526$ $\qquad$ $u_6^2 = 25.1618$

$u_7^2 = 29.2700$ $\qquad$ $u_8^2 = 21.9586$

$u_9^2 = 21.7801$

Put n = 2 and carry out the iterations

$u_1^3 = 29.1407$ $\qquad$ $u_2^3 = 21.8940$

$u_3^3 = 21.7640$ $\qquad$ $u_4^3 = 33.6908$

$u_5^3 = 25.6763$ $\qquad$ $u_6^3 = 24.8051$

$u_7^3 = 28.9124$ $\qquad$ $u_8^3 = 21.5922$

$u_9^3 = 21.5993$

Similarly, the iterations are carried out upto required significant differences for the inner points.

2. Solve the elliptic equation $u_{xx} + u_{yy} = 0$ on the square mesh bounded by $0 \le x \le 3$, $0 \le y \le 3$. The boundary values are $u(x, 0) = 10$, $u(x, 3) = 90$, $0 \le x \le 3$ and $u(0, y) = 70$, $u(3, y) = 0$, $0 < y < 3$.

Solution:

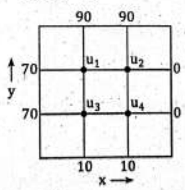
Given the elliptic equation

$$u_{xx} + u_{yy} = 0$$

Now, using the boundary values provided, the figure can be illustrated as

$u(x, 0) = 10$, $\qquad$ $u(0, 4) = 70$

$u(x, 3) = 90$, $\qquad$ $u(3, y) = 0$



Let the inner points be defined as u_1, u_2, u_3 and u_4.

Now, using standard five point formula

We have,

$$u_1 = \frac{1}{4}(70 + 90 + u_2 + u_3) = \frac{1}{4}(160 + u_2 + u_3)$$

Scanned with CamScanner

$u_2 = \frac{1}{4}(0 + 90 + u_1 + u_4) = \frac{1}{4}(90 + u_1 + u_4)$

$u_3 = \frac{1}{4}(u_1 + 70 + 10 + u_4) = \frac{1}{4}(80 + u_1 + u_4)$

$u_4 = \frac{1}{4}(0 + 10 + u_2 + u_3) = \frac{1}{4}(10 + u_2 + u_3)$

To obtain the values, let initial guess be,

$u_1 = 0, u_2 = 0, u_3 = 0$ **and** $u_4 = 0$ **then,**

Using Gauss Seidel method of iteration in tabular form,

Itn.	$u_1 = \frac{1}{4}(160 + u_2 + u_3)$	$u_2 = \frac{1}{4}(90 + u_1 + u_4)$	$u_3 = \frac{1}{4}(80 + u_1 + u_4)$	$u_4 = \frac{1}{4}(10 + u_2 + u_3)$
1	$\frac{1}{4}(160 + 0 + 0) = 40$	$\frac{1}{4}(90 + 40 + 0) = 32.5$	$\frac{1}{4}(80 + 40 + 0) = 30$	$\frac{1}{4}(10 + 32.5 + 30) = 18.125$
2	55.6250	40.9375	38.4375	22.3438
3	59.8438	43.0469	40.5469	23.3984
4	60.8984	43.5742	41.0742	23.6621
5	61.1621	43.7061	41.2061	23.7280
6	61.2280	43.7390	41.2390	23.7445
7	61.2445	43.7473	41.2473	23.7486
8	61.2486	43.7493	41.2493	23.7497
9	61.2497	43.7498	41.2498	23.7499
10	61.2499	43.7500	41.2500	23.7500
11	61.2500	43.7500	41.2500	23.7500

Hence the required values of interior points are

$u_1 = 61.25$

$u_2 = 43.75$

$u_3 = 41.25$

and, $u_4 = 23.75$

Scanned with CamScanner