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# SYLLABUS

**MTH 211.3 Engineering Mathematics IV (3 – 2 – 0)**

**Evaluation:**

| | Theory | Practical | Total |
|---|---|---|---|
| Sessional | 50 | - | 50 |
| Final | 50 | - | 50 |
| Total | 100 | | 100 |

**Course Objective:**

1. To understand complex variable.
2. To apply concepts of Fourier and Z-transform in the signal processing.
3. To study wave and diffusion equations in Cartesian, cylindrical, and polar coordinates.

| Chapter | Content | Hours |
|---|---|---|
| 1 | Complex variable:<br>i) Review of complex numbers with their properties<br>ii) De Moirves Theorem<br>iii) Function of complex variables,<br>iv) Conformal mappings<br>v) Analyticity , necessary condition of analyticity<br>vi) Cauchy integral theorem, Cauchy integral formula, Extension form of Cauchy integral formula,<br>vii) Taylor and Laurent series<br>viii) Singularities, zeros, poles, complex integration, residue theorem. | (12 Hours) |
| 2 | Z-transform:<br>i) Definition, one sided and two sided z transform<br>ii) Linear Time invariant system, Unit impulse function<br>iii) Properties of z transform, region of convergence<br>iv) Inverse Z transform by residue and partial fraction<br>v) Parseval theorem, convolution<br>vi) Application (Solution of difference equation). | (9 Hours) |
| 3 | Fourier Integral and Fourier Transform:<br>i) Fourier series in complex form<br>ii) Fourier integral, Sine integral and cosine integral<br>iii) Fourier transform, cosine transform, sine transform<br>iv) Inverse Fourier transform, Parseval identity<br>v) Convolution theorem and its applications | (7 Hours) |
| 4 | Partial Differential Equation:<br>i) Definition with examples<br>ii) Method of separation of variables<br>iii) Derivation and solutions of Wave equations (one | (14 Hours) |

| | | |
|---|---|---|
| | and two dimensional) and their applications.<br>iv) Wave equation by D Alembert's method<br>v) Derivation and solution of heat equation (one and two dimensional)and their application<br>vi) Laplacian equation [Cartesian, polar, cylindrical, spherical form(statement only)], their solutions.<br>vii) Engineering applications. | |
| 5 | Curve in space:<br>(i) Ellipsoid, hyperboloid, Paraboloid, cylinder, cone (Standard equations, their sketch)<br>(ii) Tangent line and tangent plane on the space curve | (3 Hours) |

**Text Books:**


1. E. Kreyszig, Advanced Engineering Mathematics,8th edition Wiley-Easter Publication, New Delhi
2. H. K. Dass & R. Verma, Higher Engineering Mathematics, First edition, S. Chand & Company Limited, New Delhi


**Reference Books:**


1. Digital Signal Processing: J.G. Proakis, Prentice Hall of India.
2. V Sundaran, R Bala Subramanayam, K.L. Laxminarayanam, Engineering Mathematics , Volume II
3. A.V. Oppenheim, Discrete-Time Signal Processing, Prentice Hall, India Limited, 1990.
4. K. Ogata, Discrete-Time Control System, Prentice Hall, India Limited, 1993.


*Unit 1*

# REVIEW OF COMPLEX NUMBER

**Complex Number:**

A **complex number** z is a ordered pair (x, y) where x along real axis and y along imaginary axis. Thus, $z = x + iy$.

**Polar form of Complex Number:**

Let $z = x + iy$ be a complex number. Set $x = r\cos\theta$, $y = r\sin\theta$ then

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

**Note:** $\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$.

**Statement of DeMoiver's Theorem:**

If n is any real value (may positive or negative) then

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

## EXERCISE 1.1

1. **If $z_1 = 4 + 3i$ and $z_2 = 2 - 5i$. Find each of the following in the form x + iy.**

(a) $z_1z_2$

**Solution:** Here, $z_1z_2 = (4 + 3i)(2 - 5i) = 8 - 20i + 6i + 15 = 23 - 14i$.

(b) $\frac{1}{z_1}$

**Solution:** Here, $\frac{1}{z_1} = \frac{1}{4+3i} = \frac{4-3i}{4^2-(3i)^2} = \frac{4-3i}{16+9} = \frac{4}{25} - \frac{3}{25}i = 0.16 - 0.12i.$

(c) $Re(z_1)^3$ and $(Re\ z_2)^3$

**Solution:** Since, $z_1 = 4 + 3i$.

So, $(z_1)^3 = (4 + 3i)^3 = 4^3 + 3 . 4^2 . (3i) + 3 . 4 (3i)^2 + (3i)^3$
$= 64 + 144i - 108 - 27i$
$= -44 + 117i$

Also, $z_2 = 2 - 5i$. So, $Re(z_2) = 2$. So, $(Re\ z_2)^3 = 2^3 = 8$

Thus, $Re\ (z_1)^3 = -44$ and $(Re\ z_2)^3 = 8$.

(d) $z_1\bar{z}_2$ and $\bar{z}_1z_2$

**Solution:** Here, $z_1 = 4 + 3i$ and $z_2 = 2 - 5i$. So, $\bar{z}_1 = 4 - 3i$ and $\bar{z}_2 = 2 + 5i$.

Then, $z_1\bar{z}_2 = (4 + 3i)(2 + 5i) = -7 + 26i$.

and, $\bar{z}_1z_2 = (4 - 3i)(2 - 5i) = -7 - 26i$.

Therefore, $z_1\bar{z}_2 = -7 + 26i$ and $\bar{z}_1z_2 = -7 - 26i$.

(e) $\dfrac{\bar{z}_1}{\bar{z}_2}$ and $\overline{\left(\dfrac{z_1}{z_2}\right)}$

**Solution:** Here, $z_2 = 2 - 5i \neq 0$ and $\bar{z}_2 = 2 + 5i \neq 0$. So, $\dfrac{\bar{z}_1}{\bar{z}_2}$ and $\overline{\left(\dfrac{z_1}{z_2}\right)}$ exist.

Now,

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{\overline{4+3i}}{\overline{2-5i}} = \frac{4-3i}{2+5i} = \frac{(4-3i)(2-5i)}{2^2-(5i)^2} = \frac{8-20i-6i+15i^2}{4+25} = \frac{-7-26i}{29}$$

And,

$$\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{4+3i}{2-5i}\right)} = \overline{\left(\frac{4+3i}{2-5i}\times\frac{2+5i}{2+5i}\right)} = \overline{\left(\frac{-7+26i}{4+25}\right)} = \frac{-7}{29} + i\,\frac{-26}{29}$$

**2. Express in the form x + iy.**

(a) $\dfrac{2-3i}{4-i}$

**Solution:** Here,

$$\frac{2-3i}{4-i} = \frac{2-3i}{4-i}\times\frac{4+i}{4+i} = \frac{(2-3i)(4+i)}{16+1} = \frac{8-12i+2i+3}{17} = \frac{11}{17} - i\,\frac{10}{17}$$

(b) $\dfrac{(2-8i)(7+8i)}{1+i}$

**Solution:** Here,

$$\frac{(2-8i)(7+8i)}{1+i} = \frac{14+16i-56i-64i^2}{1+i}$$
$$= \frac{14-40i+64}{1+i}$$
$$= \frac{78-40i}{1+i}\times\frac{1-i}{1-i}$$
$$= \frac{(78-40i)(1-i)}{1^2-i^2}$$
$$= \frac{78-40i-78i+40i^2}{1+1} = \frac{38-118i}{2} = 19-59i$$

(c) $\dfrac{(4+5i)^2}{(2+3i)^2}$

**Solution:** Here,

$$\frac{(4+5i)^2}{(2+3i)^2} = \frac{16+40i-25}{4+12i-9} = \frac{-9+40i}{-5+12i}\times\frac{-5-12i}{-5-12i}$$
$$= \frac{(45+480)+(108-200)i}{25+144} = \frac{525-92i}{169}$$

(d) $\dfrac{1}{(2+i)^2} - \dfrac{1}{(2-i)^2}$

**Solution:** Here,



$$\frac{1}{(2+i)^2} - \frac{1}{(2-i)^2} = \frac{(2-i)^2-(2+i)^2}{[(2+i)(2-i)]^2}$$
$$= \frac{(2-i+2+i)(2-i-2-i)}{(4+1)^2} = \frac{4(-2i)}{25} = -\frac{8i}{25}$$

**3. Express the following in the modules amplitude form:**

(a) **1 + i**

**Solution:** Here, $1 + i = \sqrt{2}\left[\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right] = \sqrt{2}\left[\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right]$.

(b) **–1 + i**

**Solution:** Here, $-1 + i = \sqrt{2}\left[-\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right] = \sqrt{2}\left[\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right]$

(c) $-1 - i\sqrt{3}$

**Solution:** Here,

$$-1 - i\sqrt{3} = 2\left[-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] = 2\left[-\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\right]$$
$$= 2\left[\cos\left(\pi + \frac{\pi}{3}\right) + i\sin\left(\pi + \frac{\pi}{3}\right)\right]$$
$$= 2\left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right]$$

(d) **1 + i tanα**

**Solution:** Here,

$1 + i\tan\alpha$

Comparing the given term with x + iy and set $x = r\cos\theta$ and $y = r\sin\theta$ then we get,

$r\cos\theta = 1$ and $r\sin\theta = \tan\alpha$

So, $r^2 = 1 + \tan^2\alpha = \sec^2\alpha \Rightarrow r = \sec\alpha$

Also, $\tan\theta = \dfrac{\tan\alpha}{1} \Rightarrow \theta = \alpha$

Thus, $1 + i\tan\alpha = \sec\alpha(\cos\alpha + i\sin\alpha)$.

(e) **1 + cosα + i sinα**

**Solution:** Here,

$$1 + \cos\alpha + i\sin\alpha = 2\cos^2\frac{\alpha}{2} + i\sin\frac{\alpha}{2}.\cos\frac{\alpha}{2} = 2\cos\frac{\alpha}{2}\left[\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right]$$

(f) **1 + sinα + i cosα**

**Solution:** Here,

$1 + \sin\alpha + i\cos\alpha$

$$= \cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}.\cos\frac{\alpha}{2} + i\left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right)$$
$$= \left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)^2 + i\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)$$
$$= \left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)\left[\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right) + i\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\right]$$

$$=\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)\left[\left(\cos\frac{\pi}{4}\cos\frac{\alpha}{2}+\sin\frac{\pi}{4}\sin\frac{\alpha}{2}\right)+i\left(\cos\frac{\pi}{4}\cos\frac{\alpha}{2}-\sin\frac{\pi}{4}\sin\frac{\alpha}{2}\right)\right]$$

$$=\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)\left[\left(\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right)+i\left(\cos\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right)\right]$$

**(g) $1-\sin\alpha+i\cos\alpha$**

**Solution**: Here,

$$1-\sin\alpha+i\cos\alpha$$

$$=\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}-2\sin\frac{\alpha}{2}.\cos\frac{\alpha}{2}+i\left(\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}\right)$$

$$=\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2+i\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)$$

$$=\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left[\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)+i\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)\right]$$

$$=\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left[\left(\cos\frac{\pi}{4}\cos\frac{\alpha}{2}-\sin\frac{\pi}{4}\sin\frac{\alpha}{2}\right)+i\left(\cos\frac{\pi}{4}\cos\frac{\alpha}{2}+\sin\frac{\pi}{4}\sin\frac{\alpha}{2}\right)\right]$$

$$=\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left[\left(\cos\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right)+i\left(\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right)\right]$$

**(h) $-3-4i$**

**Solution**: Here, $-3-4i$

Comparing the given term with x + iy and set $x = r\cos\theta$ and $y = r\sin\theta$ then we get,

$$r\cos\theta=-3 \quad \text{and} \quad r\sin\theta=-4$$

So, $r^2 = 9+16 \Rightarrow r = 5$

Also, $\tan\theta=\frac{-4}{-3} \Rightarrow \theta = 53.13°$

Now,

$$-3-4i = r(\cos\theta+i\sin\theta)$$
$$= 5(\cos 233.13 + i\sin 233.13) \quad \text{[since } -3-4i \text{ lies in 3}^{rd}\text{ quadrant]}$$

**(i) $3i, -3i$ (i.e. $\pm 3i$)**

**Solution**: Here, $\pm 3i$

Comparing the given term with x + iy and set $x = r\cos\theta$ and $y = r\sin\theta$ then we get,

$$x=r\cos\theta=0 \text{ and} \quad y=r\sin\theta=\pm 3$$

So, $r = 3$ and $\theta=\pm\tan^{-1}\left(\frac{3}{0}\right)=\pm\tan^{-1}(\infty)=\pm\frac{\pi}{2}$

Therefore, $\pm 3i = 3\left(\cos\frac{\pi}{2}\pm i\sin\frac{\pi}{2}\right)$

**(j) $\left(\frac{6+8i}{4-3i}\right)^2$**

**Solution**: Here,

$$\left(\frac{6+8i}{4-3i}\right)^2=\left[\frac{(6+8i)(4+3i)}{(4-3i)(4+3i)}\right]^2=\left[\frac{(24-24)+(18+32)i}{25}\right]^2=(2i)^2=-4.$$



Now comparing the given term $-4$ with x + iy and set $x = r\cos\theta$ and $y = r\sin\theta$ then we get,

$$r\cos\theta=-4 \quad \text{and} \quad r\sin\theta=0.$$

So, $r=\sqrt{(-4)^2}=4$ and, $\theta=\tan^{-1}\left(\frac{0}{-4}\right)=\pi$

Therefore, $\left(\frac{6+8i}{4-3i}\right)^2=4(\cos\pi+i\sin\pi)$

**(k) $\left(\frac{2+i}{5-3i}\right)$**

**Solution**: Here,

$$\left(\frac{2+i}{5-3i}\right)=\frac{(2+i)(5+3i)}{25+9}=\frac{10+6i+5i-3}{34}=\frac{7}{34}+\frac{11}{34}i$$

Now comparing the given term with x + iy and set $x = r\cos\theta$ and $y = r\sin\theta$ then we get,

$$x=r\cos\theta=\frac{7}{34} \quad \text{and} \quad y=r\sin\theta=\frac{11}{34}$$

So, $r^2=\left(\frac{7}{34}\right)^2+\left(\frac{11}{34}\right)^2=\frac{49+121}{(34)^2}=\frac{170}{(34)^2} \Rightarrow r=\sqrt{\frac{170}{34}}$

And, $\tan\theta=\frac{11/34}{7/34} \Rightarrow \theta=\tan^{-1}\left(\frac{11}{7}\right)=57.52$

Therefore, $\frac{2+i}{5-3i}=\sqrt{\frac{170}{34}}(\cos 57.52+i\sin 57.52).$

**4. Represents each of the following in the form x + iy.**

**(a) $\sqrt{8}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$**

**Solution**: Here,

$$\sqrt{8}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{8}\cos\frac{\pi}{4}+i\sqrt{8}\sin\frac{\pi}{4}=2+2i$$

**(b) $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$**

**Solution**: Here, $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=0+i=i$

**(c) $6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$**

**Solution**:

Here, $6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=6\cos\frac{\pi}{3}+i\,6\sin\frac{\pi}{3}=3+3\sqrt{3}\,i$

**(d) $\sqrt{18}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$**

**Solution**:

Here, $\sqrt{18}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=\sqrt{18}\cos\frac{3\pi}{4}+i\sqrt{18}\sin\frac{3\pi}{4}=-3+3i$

5. **Determine the principle value of the argument:** [2008 Fall Q. No. 7(a)]

(a) 1 – i

Solution: Here,

$$z = 1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$$

Thus, $|z| = r = \sqrt{2}$

so, $\arg z = -\frac{\pi}{4} \pm 2n\pi$ for n = 0, 1, 2, .....

and $\text{Arg. } z = -\frac{\pi}{4}$ (principle value).

(b) $3 \pm 4i$

Solution: Here, $z = 3 \pm 4i$

Now comparing the given term with $z = x + iy$ and set $x = r\cos\theta$ and $y = r\sin\theta$ then,

$$x = r\cos\theta = 3 \quad \text{and} \quad y = r\sin\theta = \pm 4$$

So, $r^2 = 9 + 16 \Rightarrow r = 5$

Also, $\tan\theta = \pm\frac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\pm\frac{4}{3}\right) = (\pm 53.13)$

Thus,

$$3 \pm 4i = 5[\cos(\pm 53.13) + i\sin(\pm 53.13)]$$

Thus,

$|z| = 5$ and $\arg z = \pm 53.13 \pm 2n\pi$ for n = 0, 1, 2, .....

Now,

$\text{Arg } z = \pm 53.13$ (Principle value).

(c) $-\pi - \pi i$

Solution: Let, $z = -\pi - \pi i$

So, $r^2 = (-\pi)^2 + (-\pi)^2 \Rightarrow r = \pi\sqrt{2}$

And, $\tan\theta = \frac{-\pi}{-\pi} \Rightarrow \tan\theta = \tan\left(-\frac{3\pi}{4}\right) \Rightarrow \theta = -\frac{3\pi}{4}$

So,

$$-\pi - \pi i = \pi\sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$$

So, $|z| = \pi\sqrt{2}$

and, $\arg. z = -\frac{3\pi}{4} \pm 2n\pi$ for n = 0, 1, 2, ......

Hence, $\text{Arg. } z = -\frac{3\pi}{4}$ (Principle value)

## OTHER IMPORTANT QUESTION FROM FINAL EXAM FOR PRACTICE

**2008 Fall Q. No. 7(b)**

Write down the equation of the circle in complex form whose centre is at 1 + i and of radius 1.

**2009 Fall Q. No. 7(a)**

Find the principal argument of $z = -2 - 3i$.

□□□

# Unit 2
# DERIVATIVE AND ANALYTIC FUNCTION

**Limit of a function of a complex variable:**

A single valued function f(z) of a complex variable z is said to has limit $l = \alpha + i\beta$ if $\lim_{z\to z_0} f(z) = l$.

**Continuity of a function of a complex variable:**

A single valued function f(z) of a complex variable z is called continuous at a point $z = z_0$ if $\lim_{z\to z_0} f(z) = f(z_0)$.

**Differentiability of a complex function:**

A single valued function f(z) of a complex variable z is called differentiable at a function $z = z_0$ if $\lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ where $\Delta z = z - z_0$ exists. That is, if $\lim_{z\to z_0} \frac{f(z - z_0) - f(z_0)}{z - z_0}$ exists. It is denoted by $f'(z_0)$.

**OR**

A complex valued function f(z) is called differentiable at $z = z_0$ if

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z - z_0) - f(z_0)}{z - z_0}.$$

**Analytic function:**

A complex valued function f(z) is called analytic at a point $z = z_0$ in the domain D if f(z) is defined and differentiable at each point in a neighborhood of $z_0$.

A complex valued function f(z) is called analytic in the domain D if f(z) is defined and differentiable at each point in D.

**Cauchy–Riemann's (C–R) Equation:**

A function $f(z) = u + iv$ where $u = u(x, y)$ and $v = v(x, y)$, is analytic in a domain D. Then we say f(z) satisfies the Cauchy–Riemann's (i.e. C–R) equation if the partial derivatives $u_x, u_y, v_x, v_y$ exist and

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

**Cauchy–Riemann's (C–R) Equation in Polar Form:**

A function $f(z) = u + iv$ where $u = u(x, y)$ and $v = v(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, is analytic in a domain D. Then we say f(z) satisfies the Cauchy–Riemann's (i.e. C–R) equation if the partial derivatives $u_r, u_\theta, v_r, v_\theta$ exist and

$$u_r = \frac{1}{r}v_\theta \quad \text{and} \quad u_\theta = -r\,v_r.$$

***Note:*** A function f(z) is analytic in D if and only if it satisfies the C–R equation.

**Laplace Equation:**

A function $f(z) = u + iv$ where $u = u(x, y)$ and $v = v(x, y)$, is analytic in a domain D. Then we say the function u satisfies the Laplace equation if

$$\nabla^2 u = u_{xx} + u_{yy} = 0.$$

As similar, the function v satisfies the Laplace equation if

$$\nabla^2 v = v_{xx} + v_{yy} = 0.$$

**Harmonic function:**

Let function $f(z) = u + iv$ where $u = u(x, y)$ and $v = v(x, y)$, is analytic in a domain D. Then we say the function u is harmonic function if it satisfies its Laplace equation i.e. $\nabla^2 u = u_{xx} + u_{yy} = 0$.

In this case v is called complex conjugate of u.

As similar, the function v is harmonic function if it satisfies its Laplace equation i.e. $\nabla^2 v = v_{xx} + v_{yy} = 0$.

In this case u is called complex conjugate of v.

## Theorem (Cauchy–Riemann equation)

**(Necessary condition for analyticity of a function)**

*Let* $f(z) = u(x, y) + i\,v(x, y)$ *be defined and continuous in some neighborhood of a point* $z = x + iy$ *and differentiable at z itself. Then at that point,*

$$u_x = v_y \quad \textit{and} \quad u_y = -v_x.$$

*Hence, if* $f(z) = u + iv$ *is analytic in a domain D, those partial derivatives exists and satisfy* $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ *at all points of D.*

**2015 Spring Q. No. 1(a)**

Show that the necessary condition for analyticity of $f(z) = u + iv$, is $u_x = v_y$ and $u_y = -v_x$.

**2016 Fall Q. No. 1(a)**

Show that if the function f(z) is analytic then show that $u_x = v_y$ and $u_y = -v_x$.

**Proof:** We have $f(z) = u + iv$ is differentiable. Then $f'(z)$ exists at z itself.

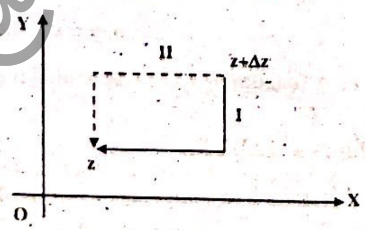
where,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)] - [u(x, y) + iv(x, y)]}{\Delta x + i\Delta y} \quad \dots (1)$$

Here $\Delta z$ approach to zero along any path in a neighborhood of z. Thus we may choose the two paths I and II in figure and then equate the results.

Choose path I. on this path we observe x is variate and y is constant. So, $\Delta x \to 0$ and $\Delta y = 0$. Then the equation (1) becomes






$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}$$

$$\Rightarrow \quad f'(z) = u_x + iv_x \quad \dots (2)$$

by definition of partial derivatives.

Next, choose path II. on this path we observe y is variate and x is constant. So, $\Delta y \to 0$ and $\Delta x = 0$. Then the equation (1) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{i\Delta y} + i \lim_{\Delta x \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y}$$

$$\Rightarrow \quad f'(z) = \frac{1}{i} u_y + v_y = -iu_y + v_y$$

$$\Rightarrow \quad f'(z) = v_y - iu_y \quad \dots (3)$$

We have $f'(z)$ exists, implies from (2) and (3)

$$u_x + iv_x = v_y - iu_y$$

Comparing the real and imaginary value we get

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

Thus, we get if f(z) is analytic then $u_x = v_y$ and $u_y = -v_x$.

## Theorem (Cauchy–Riemann equation)

### Sufficient Condition for Analyticity of a Function

*The single valued continuous function* $f(z) = u + iv$ *is analytic in a region R of the z–plane, if the four partial derivatives* $u_x$, $v_x$, $u_y$, $v_y$ *exist, continuous and satisfy the Cauchy–Riemann equation* $u_x = v_y$ *and* $u_y = -v_x$ *at each point of R.*

**Proof:** Here, $\Delta u = u(x + \Delta x, y + \Delta y) - u(x, y)$

$$= u(x + \Delta x, y + \Delta y) - u(x + \Delta x, y) + u(x + \Delta x, y) - u(x, y)$$

$$= \Delta y\, u_y(x + \Delta x, y + \theta_1 \Delta y) + \Delta x\, u_x(x + \theta_2 \Delta x, y)$$

where $0 < \theta_1 < 1$, $0 < \theta_2 < 1$ (by mean value theorem)

$$= \Delta y (u_y + \epsilon_1) + \Delta x (u_x + \epsilon_2),$$

where $\epsilon_1$ and $\epsilon_2$ both tends to zero as $\Delta z \to 0$.

$$= \Delta y(-v_x + \epsilon_1) + \Delta x(u_x + \epsilon_2),$$

since $u_y = v_x$ similarly we get.

$$\Delta v = \Delta y(v_y + \epsilon_3) + \Delta x(v_x + \epsilon_4)$$

$$\Rightarrow \quad \Delta v = \Delta y(u_x + \epsilon_3) + \Delta x(v_x + \epsilon_4) \quad \text{since } v_y = u_x$$

where $\epsilon_3$, $\epsilon_4$ tends to zero if $\Delta z \to 0$.

Therefore,

$$\Delta u + i\Delta v = (u_x + iv_x)(\Delta x + i\Delta y) + \eta \Delta x + \eta' \Delta y$$

Dividing both side by $\Delta x + i\Delta y$, we get

$$\frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} = (u_x + iv_x) + \frac{(\eta\Delta x + \eta'\Delta y)}{\Delta x + i\Delta y}$$

$$\Rightarrow \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} = (u_x + iv_x) + \frac{(\Delta x + i\Delta y) + \eta\,\Delta x + \eta'\Delta y}{\Delta x + i\Delta y}$$

$$\Rightarrow \frac{\Delta w}{\Delta z} = (u_x + iv_x) + \eta\frac{\Delta x}{\Delta z} + \eta'\frac{\Delta y}{\Delta z} \quad \text{.......... (i)}$$

We have $|\Delta x| \le |\Delta z|$

$$\Rightarrow \left|\frac{\Delta x}{\Delta z}\right| \le 1 \quad \text{and} \quad \left|\frac{\Delta y}{\Delta z}\right| \le 1.$$

So. $\left|\eta\frac{\Delta x}{\Delta z} + \eta'\frac{\Delta y}{\Delta z}\right| \le |\eta| + |\eta'| \to 0$

Thus we get,

$$\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} = \lim_{\Delta z\to 0}[u_x + iv_x] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Hence, $\frac{dw}{dz}$ exists because $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ exists.

Therefore, the sufficient condition for the function f(z) to be analytic require the continuity of the four partial derivatives of u and v.

## Exercise – 2.1

**1. Express the following function of z in the form u + iv.**

**(a) f(z) = sinz** **[2007 Fall Q. No. 7(a); 2012 Fall Q. No. 7(a)]**

**Solution:** Here,

$$f(z) = \sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{ix-y} - e^{-ix+y}}{2i}$$

$$= \frac{e^{-y}}{2i}(\cos x + i\sin x) - \frac{e^{y}}{2i}(\cos x - i\sin x)$$

$$= i^2\cos x\,\frac{e^{y} - e^{-y}}{2i} + i\sin x\,\frac{e^{y} + e^{-y}}{2i}$$

$$= i\cos x\sinh y + \sin x\cosh y$$

**(b) f(z) = cosz**

**Solution:** Here,

$$f(z) = \cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{ix-y} + e^{-ix+y}}{2}$$

$$= \frac{e^{ix}(\cosh y - \sinh y) + e^{-ix}(\cosh y + \sinh y)}{2}$$

$$= \frac{e^{ix} + e^{-ix}}{2}\cosh y - i\,\frac{e^{ix} - e^{-ix}}{2i}\sinh y$$

$$= \cos x\cosh y - i\sin x\sinh y$$



**(c) f(z) = e^z**

**Solution:** Here,

$$f(z) = e^z = e^{x+iy} = e^x(\cos y + i\sin y) = e^x\cos y + i\,e^x\sin y$$

**(d) $f(z) = \frac{1}{z}$**

**Solution:** Here,

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy}\times\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$$

**2. Find the values of Re(f) and Im(f) at the indicated point.**

**(a) $f(z) = \frac{1}{1-z}$ at z = 7 + 2i**

**Solution:** Here,

$$f(z) = \frac{1}{1-z} = \frac{1}{-6-2i} = \frac{1}{-6-2i}\times\frac{-6+2i}{-6+2i} = \frac{-6+2i}{36+4} = \frac{-6+2i}{40} = \frac{-3}{20} + i\frac{1}{20}$$

Thus $Re(f) = \frac{-3}{20}$ and $Im(f) = \frac{1}{20}$.

**(b) $f(z) = z^2 + 3z$ at z = 1 + 3i**

**Solution:** Here,

$$f(z) = z^2 + 3z = (1+3i)^2 + 3(1+3i) = 1 - 9 + 6i + 3 + 9i = -5 + 15i$$

Thus Re(f) = –5 and Im(f) = 15.

**(c) $f(z) = 2iz + 6\bar{z}$ at $z = \frac{1}{2} + 4i$**

**Solution:** Here,

$$f(z) = 2iz + 6\bar{z} = 2i\left(\frac{1}{2} + 4i\right) + 6\left(\frac{1}{2} - 4i\right) = i - 8 + 3 - 24i = -5 - 23i$$

Thus Re(f) = –5 and Im(f) = –23.

## Exercise – 2.2

**1. Show that $\lim_{z\to 0} f(z)$ does not exists if $f(z) = \frac{xy}{x^2 + 4y^2}$ and f(0) = 0.**

**Solution:** Given that,

$$f(z) = \frac{xy}{x^2 + 4y^2} \text{ and } f(0) = 0$$

Suppose y = mx where m be any variable. Then as z→0, we get x→0, then

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}\frac{xy}{x^2 + 4y^2}$$

$$= \lim_{x\to 0}\frac{mx^2}{x^2 + 4m^2x^2} = \lim_{x\to 0}\frac{m}{1 + 4m^2} = \frac{m}{1 + 4m^2}$$

Since m is a variable, so it does not give any fixed value. And we have f(0) = 0.

This shows that $\lim_{z\to 0} f(z)$ does not exist.

Therefore f(z) is not continuous at z = 0.

**2. Show that the function f(z) is not continuous at z = 0 if** $f(z) = \begin{cases} \dfrac{xy}{2x^2+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$

**Solution:** Given that,

$$f(z) = \begin{cases} \dfrac{xy}{2x^2+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

Here, f(0) = 0 exists.

Since y is a variable, suppose y = mx where m be any variable. Then, $x \to 0$ as $z \to 0$.

Now,

$$\lim_{z\to0} f(z) = \lim_{z\to0} \frac{xy}{2x^2+y^2} = \lim_{x\to0} \frac{mx^2}{2x^2+m^2x^2} = \lim_{x\to0} \frac{m}{2+m^2} = \frac{m}{2+m^2}$$

Since, m is a variable and $\lim_{z\to0} f(z)$ depends upon m, so it has no fixed value. So, $\lim_{z\to0} f(z)$ does not exists.

Therefore f(z) is not continuous at z = 0.

**3. Show that the function f(z) is not continuous at z = 0 if** $f(z) = \begin{cases} \dfrac{xy(x-2y)}{x^3+y^3} & z \neq 0 \\ 0 & z = 0 \end{cases}$

**Solution:** Given that,

$$f(z) = \begin{cases} \dfrac{xy(x-2y)}{x^3+y^3} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

Here, f(0) = 0 exists.

Suppose y = mx where m is a variable, exists. Then $x \to 0$ as $z \to 0$.

Now,

$$\lim_{z\to0} f(z) = \lim_{x\to0} \frac{mx^2(x-2mx)}{m^3x^3+x^3} = \lim_{x\to0} \frac{m(1-2m)}{1+m^3} = \frac{(1-2m)m}{1+m^3}$$

Since, m is a variable and $\lim_{z\to0} f(z)$ has value in terms of m. So, $\lim_{z\to0} f(z)$ does not have fixed value. Thus, $\lim_{z\to0} f(z)$ does not exist.

Therefore f(z) is not continuous at z = 0.

**4. Show that the function f(z) is not continuous at z = 0 if** $f(z) = \begin{cases} \dfrac{2xy^2}{x^2+3y^4} & z \neq 0 \\ 0 & z = 0 \end{cases}$

**Solution:** Given that,

$$f(z) = \begin{cases} \dfrac{2xy^2}{x^2+3y^4} & z \neq 0 \\ 0 & z = 0 \end{cases}$$



Since, f(0) = 0 exists. Since, y be a variable, so we may suppose $x = m^2y^2$. Then, $y \to 0$ as $z \to 0$.

Now,

$$\lim_{z\to0} f(z) = \lim_{y\to0} \frac{2m^2y^4}{m^4y^4+3y^4} = \lim_{y\to0} \frac{2m^2}{m^4+3} = \frac{2m^2}{m^4+3}$$

This shows that $\lim_{z\to0} f(z)$ has value in terms of m and m is a variable. So, $\lim_{z\to0} f(z)$ has no fixed value. Therefore, $\lim_{z\to0} f(z)$ does not exist.

Hence, f(z) is not continuous at z = 0.

**5. Show that the function f(z) is continuous at z = 0 if** $f(z) = \dfrac{xy(x-2y)}{x^2+y^2}$ **for** $z \neq 0$ **and f(0) = 0.**

**Solution:** Given that,

$$f(z) = \frac{xy(x-2y)}{x^2+y^2}$$

Given that, f(0) = 0.

Also, for $z \neq 0$, suppose y = mx, where m is a variable, being y is a variable. Then, $x \to 0$ as $z \to 0$.

Now,

$$\lim_{z\to0} f(z)\,dz = \lim_{x\to0} \frac{mx^2(x-2mx)}{x^2+m^2x^2}$$

$$= \lim_{x\to0} \frac{mx(1-2m)}{1+m^2} = 0$$

Thus, $\lim_{z\to0} f(z) = 0 = f(0)$. Therefore, f(z) is continuous at z = 0.

**6. Show that the function** $f(z) = \dfrac{x^4y(y-x)}{(x^8+y^2)(x+y)}$ **is not continuous at (0, 0) such that f(0, 0) = 0.**

**Solution:** Given that,

$$f(z) = \frac{x^4y(y-x)}{(x^8+y^2)(x+y)}$$

Given that, f(0, 0) = 0.

Since y is a variable. Let $y = m(x^4)$ where m is a variable. Then, $x \to 0$ as $z \to 0$.

Now,

$$\lim_{z\to0} f(z) = \lim_{x\to0} \frac{m^4x^8(m^4x^4-x)}{(x^8+y^8m^8)(x+x^4m^4)}$$

$$= \lim_{x\to0} \frac{m^4[(mx)^3-1]}{(1+m^8)(1+m^4x^3)}$$

$$= \frac{-m^4}{1+m^8}$$

This shows that the value of $\lim_{z\to0} f(z)$ depends upon a variable m, so it does not have a fixed value. So, $\lim_{z\to0} f(z)$ does not exist.

**7. Find to whether f(z) is continuous at z = 0 if f(0) = 0 and for z ≠ 0 the function f is equal to**

**(a)** $\dfrac{Im(z)}{|z|}$

**Solution:** Let, f(0) = 0 and for z ≠ 0, $f(z) = \dfrac{Im(z)}{|z|}$

Given that, f(0) = 0.

And, for z ≠ 0,

$$f(z) = \frac{Im(z)}{|z|} = \frac{y}{\sqrt{x^2+y^2}}$$

Suppose, y = mx, where m is a variable. Then, x→0 as z→0.

Now,

$$\lim_{z\to 0} f(z) = \lim_{x\to 0}\frac{mx}{\sqrt{x^2+m^2x^2}} = \lim_{x\to 0}\frac{m}{\sqrt{1+m^2}} = \frac{m}{\sqrt{1+m^2}}$$

Here, f(z) depends upon m and m is a variable. So, $\lim_{z\to 0} f(z)$ does not exist. Therefore, f(z) is not continuous at z = 0.

**(b)** $\dfrac{Re(z)}{1+|z|}$

**Solution:** Let, f(0) = 0 and for z ≠ 0, $f(z) = \dfrac{Re(z)}{1+|z|}$

Given that, f(0) = 0.

And, for z ≠ 0,

$$f(z) = \frac{Re(z)}{1+|z|} = \frac{x}{1+\sqrt{x^2+y^2}}$$

Suppose y = mx where m is a variable. Then x→0 as z→0.

Now,

$$\lim_{z\to 0} f(z) = \lim_{x\to 0}\frac{x}{1+\sqrt{x^2+m^2x^2}} = \frac{0}{1} = 0.$$

Thus, $\lim_{z\to 0} f(z) = 0 = f(0)$. Hence, f(z) is continuous at z = 0.

**8. Find the value of derivative of**

**(a)** $\dfrac{z-i}{z+i}$ at $z_0 = i$

**Solution:** Here, $f(z) = \dfrac{z-i}{z+i}$ at $z_0 = i$

Since, $f'(z_0) = \lim_{z\to z_0}\dfrac{f(z)-f(z_0)}{z-z_0}$

So, $f'(i) = \lim_{z\to i}\left(\dfrac{1}{z-i}\left[\dfrac{z-i}{z+i} - \dfrac{i-i}{i+i}\right]\right)$



$$= \lim_{z\to i}\left(\frac{z-i}{(z-i)(z+i)}\right)$$

$$= \lim_{z\to i}\left(\frac{z-i}{z^2-i^2}\right)$$

$$= \lim_{z\to i}\left(\frac{z-i}{z^2+1}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$\lim_{z\to i}\left(\frac{1}{2z}\right) = \frac{1}{2i} = -\frac{i^2}{2i} = -\frac{i}{2}$$

**(b)** $\dfrac{5+3i}{z^3}$ **at $z_0 = 2 + i$**

**Solution:** Here, $f(z) = \dfrac{5+3i}{z^3}$ at $z_0 = i$

Since, $f'(z_0) = \lim_{z\to z_0}\dfrac{f(z)-f(z_0)}{z-z_0}$

So, $f'(2+i) = (5+3i)\lim_{z\to(2+i)}\left(\dfrac{1}{z-(2+i)}\left[\dfrac{1}{z^3} - \dfrac{1}{(2+i)^3}\right]\right)$

$$= (5+3i)\lim_{z\to(2+i)}\left(\frac{(2+i)^3 - z^3}{[z-(2+i)]\,[z^3(2+i)^3]}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= (5+3i)\lim_{z\to(2+i)}\left(\frac{-3z^2}{z^3(2+i)^3 + [z-(2+i)]\,(2+i)^3.3z^2}\right)$$

$$= \frac{(5+3i)(-3)(2+i)^2}{(2+i)^6}$$

$$= \frac{-3(5+3i)}{(2+i)^4}$$

$$= -\frac{3(5+3i)}{(-7+24i)}$$

$$= -\frac{3(5+3i)(-7-24i)}{(-7)^2-(24i)^2} = \frac{-111+423i}{49+576} = \frac{-111+423i}{625}$$

**(c)** $z^4 + \dfrac{1}{z^4}$ **at $z_0 = -1 - i$**

**Solution:** Here, $f(z) = z^4 + \dfrac{1}{z^4}$ at $z_0 = -1 - i$.

Since, $f'(z_0) = \lim_{z\to z_0}\dfrac{f(z)-f(z_0)}{z-z_0}$

So, $f'(-1-i) = \lim_{z\to(-1-i)}\left[\dfrac{z^4+\frac{1}{z^4} - (1+i)^4 - \frac{1}{(1+i)^4}}{z+1+i}\right] \qquad \left[\text{in } \dfrac{0}{0} \text{ form}\right]$

$$= \lim_{z\to(-1-i)}\left[\frac{4z^3 + \frac{-4}{z^5}}{1}\right]$$

$$= \left[4(-1-i)^3 + \frac{-4}{(-1-i)^5}\right]$$

$$= \frac{4[(-1-i)^8 - 1]}{(-1-i)^5}$$

$$= \frac{-4[(2i)^4 - 1]}{(2i)^2 (1+i)} \quad [\text{Since } (1+i)^2 = 2i]$$

$$= \frac{-4(16-1)}{-4(1+i)}$$

$$= 15\left(\frac{1}{1+i} \times \frac{1-i}{1-i}\right) = 15\frac{(1-i)}{2}$$

**(d)** $f(z) = \frac{iz+2}{3z-6i}$ **at any** $z = z_0$

**Solution:** Here,

$$f(z) = \frac{iz+2}{3z-6i} \text{ at any } z = z_0$$

Since

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \frac{\left[\frac{iz+2}{3z-6i} - \frac{iz_0+2}{3z_0-6i}\right]}{z - z_0} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{z \to z_0} \left[\frac{\frac{(3z-6i)\, i - (iz+2)3}{(3z-6i)^2}}{1}\right]$$

$$= \frac{(3z_0 - 6i)\, i - (iz_0 + 2)\, 3}{(3z_0 - 6i)^2}$$

$$= \frac{3iz_0 + 6 - 3iz_0 - 6}{(3z_0 - 6i)^2} = \frac{0}{(3z_0 - 6i)^2} = 0.$$

## Exercise – 2.3

**1. Check for analyticity by using Cauchy–Riemann equation.**

**NOTE: To check the analyticity of f(z) it sufficient to show that f(z) satisfies the C-R equation i.e. for $f(z) = u + iv$ and $z = x + iy$ then**

**(i) f(z) is analytic if $u_x = v_y$ and $u_y = -v_x$. Otherwise f(z) is not analytic.**

**(ii) f(z) is analytic if $u_r = \frac{1}{r} v_\theta$ and $u_\theta = r v_r$. Otherwise f(z) is not analytic.**

**(iii) f(z) is everywhere continuous if $u_x$, $u_y$, $v_x$, $v_y$ exist and has finite value.**

**a.** $f(z) = z^6$

**Solution:** Here, $f(z) = z^6 = (r(\cos\theta + i\sin\theta))^6$ $\quad [\because z = r(\cos\theta + i\sin\theta)]$

$= r^6(\cos 6\theta + i \sin 6\theta)$ $\quad [\because$ using DeMoiver's theorem$]$



Comparing it with $f(z) = u + iv$, then we get,

$u = r^6 \cos 6\theta$, $\quad v = r^6 \sin 6\theta$

So, $u_r = 6r^5 \cos 6\theta$ $\quad$ so, $v_r = 6r^5 \sin 6\theta$

$u_\theta = -6r^6 \sin 6\theta$, $\quad v_\theta = 6r^6 \cos 6\theta$

Now,

$$u_r = 6r^5 \cos 6\theta = \frac{1}{r} 6r^6 \cos 6\theta = \frac{1}{r} v_\theta$$

And, $u_\theta = -6r^6 \sin 6\theta = -r\, 6r^5 \sin 6\theta = -r v_r$

Thus, the C-R equation is satisfied. Therefore, the function f(z) is analytic.

**b.** $f(z) = e^x(\cos y + i \sin y)$

**Solution:** Here, $f(z) = e^x(\cos y + i \sin y)$

Comparing it with $f(z) = u + iv$ then we get,

$u = e^x \cos y$ and $v = e^x \sin y$

So that, $u_x = e^x \cos y$ $\quad v_x = e^x \sin y$

$u_y = -e^x \sin y$ $\quad v_y = e^x \cos y$

Now,

$u_x = e^x \cos y = v_y$, $\quad u_y = -e^x \sin y = -v_x$

Thus, the C–R equation is satisfied. Therefore, the function f(z) is analytic.

**c.** $f(z) = z\bar{z}$

**Solution:** Here, $f(z) = z\bar{z} = x^2 + y^2$.

Comparing it with $f(z) = u + iv$, then we get,

$u = x^2 + y^2$, $\quad v = 0$

So that, $u_x = 2x$ $\quad v_y = 0$

This shows that $u_x \neq v_y$. That is, f(z) does not satisfy the C–R equation. Therefore, the function f(z) is not analytic.

**d.** $f(z) = \log|z| + i\,\text{Arg}(z)$

**Solution:** Here, $f(z) = \log|z| + i\,\text{Arg}(z) = \log(r) + i\theta$.

Comparing it with $f(z) = u + iv$ then we get,

$u = \log(r)$, $\quad v = \theta$

Then, $u_r = \frac{1}{r}$, $u_\theta = 0$ and $v_r = 0$, $v_\theta = 1$.

Now,

$$u_r = \frac{1}{r} v_\theta \Rightarrow \frac{1}{r} = \frac{1}{r} \cdot 1 \quad \text{and} \quad u_\theta = r\, v_r \Rightarrow 0 = r.0 = 0$$

Thus, f(z) satisfies the C-R equation. Therefore, the function f(z) is analytic.

e. $f(z) = \frac{\text{Re}(z)}{\text{Im}(z)}$

**Solution:** Here, $f(z) = \frac{\text{Re}(z)}{\text{Im}(z)} = \frac{x}{iy} = \frac{ix}{-y}$.

Comparing it with $f(z) = u + iv$ then we get,

$$u = 0 \qquad \text{and} \qquad v = \frac{-x}{y}$$

Then, $u_y = 0$, $v_x = \frac{-1}{y}$

This shows that $u_y \neq -v_x$. This means f(z) does not satisfy the C-R equation. Therefore, the function f(z) is not analytic.

f. $f(z) = \text{Re}(z)^3$

**Solution:** Here,
$$f(z) = \text{Re}(z)^3 = \text{Re}[x^3 + 3x^2.iy + 3x(iy)^2 + (iy)^3]$$
$$= \text{Re}[x^3 - 3xy^2 + i(3x^2 - y^3)]$$
$$= x^3 - 3xy^2$$

Comparing it with $f(z) = u + iv$ then,

$$u = x^3 - 3xy^2 \qquad v = 0$$

So, $u_x = 3x - 3y^2$, $v_y = 0$

This shows that $u_x \neq v_y$. This means f(z) does not satisfy the C-R equation. Therefore, the function f(z) is not analytic.

g. $f(z) = |z^2| = x^2 - y^2$

**Solution:** Here, $f(z) = x^2 - y^2$.

Comparing it with $f(z) = u + iv$ then,

$$u = x^2 - y^2, \qquad v = 0$$

So, $u_x = 2x$, $v_y = 0$

This shows that $u_x \neq v_y$. This means f(z) does not satisfy the C-R equation. Therefore, the function f(z) is not analytic.

**2. Show that the function f(z) = xy+iy is everywhere continuous but not analytic.**

**Solution:** Here, $f(z) = xy + iy$, comparing it with $f(z) = u + iv$ then,

$u = xy$, $v = y$

Then, $u_x = y$, $v_x = 0$

$u_y = x$, $v_y = 1$

This shows the partial derivatives of f(z) are exist, so f(z) is continuous everywhere.

But $u_x \neq v_y$. This means f(z) does not satisfy the C–R equation. So, the function f(z) is not analytic.

**3. Show that sinh z is an analytic function.**

**Solution:** Here, $f(z) = \sinh z = \frac{e^z - e^{-z}}{2} = \frac{e^{x+iy} - e^{-x-iy}}{2}$



$$= \frac{e^x(\cos y + i \sin y) - e^{-x}(\cos y - i \sin y)}{2}$$
$$= \frac{e^x - e^{-x}}{2}\cos y + i\frac{e^x + e^{-x}}{2}\sin y$$
$$= \sinh x \cos y + i \sin y \cosh x$$

Comparing it with $f(z) = u + iv$ then,

$u = \sinh x \cos y$, $v = \cosh x \sin y$

Then $u_x = \cosh x \cos y$, $v_x = \sinh x \sin y$

$u_y = -\sinh x \sin y$, $v_y = \cosh x \cos y$

Now, $u_x = v_y$ and $u_y = -v_x$. Thus, f(z) satisfies the C – R equation. Therefore, f(z) is analytic.

**4. Show that $f(z) = z^2$ is analytic and show that $f'(z) = 2z$.**

**Solution:** Here, $f(z) = z^2 = x^2 - y^2 + 2ixy$

Comparing it with $f(z) = u + iv$ then,

$$u = x^2 - y^2 \qquad \text{and} \qquad v = 2xy$$

So, $u_x = 2x$, $u_y = -2y$, $v_x = 2y$, $v_y = 2x$

Thus, $u_x = v_y$ and $u_y = -v_x$. That means f(z) satisfies the C–R equation. Therefore, the function f(z) is analytic.

Also,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{z^2 + 2\Delta z + (\Delta z)^2 - z^2}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{\Delta z(\Delta z + 2z)}{\Delta z} = \lim_{\Delta z \to 0} (\Delta z + 2z) = 2z$$

**5. Show that $v = 2xy - \frac{y}{x^2 + y^2}$ is a harmonic function. Find harmonic conjugate u of v.** **2009 (Spring) Q. No. 1(b)**

**Solution:** Here, $v = 2xy - \frac{y}{x^2 + y^2}$

So, $v_x = 2y + \frac{2xy}{(x^2 + y^2)^2}$,

$$v_y = 2x - \frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)} = 2x - \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

And $v_{xx} = \frac{(x^2 + y^2)\,2x - 2.2xy\,(x^2 + y^2).2x}{(x^2 + y^2)^4} = 2y\frac{y^2 - 3x^2}{(x^2 + y^2)^3}$

$$v_{yy} = -\frac{(x^2 + y^2)(-2y) - 2(x^2 + y^2)(x^2 - y^2).2y}{(x^2 + y^2)^4} = \frac{2y(3x^2 - y^2)}{(x^2 + y^2)^3}$$

Now, $v_{xx} + v_{yy} = 0$. Thus, by definition, v is a harmonic function.

Suppose that f(z) = u + iv be analytic. So. f(z) satisfies the C–R equation. Therefore,

$$u_y = -v_x = -2y + \frac{2xy}{(x^2+y^2)^2}$$

Integrating w. r. t. y we get,

$$u = -y^2 + \frac{x}{x^2+y^2} + h(x) \quad \ldots\ldots\ldots\ldots (i)$$

Then, $u_x = \frac{y^2-x^2}{(x^2+y^2)^2} + h'(x)$

Since, $u_x = v_y$. So. $h'(x) = 2x$. Then, $h(x) = x^2 + c$. So, by (i)

$$u = -y^2 + x^2 + \frac{x}{x^2+y^2} + c$$

6. **Show that $u = e^{2x}$ (x cos2y − y sin2y) is a harmonic function. Find an analytic function for which u(x, y) is the real part.**

**Solution:** We know that a function u(x, y) is harmonic if $u_{xx} + u_{yy} = 0$.

Here,

$$u = e^{2x}(x\cos 2y - y\sin 2y) \quad \ldots\ldots\ldots\ldots (i)$$

Then, $u_x = 2e^{2x}(x\cos 2y - y\sin 2y) + e^{2x}\cos 2y$

And, $u_{xx} = 4e^{2x}(x\cos 2y - y\sin 2y) + 4e^{2x}\cos 2y$

Also, $u_y = e^{2x}[-2x\sin 2y - \sin 2y - 2y\cos 2y]$

And, $u_{yy} = e^{2x}[-4x\cos 2y - 2\cos 2y - 2\cos 2y + 4y\sin 2y]$.

Now,

$$u_{xx} + u_{yy} = e^{2x}[4x\cos 2y - 4y\sin 2y + 4\cos 2y - 4x\cos 2y - 4\cos 2y + 4y\sin 2y] = 0.$$

This shows that the function u is a harmonic function.

Suppose that f(z) = u + iv be analytic function, where u is given in (i). So, f(z) satisfies the C–R equation. That is,

$$u_x = v_y, \quad u_y = -v_x$$

Here, $v_y = u_x = e^{2x}[2x\cos 2y + \cos 2y - y\sin 2y]$

Integrating w. r. t. y we get,

$$v = e^{2x}[y\cos 2y + x\sin 2y] + h(x) \quad \ldots\ldots\ldots\ldots (ii)$$

Then, $v_x = e^{2x}[2y\cos 2y + 2x\sin 2y + \sin 2y] + h'(x)$

Since, $u_y = -v_x$. So,

$$h'(x) = 0$$

Then, h(x) = c. So that (i) becomes,

$$v = e^{2x}(y\cos 2y + x\sin 2y) + c.$$

7. **Show that $e^x$(x cosy − y siny) is a harmonic function. Find the analytic function for which the given function is the imaginary part.**

**Solution:** We know that a function u(x, y) is harmonic if $u_{xx} + u_{yy} = 0$.

Here,

$$v = e^x(x\cos y - y\sin y) \quad \ldots\ldots\ldots\ldots (i)$$



Then, $v_x = e^x(x\cos y - y\sin y + \cos y)$

and $v_{xx} = e^x[x\cos y - y\sin y + 2\cos y]$

Also, $v_y = e^x[-x\sin y - \sin y - y\cos y]$

and $v_{yy} = e^x[-x\cos y - 2\cos y + y\sin y]$

Now,

$$v_{xx} + v_{yy} = e^x[x\cos y - y\sin y + 2\cos y] + e^x[-x\cos y - 2\cos y + y\sin y]$$
$$= e^x[x\cos y - y\sin y + 2\cos y - x\cos y - 2\cos y + y\sin y]$$
$$= 0.$$

Thus, the function v is a harmonic.

Suppose that f(z) = u + iv be an analytic function for which v is given in (i). Then, the function f(z) satisfies the C–R equation i.e. $u_x = v_y$ $u_y = -v_x$

So that,

$$u_x = e^x[-x\sin y - \sin y - y\cos y]$$

Then, $u = -e^x x\sin y - e^x y\cos y + h(y) \quad \ldots\ldots\ldots\ldots (2)$

So, $u_y = -e^x x\cos y - e^x\cos y + e^x y\sin y + h(y)$

Since, $u_y = -v_x$. So that, $h'(y) = 0$ then,

$$h(y) = c.$$

Hence, by (2),

$$u = -e^x[x\sin y + y\cos y] + c$$

Thus, $f(z) = u + iv = -e^x[x\sin y + y\cos y - ix\cos y + iy\sin y] + c$

$$= -e^x[ye^{iy} - ixe^{iy}] + c$$
$$= e^{x+iy}\, i\,[x + iy] + c$$
$$= ize^z + c.$$

8. **Given that $u = x^2 - y^2$ and $v = -\frac{y}{x^2+y^2}$, prove that both u and v are harmonic function but u + iv is not analytic function of z.**

**Solution:** Let f(z) = u + iv where, $u = x^2 - y^2$ and $v = -\frac{y}{x^2+y^2}$

So, $u_x = 2x$, $u_{xx} = 2$ and $u_y = -2y$, $u_{yy} = -2$.

Therefore, $u_{xx} + u_{yy} = 2 - 2 = 0$. Thus, u is a harmonic function

Also,

$$v_x = \frac{2xy}{(x^2+y^2)^2},$$

$$v_{xx} = \frac{(x^2+y^2)2y - 2xy.2(x^2+y^2).2x}{(x^2+y^2)^4} = \frac{2y[x^2+y^2-4x^2]}{(x^2+y^2)^3}$$

And $v_y = \frac{(x^2+y^2) + y.2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$$v_{yy} = \frac{(x^2+y^2)^2.2y - (y^2-x^2).2(x^2+y^2).2y}{(x^2+y^2)^4} = \frac{2y[3x^2-y^2]}{(x^2+y^2)^3}$$

Thus, $v_{xx} + v_{yy} = 0$. Therefore, v is also a harmonic function.

Next, $u_x = 2x \neq \frac{y^2 - x^2}{(x^2 + y^2)^2} = v_y$

This shows that f(z) does not satisfy the C–R equation. Therefore, the function f(z) = u + iv is not analytic.

**9. Determine whether the following functions are harmonic. If your answer is yes, find a corresponding analytic function f(z) = u + iv.**

**(a)** $u = \frac{x}{x^2 + y^2}$

**Solution:** Here, $u = \frac{x}{x^2 + y^2}$

So, $u_x = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$

And $u_{xx} = \frac{(x^2 + y^2)(-2x) - (y^2 - x^2).2(x^2 + y^2).2x}{(x^2 + y^2)^4}$

$= \frac{-2x[x^2 + y^2 + 2y^2 - 2x^2]}{(x^2 + y^2)^3}$

$= \frac{-2x[3y^2 - x^2]}{(x^2 + y^2)^3}$

Also, $u_y = \frac{2xy}{(x^2 + y^2)^2}$

And, $u_{yy} = \frac{(x^2 + y^2)^2 2x - 2xy.2(x^2 + y^2).2y}{(x^2 + y^2)^4}$

$= \frac{2x[x^2 + y^2 - 4y^2]}{(x^2 + y^2)^3}$

$= \frac{2x[x^2 - 3y^2]}{(x^2 + y^2)^3}$

Now $u_{xx} + u_{yy} = \frac{-2x[3y^2 - x^2]}{(x^2 + y^2)^3} + \frac{2x[x^2 - 3y^2]}{(x^2 + y^2)^3} = 0$

Thus, the function u is a harmonic function. Therefore f(z) = u + iv be an analytic function. So, it satisfies the C–R equation, i.e.

$u_x = v_y$ and $u_y = -v_x$

Since, $u_x = v_y$, that means,

$v_y = \frac{y^2 - x^2}{(x^2 + y^2)^2} = -\frac{(x^2 + y^2) - y.2y}{(x^2 + y^2)^2}$

Therefore, $v = -\frac{y}{x^2 + y^2} + h(x)$

So that, $v_x = \frac{2xy}{x^2 + y^2} + h'(x)$

Since, $u_y = -v_x$. So, h'(x) = 0. This gives,

h(x) = c



Hence.

$f(z) = u + iv = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} + ic$

$= \frac{x - iy}{x^2 + y^2} + ic = \frac{\bar{z}}{z\bar{z}} + ic = \frac{1}{z} + ic$

**b.** $v = -e^{-x}\sin y$

**Solution:** Here, $v = -e^{-x}\sin y$

So, $v_x = e^{-x}\sin y$, $v_{xx} = -e^{-x}\sin y$

And, $v_y = -e^{-x}\cos y$ $v_{yy} = e^{-x}\sin y$

Clearly, $v_{xx} + v_{yy} = 0$. Therefore, v is a harmonic function.

Therefore, f(z) = u + iv is analytic function. So, it satisfies the C–R equation i.e. $u_x = v_y$ and $u_y = -v_x$.

Therefore,

$u_x = -e^{-x}\cos y$

Integrating w.r.t. x then

$u = e^{-x}\cos y + h(y)$

Then, $u_y = -e^{-x}\sin y + h'(y)$

Since, $u_y = -v_x$. So, h'(y) = 0. Therefore, h(y) = c

Hence,

$f(z) = u + iv = e^{-x}[\cos y - i\sin y] + c = e^{-x}.e^{-iy} + c = e^{-z} + c$

**c.** $u = \log|z|$

**Solution:** Here, $u = \log|z| = \log\left(\sqrt{x^2 + y^2}\right)$

So, $u_x = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2}(x^2 + y^2)^{-1/2}.2x = \frac{x}{x^2 + y^2}$

And, $u_{xx} = \frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$

Also, $u_y = \frac{y}{x^2 + y^2}$ and $u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$

Now,

$u_{xx} + u_{yy} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0$

Therefore, u is harmonic. So, f(z) = u + iv is analytic and it satisfies the C–R equation, , i.e.

$u_x = v_y$ and $u_y = -v_x$

Therefore, $v_y = u_x = \frac{x}{x^2 + y^2}$

Integrating w. r. t. y then,

$v = x\int\frac{dy}{x^2 + y^2} + h(x) = \frac{x}{x}\tan^{-1}\left(\frac{y}{x}\right) + h(x) = \tan^{-1}\left(\frac{y}{x}\right) + h(x)$

Then,

$$v_x = \left(\frac{1}{1+\frac{y^2}{x^2}}\right)\left(\frac{-y}{x^2}\right) + h'(x) = \frac{-y}{x^2+y^2} + h'(x)$$

By C–R equation.

$$v_x = -u_y \Rightarrow \frac{-y}{x^2+y^2} + h'(x) = -\frac{y}{x^2+y^2} \Rightarrow h'(x) = 0.$$

Integrating, $h(x) = c$

Hence, $v = \tan^{-1}\left(\frac{y}{x}\right) + c.$

Thus, $f(z) = u + iv$

$$= \log|z| + i\tan\left(\frac{y}{x}\right) + c = \log|z| + i\,\mathrm{Arg}z + c \quad \left[\because \theta = \tan^{-1}\left(\frac{y}{x}\right)\right]$$

**d.** $v = x^3 - 3x^2$

**Solution:** Here, $v = x^3 - 3x^2$

So, $v_x = 3x^2 - 6x$ and $v_{xx} = 6x - 6$

Also, $v_y = 0$ and $v_{yy} = 0$

Then, $u_{xx} + u_{yy} \neq 0$. This means the function is not harmonic.

**10. Determine a and b such that the given function is harmonic and find a conjugate harmonic.**

**a.** $ax^3 + by^3$

**Solution:** Here, $u = ax^3 + by^3$

Then, $u_x = 3ax^2$ and $u_{xx} = 6ax$

And, $u_y = 3by^2$ and $u_{yy} = 6by$

Since, u is harmonic. So,

$$u_{xx} + u_{yy} = 0 \Rightarrow 6ax + 6by = 0$$
$$\Rightarrow ax + by = 0$$

The condition will satisfy only if a = 0, b = 0.

Therefore, u = 0.

By given hypothesis u is harmonic. So, the function $f(z) = u + iv$ is analytic and therefore f(z) satisfies its C–R equation i.e. $u_x = v_y$ and $u_y = -v_x$.

Since u = 0. So, $v_y = u_x = 0$.

Integrating w. r. t. y we get, $v = h(x)$

Then, $v_x = h'(x)$

Since, u satisfies C–R equation so, $u_y = -v_x$

So, $h'(x) = v_x = -u_y = 0$

Integrating w. r. t. x then $h(x) = c$.

Hence, the harmonic conjugate v of u is v = c.



**(b)** $u = e^{ax}\cos 5y$

**Solution:** Here, $u = e^{ax}\cos 5y$

So, $u_x = ae^{ax}\cos 5y$, $u_{xx} = a^2e^{ax}\cos 5y$

And $u_y = -5e^{ax}\sin 5y$, $u_{yy} = -25e^{ax}\cos 5y$

Since, the function u is harmonic. So,

$$u_{xx} + u_{yy} = 0 \Rightarrow a^2 - 25 = 0 \;[\text{Being } e^{ax}\cos 5y \neq 0]$$
$$\Rightarrow a = \pm 5$$

Since u is harmonic function. Therefore, the function $f(z) = u + iv$ is analytic. So, it satisfies the C–R equation i.e.

$u_x = v_y$, $u_y = -v_x$

Then, $v_x = -u_y = 5e^{\pm 5x}\sin 5y$

So, $v = \pm e^{\pm 5x}\sin 5y + h(y)$

And, $v_y = \pm 5e^{\pm 5x}\cos 5y + h'(y)$

Since, $u_x = v_y \Rightarrow \pm 5e^{\pm 5x}\cos 5y = \pm 5e^{\pm 5x}\cos 5y + h'(y)$

$\Rightarrow h'(y) = 0.$

Integrating we get, $h(y) = c$

Hence, the harmonic conjugate of u is,

$$v = \pm e^{\pm 5x}\sin 5y + c.$$

**11. Show that the function u = cosx coshy is harmonic and find its harmonic conjugate.**

**Solution:** Here, u = cosx coshy

So, $u_x = -\sin x\cosh y$, $u_{xx} = -\cos x\cosh y$

And, $u_y = \cos x\sinh y$ $u_{yy} = \cos x\cosh y$

Now,

$$u_{xx} + u_{yy} = -\cos x\cosh y + \cos x\cosh y = 0.$$

Thus, u is a harmonic function. Then $f(z) = u + iv$ is analytic where v is harmonic conjugate of u. So, f(z) satisfies the C–R equation i.e.

$u_x = v_y$ $u_y = -v_x$

Since, $u_x = v_y \Rightarrow v_y = -\sin x\cosh y.$

Then, $v = -\sin x\sinh y + h(x)$

So, $v_x = -\cos x\sinh y + h'(x)$

Since $u_y = -v_x \Rightarrow -\cos x\sinh y = -\cos x\sinh y + h'(x)$

$\Rightarrow h'(x) = 0.$

Integrating we get, $h(x) = c$

Hence, the harmonic conjugate of u is,

$$v = -\sin x\sinh y + c.$$

**12. Prove that $u = y^3 - 3x^2y$ is a harmonic function. Determine its harmonic conjugate and find the corresponding analytic function f(z) in terms of z**

**Solution:** Here, $u = y^3 - 3x^2y$

So, $u_x = -6xy$, $u_{xx} = -6y$

And, $u_y = 3y^2 - 3x^2$ $u_{yy} = 6y$

Clearly, $u_{xx} + u_{yy} = 0$. So, u is a harmonic function. Therefore $f(z) = u + iv$ is analytic. So, f(z) satisfies C–R equation i.e.

$$u_x = v_y \text{ and } u_y = -v_x$$

Since, $u_x = v_y \Rightarrow v_y = -6xy$.

Integrating w. r. t. y we get,

$$v = -3xy^2 + h(x) \quad \ldots\ldots (i)$$

So, $v_x = -3y^2 + h'(x)$

Since, $u_y = -v_x \Rightarrow -3y^2 = -3y^2 + h'(x) \Rightarrow h'(x) = 3x^2$.

Then, $h(x) = x^3 + c$.

Hence (i) becomes, $v = -3xy^2 + x^3 + c$

Thus,

$$f(z) = u + iv = y^3 - 3x^2y - i\,3xy^2 + i\,x^3 + ic$$
$$= i\,[x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3] + ic$$
$$= i\,(x + iy)^3 + ic$$
$$= i\,z^3 + ic$$

**13. If $u = x^3 - 3xy^2$, show that there exists a function v(x, y) such that w = u + iv is analytic in a finite region.**

**Solution:** Here, $u = x^3 - 3xy^2$ ................. (i)

So, $u_x = 3x^2 - 3y^2$, $u_{xx} = 6x$

And, $u_y = -6xy$ $u_{yy} = -6x$

Clearly, $u_{xx} + u_{yy} = 0$.

Thus, u is harmonic function. Therefore, w = u + iv is analytic and satisfies the C–R equation, i.e. $u_x = v_y$ and $u_y = -v_x$.

Here, $v_y = u_x = 3x^2 - 3y^2$

Integrating w. r. t y then,

$$v = 3x^2y - y^3 + h(x) \quad \ldots\ldots (ii)$$

Then, $v_x = 6xy + h'(x)$

And, $u_y = -v_x \Rightarrow 6xy + h'(x) = 6xy$

$\Rightarrow h'(x) = 0$

Integrating w. r. t. x then $h(x) = c$.

Thus, (ii) becomes, $v = 3x^2y - y^3 + c$

Hence, $w = u + iv = x^3 - 3xy^2 + i\,(3x^2y - y^3 + c)$.



## OTHER IMPORTANT QUESTION FROM FINAL EXAM FOR PRACTICE

**2002 Q. No. 1(a); 2003 (Fall) Q. No. 1(a)**

**Define analytic function. Show that real and imaginary part of an analytic function, f(z) = u + iv which have continuous second order partial derivatives satisfy Laplace's equation.**

**OR**

**2004 (Fall) Q. No. 1(a)**

**What is meant by analyticity of the function f(z) at the point $z = z_0$? Show that Cauchy–Riemann equations are the necessary conditions for the function f(z) to be analytic.**

**OR**

**2011 (Spring) Q. No. 1(a)**

**Define analyticity of the complex valued function f(z). Show that the necessary condition for analyticity of the complex valued function f(z) = u + iv, is $u_x = v_y$ and $u_y = -v_x$ at any point (x, y).**

**Solution: Definition of analyticity of f(z):**

A complex valued function f(z) is called analytic at $z = z_0$ in D if f(z) is analytic in a neighborhood of $z_0$. A complex valued function f(z) is called analytic in D if it is defined and differentiable at each point of D.

**Second Part:** See theory part of this chapter.

**2004 (Spring) Q. No. 1(a)**

**Define derivative of an analytic function. Show that**

**(i) $\frac{1}{z-3}$, $(z \neq 3)$ is analytic. (ii) Re(z) is not analytic.**

**2004 (Fall) 1(a) OR**

**Define harmonic function. If u = sinhx siny, show that u is harmonic. Also find its harmonic conjugate and the corresponding analytic function.**

**Solution: Definition of harmonic function:**

Let, f(x) = u + iv. Then u is called harmonic function if the condition $u_{xx} + u_{yy} = 0$. In this case v is called harmonic conjugate of u(x, y). And, vice–versa.

**Problem Part:** Here, u = sin y sinh x

So, $u_x$ = siny coshx, $u_{xx}$ = siny sinh x

And, $u_y$ = cosy sinh x $u_{yy}$ = – sin y sinh x

Now,

$$u_{xx} + u_{yy} = \sin y \sinh x - \sin y \sinh x = 0.$$

Thus, u is a harmonic function. Therefore, f(z) = u + iv is analytic. So, it satisfies the C–R equation i.e.

$$u_x = v_y \quad u_y = -v_x$$

Since, $u_x = v_y$.

$\Rightarrow v_y = \sin y \cosh x$.

Then, $v = -\cos y \cosh x + h(x)$

So, $v_x = -\cos y \sinh x + h'(x)$

Since, $u_y = -v_x \Rightarrow \cos y \sinh x = \cos y \sinh x - h'(x)$

$\Rightarrow h'(x) = 0$.

Therefore, $h(x) = c$.

Thus the harmonic conjugate of u is,

$$v = -\cos y \cosh x + c.$$

**2005 (Fall) Q. No. 1(a)**

**Define analytic function f(z). Show that the necessary condition for analyticity of a function f(z) = u + iv at z = x + iy is to satisfy the Cauchy Riemann equation. Check analyticity of $f(z) = \bar{z}$.**

**Solution: First Part: See exam question solution 2002.**

**Problem Part:** Let $z = x + iy$. Then $f(z) = \bar{z} = x - iy$. $\Rightarrow u + iv = x - iy$.

Comparing the real and imaginary parts we get,

$u = x$ and $v = -y$.

So, $u_x = 1$ and $v_y = -1$.

Here, $u_x \neq u_y$. So, $f(z) = \bar{z}$ does not satisfy the C–R equation and so f(z) is not analytic function.

**2005 (Spring) Q. No. 1(a)**

**Define analyticity of a complex valued function f(z). Show that $u = x^3 - 3xy^2$ is harmonic and find harmonic conjugate of u.**

**Solution: Definition of harmonic function:**

Let, f(x) = u + iv. Then u is called harmonic function if the condition $u_{xx} + u_{yy} = 0$. In this case v is called harmonic conjugate of u(x, y) and vice–versa.

**Problem Part: See Exercise 2.3 Q. No. 14.**

**2006 (Fall) Q. No. 1(a)**

**Define Laplace equation and harmonic function. Determine a and b such that $u = ax^3 + by^3$ is harmonic and find harmonic conjugate.**

**Solution: Definition of harmonic function:**

Let, f(x) = u + iv. Then u is called harmonic function if the condition $u_{xx} + u_{yy} = 0$. In this case v is called harmonic conjugate of u(x, y) and vice–versa.

**Problem Part: See Exercise 2.3 Q. No. 11(a).**



**2006 (Spring) Q. No. 1(a)**

**Define analytic function. Show that the function $u(x, y) = 3x^2y + x^2 - y^3 - y^2$ is a harmonic function. Find a function v(x, y) such that u + iv is an analytic function.**

**Solution: Definition of harmonic function:**

Let, f(x) = u + iv. Then u is called harmonic function if the condition $u_{xx} + u_{yy} = 0$. In this case v is called harmonic conjugate of u(x, y) and vice–versa.

**Problem Part:** Solution: Here, $u = 3x^2y + x^2 - y^3 - y^2$

So, $u_x = 6xy + 2x$, $u_{xx} = 6y + 2$

And, $u_y = 3x^2 - 3y^2 - 2y$ $u_{yy} = -6y - 2$.

Clearly, $u_{xx} + u_{yy} = 0$. So, u is a harmonic function.

Therefore f(z) = u + iv be analytic. So, f(z) satisfies C–R equation i.e.

$u_x = v_y$ and $u_y = -v_x$

Since, $u_x = v_y \Rightarrow v_y = 6xy + 2x$.

Integrating w. r. t. y we get,

$v = 3xy^2 + 2xy + h(x)$ ...... (i)

So, $v_x = 3y^2 + 2y + h'(x)$

Since, $u_y = -v_x \Rightarrow 3x^2 - 3y^2 - 2y = -3y^2 - 2y - h'(x)$.

$\Rightarrow h'(x) = -3x^2$.

Integrating w.r.t. x then, $h(x) = -x^3 + c$

Then (i) becomes, $v = -3xy^2 - x^3 + c$

Thus,

$$f(z) = u + iv = 3x^2y + x^2 - y^3 - y^2 - i\,3xy^2 - i\,x^3 + ic$$

**2007 (Fall) Q. No. 1(a); 2016 Spring Q. No. 1(a)**

**Define analyticity of the complex valued function f(z). If $f(z) = z + \frac{1}{z}$, check analyticity of f(z) by using Cauchy Riemann equation.**

**Hint: Define analytic function. Verify the C–R equation by f(z).**

**2007 (Spring) Q. No. 1(a)**

**Define harmonic function. Verify that $u = x^2 - y^2 - y$ is harmonic and find conjugate harmonic v of u. Also find the corresponding analytic function.**

**Solution: Definition of harmonic function:**

Let, f(x) = u + iv. Then u is called harmonic function if the condition $u_{xx} + u_{yy} = 0$. In this case v is called harmonic conjugate of u(x, y) and vice–versa.

**Problem Part:** Here, $u = x^2 - y^2 - y$

So, $u_x = 2x$, $u_{xx} = 2$

And, $u_y = -2y - 1$  $u_{yy} = -2$.

Clearly, $u_{xx} + u_{yy} = 0$. So, u is a harmonic function.

...refore f(... u + iv is analytic. So, f(z) satisfies C–R equation i.e.

..., and $u_y = -v_x$

$\Rightarrow v_y = 2x$.

...n... y we get,

$v = ... h(x)$ …… (i)

So, $v_x = 2y + h'(x)$

Since, $u_y = -v_x \Rightarrow -2y - 1 = -2y - h'(x) \Rightarrow h'(x) = 1$.

Integrating, $h(x) = x + c$

Hence (i) becomes, $v = 2y + x + c$

Thus,

$$f(z) = u + iv = x^2 - y^2 - y + i\,3xy^2 - i\,2y + ix + ic.$$

**2008 (Fall) Q. No. 1(a)**

**State Cauchy Riemann equations and hence show that the function $f(z) = |z|^2$ is nowhere analytic.**

**Hint:** See theorem for first part.

**Second Part:** Here, $f(z) = |z|^2 = x^2 - y^2$.

Comparing it with $f(z) = u + iv$ we get, $u = x^2 - y^2$ and $v = 0$

So, $u_x = 2x$,  $v_y = 0$.

Thus $u_x \neq v_y$. This means f(z) does not satisfy the C–R equation, so f(z) is nowhere analytic.

**2008 (Fall) Q. No. 1(a) OR**

**Define a harmonic function and its harmonic conjugate. Show that u(x, y) = sinx coshy is harmonic and find its harmonic conjugate v(x, y) so that the function f(x, y) = u(x, y) + iv(x, y) is analytic.**

**Solution: Definition of harmonic function:**

Let, f(x) = u + iv. Then u is called harmonic function if the condition $u_{xx} + u_{yy} = 0$. In this case v is called harmonic conjugate of u(x, y) and vice–versa.

**Problem Part:** Here, u = sinx cosh x

So, $u_x = \cos x \cosh y$,  $u_{xx} = -\sin x \cosh y$

And, $u_y = \sin x \sinh y$  $u_{yy} = \sin x \cosh xy$

Now,

$$u_{xx} + u_{yy} = \sin x \cosh y - \sin x \cosh y = 0.$$

Thus, u is a harmonic function. Therefore f(z) = u + iv is analytic. So, f(z) satisfies the C–R equation i.e.

$u_x = v_y$  $u_y = -v_x$

Since, $u_x = v_y$. $\Rightarrow v_y = \cos x \cosh y$.



...tegrating w.r.t. y then,

$v = \cos x \sinh y + h(x)$

...fferentiating, $v_x = -\sin x \sinh y + h'(x)$

Since, $u_y = -v_x \Rightarrow \sin x \sinh y = \sin x \sinh y + h'(x)$.

$\Rightarrow h'(x) = 0$.

Integrating $h(x) = c$.

Therefore, $v = -\cos x \sinh y + c$.

Thus, $f(z) = u + iv = \sin x \cosh x - i\cos x \sinh y + ic$.

**2008 (Spring) Q. No. 1(a)**

**Define harmonic function. Prove that $u = y^3 - 3x^2y$ is a harmonic function. Determine its harmonic conjugate v of function f(z).**

**Solution: Definition of harmonic function:**

Let, f(x) = u + iv. Then u is called harmonic function if the condition $u_{xx} + u_{yy} = 0$ and such case v is called harmonic conjugate of u(x, y). And, vice–versa for v.

**Problem Part: See Exercise 2.3 Q. No. 13.**

**2009 (Fall) Q. No. 1(a)**

**Define harmonic function and its harmonic conjugate. Show that the function $u(x, y) = x^2 - y^2 - y$ is harmonic and find a harmonic conjugate v of u.**

**Solution: Definition of h... ...ction:**

Let, f(x) = u ... is called harmonic function if the condition $u_{xx} + u_{yy} = 0$. ...s called harmonic conjugate of u(x, y) and vice–versa.

**Problem Part: Se... ...lu... ...n of 2007 Spring.**

**2011 (Fall) Q. No. 1(a)**

**State and prove the necessary condition for a function f(z) = u + iv to be analytic.**

**Solution: See Problem Part of 2002.**

**2012 (Fall) Q. No. 1(a)**

**State Cauchy Riemann equation. Using these equations, show that the function $f(z) = z^3$ is everywhere analytic.**

**Hint: Solve as in 2008.**

**2015 (Fall) Q. No. 1(b)**

**Define Laplace equation. Test u = cosx cosh y is harmonic or not. If yes, find the harmonic function and the corresponding analytic function f(z).**

**Hint: Solve as in 2004 fall Q. No. 1(a) OR.**

**2017 (Fall) Q. No. 1(a)**

**Define harmonic function. If v = argz is harmonic? If yes, find a corresponding harmonic conjugate.**

**Hint: See definition and ex. 2.3 Q. 1(d).**

## SHORT QUESTIONS

**2004 Spring Q. No. 7(a)**

If $u = x^3 - 3xy^2$ show that u is harmonic.

Hint: See part of Ex 2.3 Q. 14.

**2006 Fall Q. No. 7(d)**

Check analyticity of $f(z) = z^2$.

Hint: See problem part of 2008 fall.

**2007 Spring Q. No. 7(b)**

Show that $\bar{z}$ is not analytic.

Hint: See part of solution of 2005 Fall.

**2008 Spring Q. No. 7(d)**

Find real part of $e^z$.

**Solution:** Here,

$$f(z) = e^z = e^{x+iy} = e^x (\cos y + i \sin y) = e^x \cos y + i\, e^x \sin y$$

Thus, $Re(e^z) = Re(e^x \cos y + i\, e^x \sin y) = e^x \cos y$.

**2009 Spring Q. No. 7(a)**

Express the function $f(z) = \sinh z$ in the form of $u + iv$.

**Solution:** Here,

$$f(z) = \sinh z = \frac{e^z - e^{-z}}{2} = \frac{e^{x+iy} - e^{-x-iy}}{2}$$

$$= \frac{e^x}{2}(\cos y + i \sin y) - \frac{e^{-x}}{2}(\cos y - i \sin y)$$

$$= \cos y \frac{e^x - e^{-x}}{2} + i \sin y \frac{e^x + e^{-x}}{2}$$

$$= \cos y \sinh x + i \sin y \cosh x$$

**2009 Spring Q. No. 7(b)**

Find the derivative of $f(z) = \frac{z - 2i}{z + 2i}$ at $z = i$

**2009 Spring Q. No. 7(b)**

Verify $u = e^x \sin y$ satisfies two dimensional Laplace.

**2016 Fall Q. No. 7(b)**

Examine whether $\bar{z}$ is analytic or not?

**2016 Spring Q. No. 7(a)**

If $z = u + iv$ is an analytic function then prove that u and v both satisfy Laplace equation.

□□□

*Unit 3*

# CONFORMAL MAPPING

## Exercise – 3.1

1. **Transform the rectangular region ABCD in z plane bounded by $x = 1$, $x = 3$, $y = 0$, and $y = 3$ under the transformation $f(z) = w = z + 2 + i$.**

**Solution:** Given, function is,

$$f(z) = w = z + 2 + i = x + iy + 2 + i = (x + 2) + i(y + 1)$$

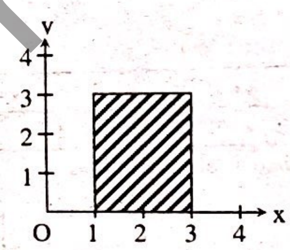
Since $f(z) = u + iv$ and $z = x + iy$. So, we get,

$$u = x + 2,\ v = y + 1$$

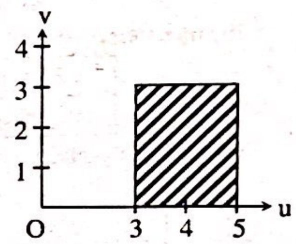
The transformation is as

| in z–plane | x = 1 | x = 3 | y = 0 | y = 3 |
|---|---|---|---|---|
| in w–plane | u = 3 | u = 5 | v = 1 | v = 4 |

Thus, the region in z–plane is mapped into w–plane as in figure.




z-plane




W-plane

2. **If $u = 2x^2 + y^2$ and $v = \frac{y^2}{x}$, show that the curves u = constant and v = constant cuts orthogonally at all intersections but that the transformation $w = u + iv$ is not conformal.**

**Solution:** Given, function is,

$$u = 2x^2 + y^2 \text{ and } v = \frac{y^2}{x}$$

Then, $u_x = 4x$, $u_y = 2y$, $v_x = -\frac{y^2}{x^2}$, $v_y = \frac{2y}{x}$

Here, $u_x = 4y \neq v_y$. So, by Cauchy–Riemann equation for analytic in a complex plane, the given function $f(z) = u + iv$ with $u = 2x^2 + y^2$, $v = \frac{y^2}{x}$, is not analytic. This implies $w = f(z)$ is not conformal mapping.

3. **For the conformal transformation $w = z^2$, find the magnification and the angle of rotation at $z = 1 + i$.**

**Solution:** Given that $w = z^2$ is conformal. So, $w = z^2$ is analytic. Therefore, w' exists and $w' = 2z$. At the point $z = 2 + i$,

$$w' = 2(2 + i) = 4 + 2i$$

Then, the magnification of the conformal mapping at z = 2 + i is,

$$|(w')_{at\ z=2+i}| = |4 + 2i| = \sqrt{16+4} = \sqrt{20}$$

And angle of rotation at z = 2 + i is,

$$\phi = \text{amp}\,(w'(2+i)) = \tan^{-1}\left(\frac{v}{u}\right)\Big|_{at\ z=2+i}$$

$$= \tan^{-1}\left(\frac{2}{4}\right) = \tan^{-1}\left(\frac{1}{2}\right).$$

**4. Determine the region of the mapping w = iz in the w plane corresponding to the region bounded by x = 0, y = 1, y = x.**

**Solution:** Given function is

$$f(z) = w = iz = i\,(x + iy) = -y + ix.$$

Since we know

$$f(z) = u + iv \quad \text{and} \quad z = x + iy.$$

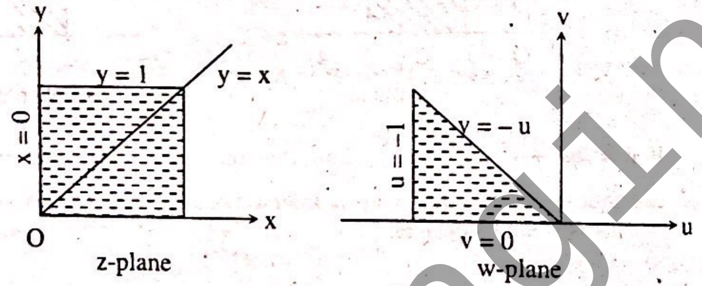
Then we observe

$$u = -y \quad \text{and} \quad v = x.$$

The transformation is as in table:

| in z–plane | x = 0 | y = 1 | y = x |
|---|---|---|---|
| in w–plane | v = 0 | u = –1 | v = –u |

The region is z–plane is mapped into w–plane as in figure.




**5. Determine the region of transformation w = 2z in the w plane where the region is z–plane enclosed by the lines x = 0, y = 0, x + y = 1.**

**Solution:** Given function is,

$$f(z) = w = 2z = 2(x + iy) = 2x + 2iy$$

Since we have

$$f(z) = u + iv \quad \text{and} \quad z = x + iy$$

By observing we get,

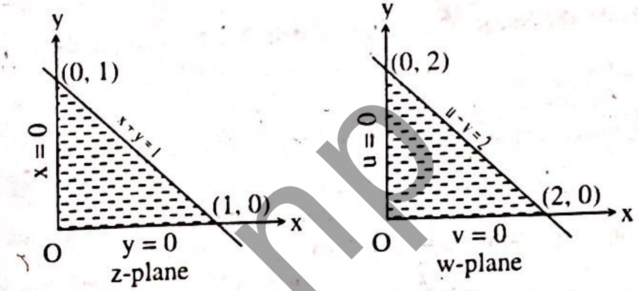
$$u = 2x, \quad v = 2y$$

The transformation is as tabled:

| in z–plane | x = 0 | y = 0 | x + y = 1 |
|---|---|---|---|
| in w–plane | u = 0 | v = 0 | u + v = 2 |

Thus, the region in z–plane is mapped into w–plane as in figure.






**6. Determine the region of transformation $w = 2ze^{i(\pi/3)}$, where the region in the z–plane be bounded by x = 0, x = 1, y = 0, y = 2.**

**Solution:** Given that

$$f(x) = w = 2ze^{i\pi/3} = 2z\,(\cos\pi/3 + i\sin\pi/3)$$

$$= 2z\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= (x + iy)\,(1 + i\sqrt{3})$$

$$= x - y\sqrt{3} + i\,(x\sqrt{3} + y)$$

Comparing it with f(z) = u + iv then,

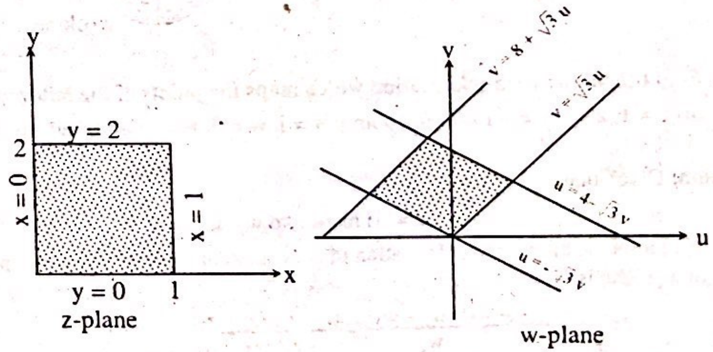
$$u = x - y\sqrt{3}, \quad v = x\sqrt{3} + y.$$

So that $x = \frac{u + \sqrt{3}\,v}{4}$.

Then the transformation form z ( = x + iy)–plane to w (= u + iv) plane is as tabled.

| in z–plane | x = 0 | x = 1 | y = 0 | y = 2 |
|---|---|---|---|---|
| in w–plane | $u = -\sqrt{3}\,v$ | $u = 4 - \sqrt{3}\,v$ | $v = \sqrt{3}\,u$ | $v = 8 + \sqrt{3}\,u$ |

Now, the region in z–plane is mapped into w–plane as in figure.




Since (i) $u = -\sqrt{3}\,v$ passes through (0, 0) and $(-\sqrt{3}, 1)$. (ii) $u = u - \sqrt{3}\,v$ passes through (4, 0) and $(0, 4/\sqrt{3})$ (iii) $v = \sqrt{3}\,u$ passes through (0, 0) and $(1, \sqrt{3})$ (iv) $v = 8 + \sqrt{3}\,u$ passes through (0, 8) and $(1, 8 + \sqrt{3})$.

**7. Obtain the image of the infinite strip $0 < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$.**

**Solution:** Given that

$$f(z) = w = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$$

Comparing it with $f(z) = u + iv$ then,

$$u = \frac{x}{x^2 + y^2},\ v = \frac{-y}{x^2 + y^2}$$

then

$$x = \frac{u}{u^2 + v^2},\ y = \frac{-v}{u^2 + v^2}$$

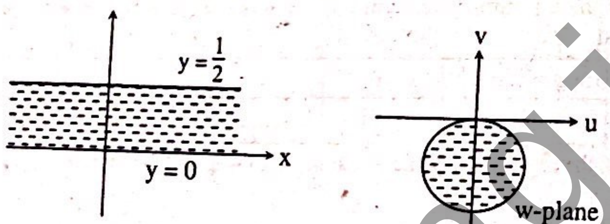
Here we have to observe the transformed image of the infinite strip $0 < y < 1/2$ under $w = 1/z$.

The transformation will as:

| in z–plane | y = 0 | y = 1/2 |
|---|---|---|
| in w–plane | v = 0 | $u^2 + (v + 1)^2 =$ |

Since $u^2 + (v + 1)^2 = 1$ is a circle with radius 1 and having center at (0, –1).

Thus, the region in z–plane and image image in w–plane is as in figure.




**8. Find the bilinear transformation which maps the points of the followings.**
**(a) z = 1, z = i, z = – i into the points w = i, w = 0, w = –i and find the image of |z| < 1.**

**Solution:** Given that

$z = 1, z = i, z = -i$ maps into $w = i, w = 0, w = -i$.

We know the bilinear transformation of z, $z_1$, $z_2$, $z_3$ in z–plane and w, $w_1$, $w_2$, $w_3$ in w–plane is

$$\frac{(w - w_1)(w_2 - w_3)}{w_2 - w_1} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \quad \dots (i)$$

Substituting the given values in (i) then

$$\frac{(w - i)(0 + i)}{(w + i)(0 - i)} = \frac{(z - 1)(i + i)}{(z + i)(i - 1)}$$



$$\Rightarrow \frac{-(w - i)}{(w + i)} = \frac{(z - 1)(2i)}{(z + i)(i - 1)}$$

$$\Rightarrow -(w - i)(z + i)(i - 1) = (w + i)(z - 1)(2i)$$

$$\Rightarrow -w[(z + i)(i - 1) + (z - 1)2i] = i(z - 1)2i - i(z + i)(i - 1)$$

$$\Rightarrow -w(zi - z - 1 - i + 2iz - 2i) = -2z + 2 - i(iz - z - 1 - i)$$

$$\Rightarrow w[z + 1 + i(3 - 3z)] = -2z + 2 + z + iz + i - 1$$

$$\Rightarrow w = \frac{(1 - z) + i(z + 1)}{(z + 1) + i(3 - 3z)} \quad \dots (i)$$

This is required bilinear transformation.

**Note:** To obtain the book's answer process with z = 1, i, –1 and w = i, 0, –i.

**Solution:** Given that the points z = 1, i, –1 maps onto the points w = i, 0, –i. Since the transformation preserves by

$$\frac{(w - w_1)(w - w_3)}{(w_1 - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_3)}$$

i.e. $$\frac{(w - i)(0 + i)}{(w + i)(0 - i)} = \frac{(z - 1)(i + 1)}{(z + 1)(i - 1)}$$

$$\Rightarrow \frac{(w - i)}{-(w + i)} = \frac{iz - i + z - 1}{iz - z + i - 1}$$

$$\Rightarrow w(iz - z + i - 1 + iz - i + z - 1) = i(iz - z + i - 1 - iz + i - z + 1)$$

$$\Rightarrow w(2iz - 2) = -2iz - 2$$

$$\Rightarrow w = \frac{-(iz + 1)}{iz - 1} = \frac{-i(z - i)}{i(z + i)} = \frac{-(z - i)}{(z + i)} = \frac{i - z}{z + 1}$$

This is required linear transformation.

And,

$$w = \frac{i - z}{z + 1} \Rightarrow z(w + 1) = i - iw$$

$$\Rightarrow z = \frac{i(1 - w)}{1 + w}$$

Given region is |z| < 1. So

$$\left| i\left(\frac{1 - w}{1 + w}\right) \right| < 1. \Rightarrow |1 - w| < |1 = w|$$

$$\Rightarrow (1 - w)(1 - \bar{w}) < (1 + w)(1 + \bar{w})$$

$$\Rightarrow 1 - w - \bar{w} + w\bar{w} < 1 + w + \bar{w} + w\bar{w}$$

$$\Rightarrow 2(w + \bar{w}) > 0$$

$$\Rightarrow 2(2u) > 0$$

$$\Rightarrow u > 0$$

Thus, the image of |z| < 1 under $w = \frac{i - z}{z + i}$ is the right half plane of w–plane.

**(b) $z = 0, z = 1, z = \infty$ into the points $w = -3, w = -1, w = 1$ and find the fixed point of the transformation.**

**Solution:** Given that

$$z = 0, 1, \infty \text{ and } w = -3, -1, 1$$

We know the bilinear transformation of $z, z_1, z_2, z_3$ in z–plane and $w, w_1, w_2, w_3$ in w–plane is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \quad \ldots \text{(i)}$$

Substituting the given values in (i) then

$$\frac{(w+3)(-2)}{(w-1)(2)} = \frac{z(1-\infty)}{(z-\infty)} = \lim_{r\to\infty} \frac{z(1-r)}{z-r} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$\Rightarrow \frac{w+3}{w-1} = \lim_{r\to\infty}\left(\frac{-z}{-1}\right) = -z$$

$$\Rightarrow w + zw = z - 3$$

$$\Rightarrow w = \frac{z-3}{z+1}$$

This is required bilinear transformation for the fixed point.

Here, $w = \dfrac{z-3}{z+1}$

For the fixed point,

$$z = \frac{z-3}{z+1} \Rightarrow z^2 + 3 = 0 \Rightarrow z = \pm i\sqrt{3}$$

Thus the fixed points are $z = \pm i\sqrt{3}$.

**(c) $z = 0, z = -1, z = i$ into the points $w = i, w = 0, w = \infty$ and find the image of the unit circle $|z| = 1$.**

**Solution:** Given that

$$z = 0, -1, i \text{ and } w = i, 0, \infty.$$

We know the bilinear transformation of $z, z_1, z_2, z_3$ in z–plane and $w, w_1, w_2, w_3$ is,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \quad \ldots \text{(i)}$$

Substituting the given values in (i) then

$$\frac{(w-i)(0-\infty)}{(w-\infty)(0-i)} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$\Rightarrow \lim_{r\to\infty} \frac{(w-i)r}{(w-r)i} = \frac{z(1+i)}{(z-i)}$$

$$\Rightarrow \frac{z(1+i)}{z-i} = \lim_{r\to\infty} \frac{(w-i)r}{(w-r)i} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{r\to\infty} \frac{(w-i)}{-i}$$

$$= \frac{(w-i)}{-i} = i(w-i) = iw + 1$$



$$\Rightarrow \frac{z(1+i)}{(z-i)} - 1 = iw$$

$$\Rightarrow w = \frac{1}{i}\left[\frac{z(1+i-1)+i}{z-i}\right] = \frac{1}{i}\left(\frac{iz+i}{z-i}\right) = \frac{z+1}{z-i}$$

This is required bilinear transformation.

Here, $w = \dfrac{z+1}{z-i} \Rightarrow z(w-1) = 1 + iw$

$$\Rightarrow z = \frac{1+iw}{w-1}$$

Given region is $|z| = 1$. So,

$$\left|\frac{1+iw}{w-1}\right| = 1 \Rightarrow |i(-i+w)| = |w-1|$$

$$\Rightarrow (-i+w)(-i+\bar{w}) = (w-1)(\bar{w}-1)$$

$$\Rightarrow -1 - i(\bar{w}+w) + w\bar{w} = w\bar{w} - w - \bar{w} + 1$$

$$\Rightarrow -2iu = -2u + 2$$

$$\Rightarrow -iu = -u + 2$$

$$\Rightarrow iu = u - 2$$

This means the image of $|z| = 1$ is $v = u - 2$.

**9. Show that the transformation $w = \dfrac{2z+3}{z-4}$ maps the circle $x^2 + y^2 - 4x = 0$ onto the straight line $4u + 3 = 0$ in w plane.**

**Solution:** Here,

$$w = \frac{2z+3}{z-4}$$

$$\Rightarrow wz - 4w = 2z + 3$$

$$\Rightarrow z = \frac{4w+3}{w-2}$$

$$\Rightarrow x + iy = \frac{4u+4iv+3}{u+iv-2} \times \frac{u-2-iv}{u-2-iv}$$

$$= \frac{4u^2 - 8u - 4iuv + 4iuv - 8iv + 4v^2 + 3u - 6 - 3iv}{(u-2)+v^2}$$

$$= \frac{4u^2 - 5u + 4v^2 - 6}{u^2 - 4u + v^2 + 4} + i\left(\frac{-11v}{u^2 - 4u + v^2 + 4}\right)$$

So,

$$x = \frac{4u^2 - 5u + 4v^2 - 6}{u^2 - 4u + v^2 + 4} \qquad y = \left(\frac{-11v}{u^2 - 4u + v^2 + 4}\right)$$

Now,

$$x^2 + y^2 - 4x = 0$$

$\Rightarrow$ $(4u^2 - 5u + 4v^2 - 6)^2 + (11v)^2 - 4(4u^2 - 5u + 4v^2 - 6)(u^2 - 4u + v^2 + 4) = 0$

$\Rightarrow$ $16u^4 + 25u^2 + 16v^4 + 36 - 40u^3 + 32u^2v^2 - 48u^2 + 60u - 40v^2u - 48v^2 + 121v^4 - 16u^4 + 64u^3 - 16u^2v^2 - 64u^2 + 20u^3 - 80u^2 + 20v^2u + 80u - 16u_2v^2 + 64uv^2 - 16v^2 - 64v^2 + 24u^2 - 96u + 24v^2 + 96 = 0.$

$\Rightarrow$ $-143u^2 + 137v^4 + 132 + 44u^3 + 44u - 20v^2u + 24v^2 + 64uv^2 = 0.$

**10. Show that the transformation $w = z + \frac{1}{z}$ transform r = constant in the z plane into a family of ellipses in the w plane. Also show that the point at which $w = z + \frac{1}{z}$ is not conformal.**

**Solution:** Given that

$$w = z + \frac{1}{z}$$

Since we have,

$$w = u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

then we get

$$u = a\cos\theta,\ v = b\sin\theta \quad \text{for } a = r + \frac{1}{r},\ b = r - \frac{1}{r}$$

Since $|z| = r$ = constant with $r \neq 1$.

Clearly a > b. This means the circle $|z| = r$ mapped onto the ellipse in w-plane.

Next,

$$w' = 1 - \frac{1}{z^2} = \frac{z^2 - 1}{z^2} = \frac{(z-1)(z+1)}{z^2}$$

Clearly $w' = 0$ at $z = \pm 1$. So, the function w is not conformal at $z = \pm 1$.

**11. Find the bilinear transformation of $w = \frac{5 - 4z}{4z - 2}$, mapping the circle $|z| = 1$ in w plane.**

**Solution:** Given that

$$w = \frac{5 - 4z}{4z - 2} \quad \dots \text{(i)}$$

$\Rightarrow$ $(4z - 2)w = 5 - 4z$

$\Rightarrow$ $4z(w + 1) = 5 + 2w$

$\Rightarrow$ $z = \frac{5 + 2w}{4(w + 1)}$

Also, given that (i) maps the circle $|z| = 1$ is w–plane, so

$$\left|\frac{5 + 2w}{4(w+1)}\right| = 1$$



$\Rightarrow$ $|5 + 2w| = |4(w + 1)|$

$\Rightarrow$ $(5 + 2w)(5 + 2\bar{w}) = 4(w + 1)\,4(\bar{w} + 1)$

$\Rightarrow$ $25 + 4w\bar{w} + 10(w + \bar{w}) = 16(w\bar{w} + 1 + w + \bar{w})$

$\Rightarrow$ $25 + 4(u^2 + v^2) + 20u = 16(u^2 + v^2 + 1 + 2u)$

$\Rightarrow$ $12u^2 + 12v^2 + 12u - 9 = 0$

$\Rightarrow$ $u^2 + v^2 + u - \frac{3}{4} = 0$

which is required transformation.

## OTHER IMPORTANT QUESTION FROM FINAL EXAM FOR PRACTICE

<u>2015 Spring 2(b)</u>

**Determine the region of $w = e^{\frac{i\pi}{4}}$ in the w-plane corresponding to the triangle region bounded by the lines x = 0, y = 0, and x + y = 1 in the z-plane.**

<u>2016 Spring 2(a) OR</u>

**Define conformal mapping. Name the type of conformal mappings. Translate the rectangular region ABCD in Z plane bounded by x = 1, x = 3, y = 0 and y = 3 under the transformation w = z + (2 + i). Illustrate with figure also.**

<u>2017 Fall 1(b) OR</u>

**Find the fixed points and the normal form of the bilinear transformation $w = \frac{z-1}{z+1}$. Also, determine the nature of this transformation.**

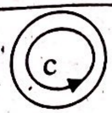
□□□

## Unit 4

# COMPLEX INTEGRATION

### Contour:

A simple closed path is a contour.




### Contour Integral

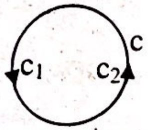
An integral over a contour, is a contour integral. If a curve c has a closed path then the integration over the curve c, is called line integration. It is denoted as

$\int_c f(z)\,dz$ where z be any point in the curve.

### Properties of Contour Integrals

(1) If the contour c is made up by two arcs $c_1$ and $c_2$ then

$$\int_c f(z)\,dz = \int_{c_1} f(z) + \int_{c_2} f(z)\,dz$$




(2) If c* be a reversed direction of a given contour c then,

$$\int_c f(z)\,dz = -\int_{c^*} f(z)\,dz$$

(3) For a and b are any constants,

$$\int_c [af_1(z) + b\,f_2(z)]\,dz = a\int_c f_1(z)\,dz + b\int_c f_2(z)\,dz$$

(4) Let $l$ be the length of the curve and the constant f(z) is bounded by a constant M i.e. $|f(z)| \le M$, for z lies in a closed contour C then,

$$\left|\int_c f(z)\,dz\right| \le lM$$

### Simply Connected Domain

A domain D is called simply connected if any curve C which lies in D is closed without having any passout of this domain.


### Multiply Connected Domain

A domain D is called multiply connected if it is not a simply connected.


### Indefinite Integration of an analytic function

Let f(z) is analytic in a simply connected domain D. Then, there is an analytic function f(z) such that F(z) = f '(z). So. $\int_c f'(z)\,dz = f(z)$.

If there is two points $z_1$ and $z_2$ in D then,

$$\int_{z_1}^{z_2} f(z)\,dz = f(z_2) - f(z_1)$$



### Green's Theorem in Plane

Let R be a closed bounded region in xy–plane whose boundary c consists of finitely many smooth curves. Let, $F_1(x, y)$ and $F_2(x, y)$ are functions that are continuous and have continuous partial derivatives everywhere in R.

Then,

$$\oint_c (F_1dx + F_2dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy$$

*Note:* (i) $\oint_c (udx - vdy) = \iint_R (-v_x - u_y)\,dxdy$

(ii) $\oint_c (udy + vdx) = \iint_R (u_x - v_y)\,dxdy$

### Cauchy's Integral Theorem

*If f(z) is analytic in a simply connected domain D then $\oint_c f(z)\,dz = 0$, for every simple closed path c in D.*

**Proof:** Let f(z) = u + iv and z = x + iy.

Then,

$$\oint_c f(z)\,dz = \oint_c (u + iv)\,(dx + idy)$$

$$= \oint_c (udx - vdy) + i\oint_c (udy + vdx) \quad \text{................ (1)}$$

Since, f(z) is analytic in D. So, it satisfies the C–R equation. That means $u_x$, $u_y$, $v_x$ and $v_y$ exist and

$u_x = v_y$ ... (2) and $u_y = -v_x$. ... (3)

By Green's theorem,

$$\oint_c (udx - vdy) = \iint_R (-v_x - u_y)\,dx\,dy = \iint_R (0)\,dx\,dy = 0$$

[Using (3)]

and, $$\oint_c (udy + vdx) = \iint_R (u_x - v_y)\,dx\,dy = \iint_R (0)\,dx\,dy = 0$$

[Using (2)]

Now, using these results in (1) then,

$$\oint_c f(z)\,dz = 0 + i.0 = 0.$$

### Independence of Paths:

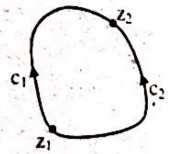
*Let, $z_1$ and $z_2$ are two points in a domain D. Then an integral of f(z) is said to have independent of path in a domain D if the value of the points depends only on the initial point and the terminal point but not a choice of the path in D.*

**Theorem:**

***If f(z) is analytic in a simply connected domain D then the Integral of f(z) is independent of path in D.***

**Proof:** Let, $z_1$ and $z_2$ are any two points in a domain D. Consider two paths $c_1$ and $c_2$ joining $z_1$ and $z_2$ in D without further points.

Assume that $c_2$* be the reversed path of $c_2$. Then the integration of a function f(z) from $z_1$ to $z_2$ over $c_1$ and from $z_2$ to $z_1$ over $c_2$*, is a simple closed path. Then by Cauchy's Integral Theorem,




$$\oint_{c_1} f(z)\,dz + \oint_{c_2^*} f(z)\,dz = 0$$

$$\Rightarrow \int_{c_1} f(z)\,dz = -\oint_{c_2^*} f(z)\,dz \qquad \text{........ (i)}$$

Since, the integration of a function over a reversed path takes negative value. Since, $c_2$* is the reversed path of $c_2$. So,

$$\int_{c_2} f(z)\,dz = -\int_{c_2^*} f(z)\,dz$$

Then, from (i),

$$\int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz.$$

This shows that the path $c_1$ and $c_2$ are independent.

### Extension of Cauchy Integral Theorem

Suppose f(z) be an analytic function in the multiply connected region R. Let R consists a simple closed curve C in R and inside it, two non–intersecting simple closed curves $c_1$ and $c_2$ are lying in c. Since, the curve c is introduced as a simple closed curve by joining the curve c and $c_1$ by a line segment and c and $c_2$ by another line segment.

Then the curve c with $c_1$ and $c_2$ becomes a closed curve without cross itself. Then, the integration becomes 0 (zero) by Cauchy Integral Theorem.

This proves that if a closed contour has simple closed curves inside it and join them original path. Then,

$$\int_{c} f(z)\,dz = \oint_{c_1} f(z)\,dz + \oint_{c_2} f(z)\,dz = 0$$

If the curve c have finitely many contours (n–contour) and connected each of them with original curve c then the contour again a simple closed and have value zero.

### Cauchy's Integral Formula

***Let f(z) be analytic in a simply connected domain D. Then, for any point $z_0$ in D and any simple closed path c in D that encloses $z_0$ such that.***

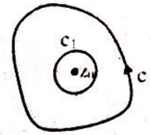
$$\oint_{c} \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0)$$

***where, the path c is taken in counterclockwise direction.***



**Proof:** Let c be a simple closed path in D. Consider a circle $c_1$ in c having radius r and centre at $z_0$.

Since, the function f(z) is analytic in D then $\frac{f(z)}{z - z_0}$ is analytic in the simply connected region bounded by c and $c_1$. Therefore by Cauchy's Integral Theorem,




$$\oint_{c} \frac{f(z)}{z - z_0}\,dz - \oint_{c_1} \frac{f(z)}{z - z_0}\,dz = 0.$$

$$\Rightarrow \oint_{c} \frac{f(z)}{z - z_0}\,dz = \oint_{c_1} \frac{f(z)}{z - z_0}\,dz \qquad \text{............ (i)}$$

Put $z - z_0 = re^{i\theta}$ on $c_1$. So, $dz = r\,ie^{i\theta}\,d\theta$. Since $c_1$ is a closed contour so, the function varies from 0 to $2\pi$. Then,

$$\oint_{c_1} \frac{dz}{z - z_0}\,dz = \oint_{0}^{2\pi} \frac{r\,i\,e^{i\theta}\,d\theta}{r\,e^{i\theta}}$$

$$= i.2\pi \qquad \text{............ (ii)}$$

Since, f(z) is analytic, so it is continuous at $z = z_0$. Then by definition, for any $\varepsilon > 0$ there exists $\delta > 0$ such that,

$$|z - z_0| < \delta \quad \Rightarrow |f(z) - f(z_0)| < \varepsilon \qquad \text{............ (iii)}$$

Since,

$$\oint_{c_1} \frac{f(z)}{z - z_0}\,dz = \oint_{c_1} \frac{f(z_0)}{z - z_0}\,dz + \oint_{c_1} \frac{f(z) - f(z_0)}{z - z_0}\,dz \qquad \text{[using (ii)]}$$

$$= 2\pi i\, f(z_0) + \oint_{c_1} \frac{f(z) - f(z_0)}{z - z_0}\,dz \qquad \text{.......... (iv)}$$

Here,

$$\left| \oint_{c_1} \frac{f(z) - f(z_0)}{z - z_0}\,dz \right| \le \oint_{c_1} \frac{|f(z) - f(z_0)|}{|z - z_0|}\,|dz| < \frac{\varepsilon}{\delta}.2\pi\delta = 2\pi\varepsilon$$

Since, ε be any arbitrary positive number. So, if we made ε is sufficiently small then,

$$\oint_{c_1} \frac{f(z) - f(z_0)}{z - z_0}\,dz \approx 0 \qquad \text{............ (v)}$$

Hence, from (i), (iv) and (v) then we get,

$$\oint_{c} \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0).$$

**Cauchy's Integral Formula for Derivative of an Analytic Function.**

**Statement:** ***Let f(z) be an analytic function on and in a simple closed curve c, enclosed in a simply connected region R. Then, if $z_0$ be a point in c, we get***

$$\oint_{c} \frac{f(z)}{(z - z_0)^{n+1}}\,dz = \frac{2\pi i}{n!}\, f^{(n)}(z_0) \qquad \text{for } n \ge 0.$$

## Exercise – 4.1

**Integrate the given integral in the given path.**

**1.** $\int_c Re(z)\,dz$**, where c is the shortest path from 1 + i to 3 + 2i.**

**Solution:** Here, the curve have a path from (1, 1) i.e. 1 + i to (3, 2) i.e. 3 + 2i.

To evaluate the line integral, we choose a path $c_1$ and $c_2$ as shown in figure.

Let z = x + iy. So, Re(z) = x and dz = dx + idy.

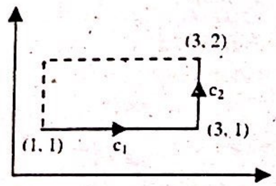
Now along the path $c_1$,

$$\int_{c_1} Re(z)\,dz = \int_{c_1} x(dx + idy)$$

$$= \int_1^3 x.dx$$

*Since in $c_1$, x varies from 1 to 3 where y remains as constant having value y = 1, so dy = 0*

$$= \left[\frac{x^2}{2}\right]_1^3 = 4.$$




And,

$$\int_{c_2} Re(z)\,dz = \int_{c_2} x\,(dx + idy)$$

$$= \int_1^2 3i\,dy,$$

*since in $c_2$, y varies from 1 to 2 where x = 3, is constant so dx = 0.*

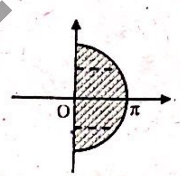
$$= 3i\int_1^2 dy = 3i\,(2 - 1) = 3i$$

Hence, $\int_c Re(z)\,dz = \int_{c_1} Re(z)\,dz + \int_{c_2} Re(z)\,dz = 4 + 3i.$

**2.** $\int_c \sin^2 z\,dz$ **where c from $-\pi i$ along $|z| = \pi$ to $\pi i$ in the right half plane.**

**Solution:** Here,

$$\int_c \sin^2 z\,dz = \int_{-\pi i}^{\pi i}\left(\frac{1-\cos 2z}{2}\right)dz$$




Since the region is symmetry from x–axis. So,

$$\int_c \sin^2 z\,dz = \int_{-\pi i}^{\pi i}\left(\frac{1-\cos 2z}{2}\right)dz$$

$$= 2\int_0^{\pi i}\left(\frac{1-\cos 2z}{2}\right)dz$$



$$= 2\left|\frac{z}{2} - \frac{\sin 2z}{4}\right|_0^{\pi i} = 2\left|\frac{\pi i}{2} - \frac{\sin 2\pi i}{4}\right| = \pi i - \frac{i\sinh 2\pi}{2}$$

**3.** $\int Re(z^2)\,dz$**, where c is the unit circle, counterclockwise.**

**Solution:** Here, $z = e^{i\theta}$ (being r = 1) then $z^2 = (e^{i\theta})^2 = e^{i2\theta} = \cos 2\theta + i\sin 2\theta$.

So, $Re(z^2) = \cos 2\theta$ and $dz = ie^{i\theta}\,d\theta$.

Since c is unit circle in which θ varies from θ = 0 to θ = 2π. Then,

$$\int_c Re(z^2)\,dz = \int_c \cos 2\theta\, e^{i\theta}\, i\,d\theta$$

$$= i\int_0^{2\pi} e^{i\theta}\cos 2\theta\,d\theta$$

$$= i\int_0^{2\pi} e^{i\theta}\left(\frac{e^{i2\theta}+e^{-i2\theta}}{2}\right)d\theta$$

$$= i\int_0^{2\pi}\frac{e^{i3\theta}+e^{-i\theta}}{2}\,d\theta$$

$$= \frac{i}{2}\left[\frac{e^{i3\theta}}{3i} - \frac{e^{-i\theta}}{i}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[\frac{e^{i6\pi}-e^0}{3} - \{e^{-2\pi i} - e^0\}\right]$$

$$= \frac{1}{2}\left[e^{i6\pi} - 1 - e^{-i2\pi} + 1\right]$$

$$= \frac{1}{2}\left[\cos 6\pi + i\sin 6\pi - (\cos 2\pi - i\sin 2\pi)\right]$$

$$= \frac{1}{2}\left[1 + 0 - 1 + 0\right]$$

$$= 0$$

**4.** $\int_c ze^{z^2}\,dz$**, where c from 1 along the axes to i.**

**Solution:** Here,

$$\int_c ze^{z^2}\,dz = \int_1^i ze^{z^2}\,dz = \left[\frac{e^{z^2}}{2}\right]_1^i = \frac{e^{-1}-e^1}{2} = -\frac{(e^1-e^{-1})}{2} = -\sinh 1$$

**5.** $\int_c \cos z\,dz$**, where c the semicircle $|z| = \pi$, $x \geq 0$, from $-\pi i$ to $\pi i$.**

**Solution:** Given that c is a semicircle $|z| = \pi$, $x \geq 0$. So the region is symmetrical about x–axis. So,

$$\int_c \cos z\,dz = \int_{-\pi i}^{\pi i}\cos z\,dz = 2\int_0^{\pi i}\cos z\,dz$$

$$= 2[\sin z]_0^{\pi i}$$

$$= 2\sin \pi i = 2i \sinh \pi \qquad [\because \sin iz = i \sinh z]$$

**6.** $\int_c \sec^2 z\, dz$**, where c be any path from** $\frac{\pi i}{4}$ **to** $\frac{\pi}{4}$ **in the unit disk.**

**Solution:** Here,

$$\int_c \sec^2 z\, dz = \int_{\pi i/4}^{\pi/4} \sec^2 z\, dz = [\tan z]_{\pi i/4}^{\pi/4}$$

$$= \tan\frac{\pi}{4} - \tan\frac{\pi i}{4}$$

$$= 1 - \tan\frac{\pi i}{4} = 1 - i\tanh\frac{\pi}{4}.$$

> **Remember that**
> **$\sin iz = i \sinh z$**
> **$\cos iz = \cosh z$**
> **$\therefore \tan iz = i \tanh z$**

## Exercise – 4.2

**1. Integrate f(z) counterclockwise around the unit circle, indicating whether Cauchy's Integral Theorem applies.**

**a.** $f(z) = e^{-z^2}$

**Solution:** Since, the exponential function is differential everywhere. So, $f(z) = e^{-z^2}$ is analytic. Therefore, by Cauchy's Integral Theorem

$$\int_c f(z)\, dz = \int_c e^{-z^2} dz = 0.$$

**b.** $f(z) = \frac{1}{|z|^2}$

**Solution:** Given, that $f(z) = \frac{1}{|z|^2}$. Clearly, this function does not exists at z = 0. Since the point z = 0 lies in the unit circle. That means the function is not analytic in the unit circle. So, the Cauchy's Integral Theorem is not applicable.

And by Cauchy's Integral formula,

$$\oint_c f(z)\, dz = \oint_c \frac{dz}{z^2} = 2\pi i f'(0) \qquad \left[\because \oint_c \frac{dz}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{f(z)}{(z-z_0)^n} dz$ with $\oint_c \frac{dz}{z^2}$ then we get, f(z) = 1. Then f '(0) = 0.

Therefore,

$$\oint_c \frac{dz}{z^2} = 0.$$



**c.** $f(z) = \frac{1}{2z-1}$

**Solution:** Given, that $f(z) = \frac{1}{2z-1}$. Clearly, f(z) does not exist at $z = \frac{1}{2}$, which lies in the unit circle. This means f(z) is not analytic in the unit circle. So, the Cauchy's Integral Theorem is not applicable.

And by Cauchy's Integral formula,

$$\oint_c f(z)\, dz = \oint_c \frac{1}{2z-1} = \frac{1}{2}\oint_c \frac{1}{z-1/2} dz$$

$$= \frac{1}{2}.\, 2\pi i \qquad \left[\because \oint_c \frac{dz}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

$$= \pi i$$

**d.** $f(z) = \frac{1}{z^4 + 1.1}$

**Solution:** Given that the closed contour C is a circle that has center at origin O and radius is 1. Since any root of 1.1 is greater than 1. This means,

$$f(z) = \frac{1}{z^4 + 1.1}$$

has no poles in C. Therefore f(z) is analytic in C. Therefore, by Cauchy Integral Theorem,

$$\oint_c f(z)\, dz = \oint_c \frac{dz}{z^4 + 1.1} = 0.$$

**e.** $f(z) = \text{Im}(z)$

**Solution:** Here, f(z) = Im z

We know, $z = e^{i\theta} = \cos\theta + i\sin\theta$. Then, $dz = ie^{i\theta} d\theta$

And, $\text{Im}(z) = \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

Given that f(z) is defined in unit circle. So, θ varies from θ = 0 to θ = 2π.

Now,

$$\oint_c f(z)\, dz = \int_0^{2\pi} \frac{e^{i\theta} - e^{-i\theta}}{2i}.\, i\, e^{i\theta} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (e^{2i\theta} - 1)\, d\theta$$

$$= \frac{1}{2}\left[\frac{e^{2i\theta}}{2i} - \theta\right]_0^{2\pi} = \frac{1}{2}\left[\frac{e^{4\pi i}}{2i} - \frac{1}{2i} - 2\pi\right]$$

$$= \frac{1}{2}\left[\frac{(\cos 4\pi + i\sin 4\pi)}{2i} - \frac{1}{2i} - 2\pi\right]$$

$$= \frac{1}{2}\left[\frac{1}{2i} - \frac{1}{2i} - 2\pi\right] = -\pi$$

**2. Evaluate: a.** $\oint_c \frac{dz}{z-3i}$ **where, c is the circle $|z| = \pi$, counterclockwise.**

**Solution:** Given that, $\oint_c \frac{dz}{z-3i}$ where, c is the circle $|z| = \pi$ moves counterclockwise.

Clearly, the given integrand function does not exist at $z = 3i$; which lies in the given circle. So, the function is not analytic in the given circle. Then, by Cauchy's Integral Formula,

$$\oint_c \frac{dz}{z-3i} = 2\pi i\, f(3i) \quad \dots (1) \qquad \left[\because \oint_c \frac{f(z)}{z-z_0}dz = 2\pi i\, f(z_0)\right]$$

Comparing $\oint_c \frac{dz}{z-3i}$ with $\oint_c \frac{f(z)}{z-z_0}dz$ then we get, $z_0 = 3i$ and $f(z) = 1$.

Then $f(3i) = 1$. Therefore (1) becomes

$$\oint_c \frac{dz}{z-3i} = 2\pi i.$$

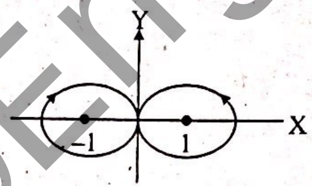
**b.** $\oint_c \frac{e^z}{z}dz$, **where c consists of $|z| = 2$, counterclockwise and $|z| = 1$ clockwise.**

**Solution:** Given that, $\oint_c \frac{e^z}{z}dz$, where c consists of $|z| = 2$ counterclockwise and $|z| = 1$ clockwise.

Clearly, the function does not exist at $z = 0$, which does not lie in region bounded by the circle $|z| = 2$ (CCW) and $|z| = 1$(CW). Thus, the function is analytic in the given region. Hence, by Cauchy's Integral Theorem,

$$\oint_c f(z)\,dz = \oint_c \frac{e^z}{z}dz = 0.$$

**c.** $\oint_c \frac{dz}{z^2-1}$, **where c is**




**Solution:** Given that

$$I = \oint_c \frac{dz}{z^2-1} \text{ where c is given in figure.}$$

$$= \oint_c \frac{dz}{(z-1)(z+1)}$$



$$= \frac{1}{2}\oint_c \left(\frac{1}{z-1} - \frac{1}{z+1}\right)dz \quad \dots (1)$$

Let, $I_1 = \oint_c \frac{dz}{z-1}$

Here the integrand function has pole at $z = 1$ of simple order. Since point $z = 1$ lies in $c_1$ but not lie in $c_2$. So, $z = 1$ lies in c. Therefore, by Cauchy integral formula,

$$I_1 = \oint_c \frac{dz}{z-1} = 2\pi i$$

Next, let

$$I_2 = \oint_c \frac{dz}{z+1}$$

Here the integrand function has pole at $z = -1$ of simple order. Since the point does not lie in $c_1$ but lie in $c_2^*$ where $c_2^*$ is negative directional contour of $c_2$. Therefore, by Cauchy Integral Formula,

$$I_2 = \oint_c \frac{dz}{z+1} = -2\pi i$$

Now, (1) becomes,

$$I = \frac{1}{2\pi i}[2\pi i - (-2\pi i)] = 2\pi i.$$

**d.** $\oint_c \frac{dz}{z^2+1}$ **where c is the circles (i) $|z + i| = 1$ (ii) $|z - i| = 1$, counterclockwise.**

**Solution:** Here, $\oint_c \frac{dz}{z^2+1} = \oint_c \frac{dz}{z^2-i^2} = \oint_c \frac{dz}{(z+i)(z-i)} = \frac{1}{2}\left[\oint_c \frac{dz}{z-i} - \oint_c \frac{dz}{z+i}\right] \dots (1)$

Clearly, the given function does not exist at $z = i$ and $z = -i$.

(i) Since, the point $z = i$ does not lie in the circle $|z + i| = 1$.

So, the function $\left(\frac{1}{z-i}\right)$ is analytic in $|z + i| = 1$. So, by Cauchy's Integral Theorem,

$$\oint_c \frac{dz}{z-i} = 0.$$

And, the point $z = -i$ lies in $|z + i| = 1$ then by Cauchy's Integral Formula,

$$\oint_c \frac{dz}{z+i} = 2\pi i\, f(-i) \quad \dots (2) \qquad \left[\because \oint_c \frac{f(z)}{z-z_0}dz = 2\pi i\, f(z_0)\right]$$

Comparing $\oint_c \frac{dz}{z+i}$ with $\oint_c \frac{f(z)}{z-z_0} dz$ then we get, $z_0 = -i$ and $f(z) = 1$.

Then $f(-i) = 1$. Therefore (2) becomes

$$\oint_c \frac{dz}{z+i} = 2\pi i.$$

Therefore (1) becomes, $\oint_c \frac{dz}{z^2+1} = \frac{1}{2}.0 - \frac{1}{2}.2\pi i = -\pi i$

(ii) Since, the point $z = -i$ does not lie in the circle $|z - i| = 1$.

So, the function $\left(\frac{1}{z+i}\right)$ is analytic in $|z - i| = 1$. Therefore by Cauchy's Integral Theorem,

$$\oint_c \frac{dz}{z+i} = 0.$$

And, the point $z = i$ lies in $|z - i| = 1$ then by Cauchy's Integral Formula,

$$\oint_c \frac{dz}{z-i} = 2\pi i\, f(i) \quad \dots (3) \qquad \left[\because \oint_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)\right]$$

Comparing $\oint_c \frac{dz}{z-i}$ with $\oint_c \frac{f(z)}{z-z_0} dz$ then we get, $z_0 = i$ and $f(z) = 1$.

Then $f(i) = 1$. Therefore (3) becomes

$$\oint_c \frac{dz}{z-i} = 2\pi i.$$

Hence (1) becomes, $\oint_c \frac{dz}{z-i} = \frac{1}{2}.2\pi i - \frac{1}{2}.0 = \pi i$.

## Exercise – 4.3

**1. Integrate $\frac{z^2}{z^4-1}$ counterclockwise around the circle**

**(a) $|z + 1| = 1$** **(b) $|z| = 0.9$**

**Solution:** Here,

$$\oint_c \frac{z^2}{z^4-1} dz = \oint_c \frac{z^2\,dz}{(z-1)(z+1)(z-i)(z+i)}$$

Clearly, the function is analytic in c except at the points $z = i, -i, 1, -1$.



**(a)** Since, we have $|z + 1| = 1$.

Then, the given function is analytic in c except at the point $z = -1$ which lies in the circle $|z + 1| = 1$. So,

$$\oint_c \frac{z^2}{z^4+1} dz = \oint_c \frac{z^2/[(z-1)(z-i)(z+i)]}{(z+1)} dz$$

By Cauchy's Integral Formula,

$$\oint_c \left(\frac{z^2/[(z-1)(z-i)(z+i)]}{(z+1)}\right) dz = 2\pi i\, f(-1) \dots (1) \quad \left[\because \oint_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)\right]$$

Comparing $\oint_c \left(\frac{z^2/[(z-1)(z-i)(z+i)]}{(z+1)}\right) dz$ with $\oint_c \frac{f(z)}{z-z_0} dz$ then we get,

$$z_0 = -1 \text{ and } f(z) = \frac{z^2}{(z-1)(z-i)(z+i)}.$$

Then $f(-1) = \frac{1}{(-2)(-1-i)(-1+i)} = \frac{1}{-(1+i)(1-i)} = \frac{1}{1+1} = \frac{1}{2}$.

Therefore (1) becomes

$$\oint_c \frac{z^2}{z^4+1} dz = \oint_c \frac{z^2/[(z-1)(z-i)(z+i)]}{(z+1)} dz = \frac{-\pi i}{2}$$

**(b)** Since, we have $|z| = 0.9$.

Clearly, no one point $z = i, -i, 1, -1$ lie in the circle $|z| = 0.9$ so, the function is analytic in $|z| = 0.9$. Therefore, by Cauchy Integral Theorem,

$$\oint_c \frac{z^2}{z^4+1} dz = 0.$$

**2. Integrate the given function counterclockwise around the unit circle.**

**(a)** $\oint_c \frac{z^3}{2z-i} dz$ **(b)** $\frac{\cosh 3z}{2z}$

**Solution: (a)** Here,

$$\frac{z^3}{2z-i} = \frac{1}{2}\cdot\frac{z^3}{z-i/2}$$

This shows that the function does not exist at $z = (i/2)$, which lies in the unit circle. Then, by Cauchy Integral Formula,

$$\oint_c \frac{z^3}{2z-i} dz = \frac{1}{2}\oint_c \frac{z^3}{z-i/2} dz$$

$$= \frac{1}{2}2\pi i\, f\left(\frac{i}{2}\right) \dots (1) \qquad \left[\because \oint_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)\right]$$

Comparing $\oint_c \frac{z^3}{z - i/2}dz$ with $\oint_c \frac{f(z)}{z - z_0}dz$ then we get, $z_0 = \frac{i}{2}$ and $f(z) = z^3$.

Then $f\left(\frac{i}{2}\right) = \left(\frac{i}{2}\right)^3 = \frac{i^3}{8} = \frac{-i}{8}$.

Therefore (1) becomes

$$\oint_c \frac{z^3}{2z - i}dz = \pi i . \frac{-i}{8} = \frac{\pi}{8}.$$

**(b)** Here,

$$f(z) = \frac{\cosh 3z}{2z} = \frac{1}{2}\frac{\cosh 3z}{z}.$$

Clearly, the function is analytic in c except at the point z = 0 which lies in the unit circle. So, the function does not analytic in the circle c. Then, by Cauchy Integral Formula,

$$\frac{1}{2}\oint_c \frac{\cosh 3z}{z}dz = \frac{1}{2}2\pi i\, f(0) \quad \ldots (1) \qquad \left[\because \oint_c \frac{f(z)}{z - z_0}dz = 2\pi i\, f(z_0)\right]$$

Comparing $\oint_c \frac{\cosh 3z}{z}dz$ with $\oint_c \frac{f(z)}{z - z_0}dz$ then we get, $z_0 = 0$ and $f(z) = \cosh 3z$.

Then $f(0) = \cosh 0 = 1$.

Therefore (1) becomes

$$\oint_c \frac{\cosh 3z}{2z}dz = \pi i . 1 = \pi i.$$

**3. Integrate the given function over the given contour c counter clockwise or as indicated.**

**(a) $\frac{1}{z^2 + 4}$, where c is the ellipse $4x^2 + (y - 2)^2 = 4$. [2015 Spring Q. No. 2(b) OR]**

**Solution:** Here,

$$f(z) = \frac{1}{z^2 + 4} = \frac{1}{(z - 2i)(z + 2i)} \quad \ldots (1)$$

This shows that the function does not exist at z = 2i, z = –2i. Also, given that the contour (i.e. ellipse) is $4x^2 + (y - 2)^2 = 4$. Clearly, only the point z = 2i lies in the ellipse. Then, by Cauchy Integral Formula,

$$\oint_c \frac{1}{(z - 2i)(z + 2i)}dz = \oint_c \frac{(z + 2i)^{-1}}{z - 2i}dz = 2\pi i . f(2i) \quad \ldots (2)$$

$$\left[\because \oint_c \frac{f(z)}{z - z_0}dz = 2\pi i\, f(z_0)\right]$$

Comparing $\oint_c \frac{(z + 2i)^{-1}}{z - 2i}dz$ with $\oint_c \frac{f(z)}{z - z_0}dz$ then we get,

$z_0 = 2i$ and $f(z) = (z + 2i)^{-1}$.



Then $f(2i) = (4i)^{-1}$.

Therefore (1) and (2) gives

$$\oint_c \frac{1}{z^2 + 4}dz = 2\pi i . (4i)^{-1} = \frac{\pi}{2}$$

**(b) $\frac{\log(z - 1)}{z - 6}$, where c is the circle $|z - 6| = 4$.**

**Solution:** Here,

$$f(z) = \frac{\log(z - 1)}{z - 6}.$$

So, the function does not exist at z = 6. Clearly, the point lies in the circle $|z - 6| = 4$. Therefore, the function is not analytic in given circle.

Then, by Cauchy Integral Formula,

$$\oint_c \frac{\log(z - 1)}{z - 6}dz = 2\pi i . f(6) \quad \ldots (1) \qquad \left[\because \oint_c \frac{f(z)}{z - z_0}dz = 2\pi i\, f(z_0)\right]$$

Comparing $\oint_c \frac{\log(z - 1)}{z - 6}dz$ with $\oint_c \frac{f(z)}{z - z_0}dz$ then we get,

$z_0 = 6$ and $f(z) = \log(z - 1)$.

Then $f(6) = \log(5)$.

Therefore (1) gives

$$\oint_c \frac{\log(z - 1)}{z - 6}dz = 2\pi i \log(5).$$

**(c) $\frac{\cosh(z^2 - \pi i)}{z - \pi i}$ where, c is the rectangle with vertices ±1, ±1 + 4i.**

**Solution:** Here,

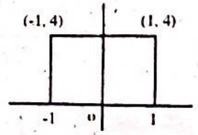
$$f(z) = \frac{\cosh(z^2 - \pi i)}{z - \pi i}$$

Clearly the function f(z) is analytic except at $z = \pi i$ ($\pi = 3.1$). Clearly, the point lies in the given rectangle with vertices ±1, ±1 + 4i.

Then, by Cauchy Integral Formula,

$$\oint_c \frac{\cosh(z^2 - \pi i)}{z - \pi i}dz = 2\pi i . f(\pi i) \quad \ldots (1) \qquad \left[\because \oint_c \frac{f(z)}{z - z_0}dz = 2\pi i\, f(z_0)\right]$$

Comparing $\oint_c \frac{\cosh(z^2 - \pi i)}{z - \pi i}dz$ with $\oint_c \frac{f(z)}{z - z_0}dz$ then we get,



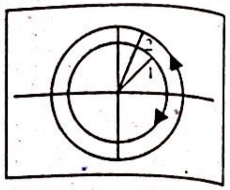

$z_0 = \pi i$ and $f(z) = \cosh(z^2 - \pi i)$.

Then $f(\pi i) = \cosh(-\pi^2 - \pi i)$.

Therefore (1) gives

$$\oint_c \frac{\cosh(z^2 - \pi i)}{z - \pi i}dz = 2\pi i \cosh(-\pi^2 - \pi i).$$

(d) $\dfrac{e^{z^2}}{z^2(z-1-i)}$ where c consists of $|z| = 2$ counterclockwise and $|z| = 1$ clockwise.



**Solution:** Here,

$$f(z) = \frac{e^{z^2}}{z^2(z-1-i)}$$

This shows that the function does not exist at $z = 0$ and $z = 1 + i$. Since, we have the contour c consists of $|z| = 2$ counterclockwise and $|z| = 1$ clockwise.

Clearly the point $z = 0$ does not lie in the region c but $z = 1 + i$ lies in that region.

Then, by Cauchy Integral Formula,

$$\oint_c \left(\frac{(e^{z^2})/z^2}{(z-1-i)}\right) dz = 2\pi i . f(1+i) \quad \ldots(1) \left[\because \oint_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)\right]$$

Comparing $\oint_c \left(\frac{(e^{z^2})/z^2}{(z-1-i)}\right) dz$ with $\oint_c \frac{f(z)}{z-z_0} dz$ then we get,

$$z_0 = 1 + i \quad \text{and} \quad f(z) = \frac{e^{z^2}}{z^2}$$

Then $\quad f(1+i) = \dfrac{e^{(1+i)^2}}{(1+i)^2} = \dfrac{e^{2i}}{2i} = \dfrac{-i\, e^{2i}}{2}$.

Therefore (1) gives

$$\oint_c \frac{e^{z^2}}{z^2(z-1-i)} dz = 2\pi i . \frac{-i\, e^{2i}}{2} = \pi e^{2i} = \pi (\cos 2 + i \sin 2).$$

4. **Show that $\oint_c \dfrac{1}{(z-z_1)(z-z_2)} dz = 0$ for a simple closed path c enclosing $z_1$ and $z_2$ which are arbitrary.**

**Solution:** Here,

$$\oint_c \frac{1}{(z-z_1)(z-z_2)} dz$$

where, c encloses $z_1$ and $z_2$. Then, the function does not exist at $z = z_1$ and $z = z_2$.

Now,

$$\oint_c \frac{1}{(z-z_1)(z-z_2)} dz = \left(\frac{1}{z_1-z_2}\right) \oint_c \left(\frac{1}{z-z_1} - \frac{1}{z-z_2}\right) dz \quad \ldots (i)$$

Since, the point $z_1$ and $z_2$ enclosed by c.

Then, by Cauchy Integral Formula,

$$\oint_c \left(\frac{1}{z-z_1} - \frac{1}{z-z_2}\right) dz = 2\pi i\, [f(z_1) - f(z_2)] \quad \ldots (2) \left[\because \oint_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)\right]$$



Comparing $\oint_c \left(\frac{1}{z-z_1} - \frac{1}{z-z_2}\right) dz$ with $\oint_c \frac{f(z)}{z-z_0} dz$ then we get,

$$z_0 = z_1,\ z_0 = z_2 \quad \text{and} \quad f(z_1) = f(z_2) = 1.$$

Therefore (2) gives

$$\oint_c \frac{1}{(z-z_1)(z-z_2)} dz = 2\pi i\, [1 - 1] = 0.$$

## Exercise – 4.4

1. **Integrate the following functions counterclockwise around the unit circle.**

(a) $\dfrac{\sinh 2z}{z^4}$

**Solution:** Given function is,

$$\frac{\sinh 2z}{z^4}$$

Clearly, the function is analytic in the unit circle except at the point $z = 0$ which lies in the unit circle, of order 3. Then, by Cauchy Integral Formula,

$$\oint_c \frac{\sinh 2z}{z^4} dz = \frac{2\pi i}{3!} f'''(0) \quad \ldots(1) \left[\because \oint_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{\sinh 2z}{z^4} dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n} dz$ then we get,

$$z_0 = 0,\ n = 4 \quad \text{and} \quad f(z) = \sinh 2z$$

Then $\quad f'''(z) = 8 \cosh 2z$. Therefore, $f'''(0) = 8$.

Therefore (1) gives

$$\oint_c \frac{\sinh 2z}{z^4} dz = \frac{2\pi i}{3!} (8) = \frac{8\pi i}{3}.$$

(b) $\dfrac{z^2}{(2z-1)^2} = \dfrac{z^2}{4(z-1/2)^2}$

**Solution:** Given function is,

$$\frac{z^2}{(2z-1)^2} = \frac{z^2}{4(z-1/2)^2}$$

Clearly, the function is analytic in the unit circle c except at $z = ½$ which lies in the unit circle, of order 1 which lies in c. Then, by Cauchy Integral Formula,

$$\oint_c \frac{z^2}{4(z-1/2)^2} dz = \frac{1}{4} \frac{2\pi i}{1!} f'\left(\frac{1}{2}\right) \ldots(1) \left[\because \oint_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{z^2}{(z-1/2)^2} dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n} dz$ then we get,

$z_0 = 1/2,\ n = 2$ and $f(z) = z^2$

Then $f'(z) = 2z$. Therefore, $f'(1/2) = 1$.

Therefore (1) gives

$$\int_c \frac{z^2}{(2z-1)^2}dz = \frac{1}{4}2\pi i\,(1) = \frac{\pi i}{2}.$$

**(c)** $\dfrac{\tan z}{(z-\pi/4)^3}$

**Solution:** Given that, $f(z) = \dfrac{\tan z}{(z-\pi/4)^3}$. Clearly, the function analytic except at $z = \frac{\pi}{4}$ which lies in the unit circle, of order 2. Then, by Cauchy Integral Formula,

$$\oint_c \frac{\tan z}{(z-\pi/4)^3}dz = \frac{2\pi i}{2!}f''\left(\frac{\pi}{4}\right) \ldots(1) \quad \left[\because \oint_c \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i}{(n-1)!}f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{\tan z}{(z-\pi/4)^3}dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}dz$ then we get,

$z_0 = \frac{\pi}{4},\ n = 3$ and $f(z) = \tan z$

Then $f''(z) = 2\sec^2 z \tan z$. Therefore, $f''\left(\frac{\pi}{4}\right) = 4$.

Therefore (1) gives

$$\int_c f(z)\,dz = \int_c \frac{\tan z}{(z-\pi/4)^3}dz = \frac{2\pi i}{2}(4) = 4\pi i.$$

**(d)** $\dfrac{\cos \pi z}{z^{2n}}$

**Solution:** Given that, $f(z) = \dfrac{\cos\pi z}{z^{2n}}$. Clearly, the function is analytic except at the point $z = 0$ which lies in the unit circle, of order $(2n-1)$.

Then, by Cauchy Integral Formula,

$$\oint_c \frac{\cos\pi z}{z^{2n}}dz = \frac{2\pi i}{(2n-1)!}f^{(2n-1)}(0) \ldots(1) \quad \left[\because \oint_c \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i}{(n-1)!}f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{\cos\pi z}{z^{2n}}dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}dz$ then we get,

$z_0 = 0,\ n = 2n$ and $f(z) = \cos\pi z.$

Because, $2n - 1$ is odd and cosine function have sine function in odd order derivative and $\sin 0 = 0$



Therefore (1) gives

$$\int_c f(z)\,dz = \int \frac{\cos\pi z}{z^{2n}}dz = \frac{2\pi i}{(2n-1)!}f^{(2n-1)}(0) = \frac{2\pi i}{(2n-1)}.0 = 0.$$

**(e)** $\dfrac{e^{3z}}{(4z-\pi i)^3}$

**Solution:** Given that, $f(z) = \dfrac{e^{3z}}{(4z-\pi i)^3}$. Clearly, the function analytic except at $z = \frac{\pi i}{4}$ which lies in the unit circle, of order 2. Then, by Cauchy Integral Formula,

$$\oint_c \frac{e^{3z}}{(4z-\pi i)^3}dz = \frac{1}{64}\frac{2\pi i}{2!}f''\left(\frac{\pi i}{4}\right) \ldots(1) \quad \left[\because \oint_c \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i}{(n-1)!}f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{e^{3z}}{(4z-\pi i)^3}dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}dz$ then we get,

$z_0 = \frac{\pi i}{4},\ n = 3$ and $f(z) = e^{3z}$

Then $f''(z) = 9e^{3z}$. Therefore, $f''\left(\frac{\pi i}{4}\right) = 9e^{3i\pi/4}$.

Therefore (1) gives

$$\oint_c \frac{e^{3z}}{(4z-\pi i)^3}dz = \frac{\pi i}{64}.9.e^{\frac{3\pi i}{4}} = \frac{9\pi i}{64}e^{\frac{3\pi i}{4}}$$

**(f)** $\dfrac{z^3 e^z}{(z-1/2)^2}$

**Solution:** Here, $f(z) = \dfrac{z^3e^z}{(z-1/2)^2}$. Clearly, the function does not exist at $z = \frac{1}{2}$ which lies in the unit circle. Then, by Cauchy Integral Formula,

$$\oint_c \frac{z^3.e^z}{(z-1/2)^2}dz = 2\pi i\,f'\left(\frac{1}{2}\right) \ldots(1) \quad \left[\because \oint_c \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i}{(n-1)!}f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{z^3e^z}{(z-1/2)^2}dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}dz$ then we get,

$z_0 = \left(\frac{1}{2}\right),\ n = 2$ and $f(z) = z^3e^z$.

Then $f'(z) = 3z^2e^z + z^3e^z$.

Therefore, $f'\left(\frac{1}{2}\right) = 3\left(\frac{1}{4}\right)e^{1/2} + \frac{1}{8}e^{1/2} = \frac{7e^{1/2}}{8}.$

Therefore (1) gives

$$\int_c f(z)\,dz = \int_c \frac{z^3 e^z}{(z-1/2)^2}\,dz = 2\pi i\,\frac{7e^{1/2}}{8} = \frac{7e^{1/2}\pi i}{4}$$

**(g) log(z)**

**Solution:** Here, f(z) = log(z). Clearly, the function does not exist at z = 0 which lies in unit circle. Then, by Cauchy's Integral Formula,

$$\int_c f(z)\,dz = \int_c \log(z)\,dz = 2\pi i$$

**2. Integrate f(z) around c counterclockwise or as indicated**

**(a) $f(z) = z^{-2}\tan\pi z$ where c be any contour enclosing zero.**

**Solution:** Here, $f(z) = z^{-2}\tan\pi z = \frac{\tan\pi z}{z^2}$

Clearly, the function does not exist at z = 0, which lies in c.

Then, by Cauchy Integral Formula,

$$\oint_c \frac{\tan\pi z}{z^2}\,dz = 2\pi i\, f'(0)\ldots(1) \quad \left[\because \oint_c \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{\tan\pi z}{z^2}\,dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}\,dz$ then we get,

$z_0 = 0,\ n = 2$ and $f(z) = \tan\pi z$

Then $f'(z) = \pi\sec^2\pi z$. Therefore, $f'(0) = \pi$.

Therefore (1) gives

$$\int_c f(z)\,dz = \int \frac{\tan\pi z}{z^2}\,dz = 2\pi i\,(\pi) = 2\pi^2 i.$$

**(b) $f(z) = \frac{\log z}{(z-2)^2}$, c: |z − 3| = 2**

**Solution:** Here, $f(z) = \frac{\log z}{(z-2)^2}$. Clearly, the function f(z) is analytic except at z = 2 and it lies in c : |z − 3| = 2. Then, by Cauchy Integral Formula,

$$\oint_c \frac{\log z}{(z-2)^2}\,dz = 2\pi i\, f'(2)\ldots(1) \quad \left[\because \oint_c \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{\log z}{(z-2)^2}\,dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}\,dz$ then we get,

$z_0 = 2,\ n = 2$ and $f(z) = \log(z)$

Then $f'(z) = \frac{1}{z}$. Therefore, $f'(2) = \frac{1}{2}$.



Therefore (1) gives

$$\int_c f(z)\,dz = \int_c \frac{\log z}{(z-2)^2}\,dz = 2\pi i\cdot\frac{1}{2} = \pi i$$

**(c) $f(z) = \frac{2z^3-3}{z\,(z-1-i)^2}$, c consists of |z| = 2 counterclockwise and |z| = 1 clockwise.**

**Solution:** Here, $f(z) = \frac{2z^3-3}{z\,(z-1-i)^2}$ and c: |z| = 2 counterclockwise and |z| = 1 clockwise. Clearly, the function f(z) is analytic in c except at z = 0 and z = 1 + i in which the point z = (1 + i) lies in c but the point z = 0 does not lie in c.

Then, by Cauchy Integral Formula,

$$\oint_c \frac{2z^3-3}{z\,(z-1-i)^2}\,dz = \oint_c \frac{(2z^3-3)/z}{(z-1-i)^2}\,dz = 2\pi i\, f'(1+i) \quad \ldots(1)$$

$$\left[\because \oint_c \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{(2z^3-3)/z}{(z-1-i)^2}\,dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}\,dz$ then we get,

$z_0 = 1 = i,\ n = 2$ and $f(z) = \frac{2z^3-3}{z}$

Then $f'(z) = \frac{z(6z^2)-(2z^3-3)}{z^2} = \frac{4z^3+3}{z^2}$.

Therefore, $f'(1+i) = \frac{4(1+i)^3+3}{(1+i)^2} = \frac{4(1-i+3i-3)+3}{2i} = \frac{8i-5}{2i}$.

Therefore (1) gives

$$\int_c f(z)\,dz = 2\pi i\cdot\frac{8i-5}{2i} = (-5+8i)\,\pi.$$

**(d) $f(z) = \frac{(1+z)\sin z}{(2z-1)^2}$ with c: |z − i| = 2.**

**Solution:** Here,

$$f(z) = \frac{(1+z)\sin z}{(2z-1)^2} \text{ with c: } |z-i| = 2.$$

Clearly f(z) is analytic in c except at $z = \frac{1}{2}$ which lie in C. Then by Cauchy Integral formula.

$$\oint_c F(z)\,dz = \oint_c \frac{(1+z)\sin z}{4(z-1/2)^2}\,dz = \frac{2\pi i}{4} f'(1/2) \quad \ldots\ldots(1)$$

Here,

$f(z) = (1+z)\sin z$

so, $f'(z) = \cos z + \sin z + z \cos z$

Then $f'(1/2) = \cos\left(\frac{1}{2}\right) + \sin\left(\frac{1}{2}\right) + \frac{\cos(1/2)}{2}$

$= 3/2 \cos(1/2) + \sin(1/2)$

Then (1) becomes,

$$\oint_c f(z)\,dz = \frac{\pi i}{4}\left[3\cos\left(\frac{1}{2}\right) + 2\sin\left(\frac{1}{2}\right)\right].$$

**(e) $f(z) = \frac{\cosh 4z}{(z-4)^3}$, c consists of $|z| = 6$ counterclockwise and $|z-3| = 2$ clockwise.**

**Solution:** Here, $f(z) = \frac{\cosh 4z}{(z-4)^3}$ and c consists of $|z| = 6$ (CCW) and $|z-3| = 2$ (CW).

Clearly f(z) is analytic except at $z = 4$, which does not lie in c.

That means f(z) is analytic in c. Therefore, by Cauchy's Integral Theorem,

$$\int_c f(z)\,dz = 0.$$

**(f) $f(z) = \frac{e^{5z}}{(z+i)^4}$, where c: $|z| = 3$.**

**Solution:** Here, $f(z) = \frac{e^{5z}}{(z+i)^4}$ and c: $|z| = 3$ (CCW).

Clearly f(z) is analytic except at $z = -i$, which lies c.

Then, by Cauchy Integral Formula,

$$\oint_c \frac{e^{5z}}{(z+i)^4}dz = \frac{2\pi i}{3!} f'''(-i) \ldots (1) \quad \left[\because \oint_c \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i}{(n-1)!}f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{e^{5z}}{(z+i)^4}dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}dz$ then we get,

$z_0 = -i$, $n = 4$ and $f(z) = e^{5z}$

Then $f'''(z) = 125e^{5z}$. Therefore, $f'''(-i) = 125e^{-5i}$.

Therefore (1) gives

$$\int_c f(z)\,dz = \frac{2\pi i}{6}.125e^{-5i} = \frac{125\pi i}{3}e^{-5i}.$$

**(g) $f(z) = \frac{z^4}{(z+1)(z-i)^2}$ where, c is $9x^2 + 4y^2 = 36$.**

**Solution:** Here, $f(z) = \frac{z^4}{(z+1)(z-i)^2}$ and c: $9x^2 + 4y^2 = 36 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1$.

Clearly f(z) is analytic except at $z = -1$ and $z = i$ which lie in c.



Here,

$$f(z) = \frac{z^4}{(z+1)(z-i)^2} = \frac{A}{(z+1)} + \frac{B}{z-i} + \frac{C}{(z-i)^2}$$

$$\Rightarrow z^4 = A(z-i)^2 + B(z+i)(z-i) + C(z+1)$$

Solving the get,

$$A = \frac{1}{(1-i)^2},\ B = \frac{i}{(1-i)^2} - \frac{i}{(1+i)},\ C = \frac{1}{1+i}$$

Now,

$$\oint_c f(z)\,dz = A.\oint_c \frac{dz}{z+1} + B\oint_c \frac{dz}{z-i} + C\oint_c \frac{dz}{(z-i)^2}$$

$$= AI_1 + BI_2 + CI_3 \quad \text{(say)} \ \ldots (1)$$

So, by Cauchy's Integral Formula,

$I_1 = 2\pi i.\,1 = 2\pi i$, $I_2 = 2\pi i.1 = 2\pi i$, $I_3 = 2\pi i.0 = 0$

Therefore, (i) becomes,

$$\oint_c f(z)\,dz = 2\pi i\,(A + B)$$

$$= 2\pi i\left[\frac{1}{(1-i)^2} + \frac{i}{(1-i)^2} - \frac{i}{(1+i)}\right]$$

$$= 2\pi i\left[\frac{1+i}{(1-i)^2} - \frac{i}{1+i}\right]$$

$$= 2\pi i\left[\frac{1+i}{-2i} - \frac{i}{1+i}\right] \quad [\because i^2 = -1]$$

$$= 2\pi i\left[\frac{(1+i)^2 + 2i^2}{-2i(1+i)}\right]$$

$$= 2\pi i\left(\frac{2i-2}{-2i(1+i)}\right)$$

$$= \frac{2\pi i.\,2(i-1)}{-2i(i+1)} \times \frac{i-1}{i-1} = \frac{2\pi(i-1)^2}{-(i^2-1)} = \frac{2\pi(i-1)^2}{2} = \pi(i-1)^2.$$

**(h) $f(z) = \frac{z+1}{z^3 - 2z^2}$, where c is the unit circle.**

**Solution:** Here, $f(z) = \frac{z+1}{z^2(z-2)}$ and c is unit circle in CCW direction.

Clearly f(z) does not exist at $z = 0$ and $z = 2$ in which $z = 0$ lies in c but $z = 2$ does not lie in c.

Then, by Cauchy Integral Formula.

$$\oint_c \frac{(z+1)/(z+2)}{z^2}\,dz = \frac{2\pi i}{1!} f'(0)\ldots(1) \left[\because \oint_c \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{(z+1)/(z+2)}{z^2}\,dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}\,dz$ then we get,

$$z_0 = 0,\ n = 2 \text{ and } f(z) = \frac{z+1}{z+2}$$

Then $f'(z) = \frac{(z+2).1-(z+1).1}{(z+2)^2} = \frac{1}{(z+2)^2}$. Therefore, $f'(0) = \frac{1}{4}$.

Therefore (1) gives,

$$\int_c f(z)\,dz = 2\pi i.\frac{1}{4} = \frac{\pi i}{2}$$

**(i) $f(z) = \frac{\cos z}{(z-\pi i)^2}$, where c is any contour enclosing the point $\pi i$ (CCW).**

**Solution:** Let $f(z) = \frac{\cos z}{(z-\pi i)^2}$ and c is any contour enclosing the point $\pi i$ (CCW).

Clearly f(z) does not exist at $z = \pi i$ which is contained in c. Then, by Cauchy Integral Formula,

$$\oint_c \frac{\cos z}{(z-\pi i)^2}\,dz = \frac{2\pi i}{1!} f'(\pi i)\ldots(1) \left[\because \oint_c \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{\cos z}{(z-\pi i)^2}\,dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}\,dz$ then we get,

$$z_0 = \pi i,\ n = 2 \text{ and } f(z) = \cos z$$

Then $f'(z) = -\sin z$. Therefore, $f'(\pi i) = -\sin(\pi i) = -i\sinh\pi$.

Therefore (1) gives

$$\int_c f(z)\,dz = 2\pi i.(-i\sinh\pi) = 2\pi\sinh\pi.$$

**(j) $f(z) = \frac{z^4-3z^2+6}{(z+i)^3}$, where c is any contour enclosing the point $z = -i$ (CCW).**

**Solution:** Let, $f(z) = \frac{z^4-3z^2+6}{(z+i)^3}$ and c is any contour enclosing the point $-i$ (CCW).

Clearly f(z) does not exist at $z = -i$ which is contained in c. Then, by Cauchy Integral Formula,

$$\oint_c \frac{z^4-3z^2+6}{(z+i)^3}\,dz = \frac{2\pi i}{2!} f''(-i)\ldots(1) \left[\because \oint_c \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{z^4-3z^2+6}{(z+i)^3}\,dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}\,dz$ then we get,



$$z_0 = -i,\ n = 3 \text{ and } f(z) = z^4 - 3z^2 + 6.$$

Then $f''(z) = 12z^2 - 6$. Therefore, $f''(-i) = 12(-i)^2 - 6 = -18$.

Therefore (1) gives

$$\int_c f(z)\,dz = \frac{2\pi i}{2!}(-18) = -18\pi i.$$

**(k) $f(z) = \frac{e^z}{(z-1)^2(z^2+4)}$ where c is any contour for which 1 lies inside and $\pm 2i$ lies outside (counterclockwise).**

**Solution:** Let, $f(z) = \frac{e^z}{(z-1)^2(z^2+4)}$ and c is any contour for which 1 lies inside and $\pm 2i$ lies outside where c has CCW direction.

Clearly f(z) analytic except at $z = 1, \pm 2i$ in which is 1 lies inside of c but $\pm 2i$ lies outside of c. Then, by Cauchy Integral Formula,

$$\oint_c \frac{e^z/(z^2+4)}{(z-1)^2}\,dz = \frac{2\pi i}{1!} f'(1)\ \ldots(1) \left[\because \oint_c \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{e^z/(z^2+4)}{(z-1)^2}\,dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}\,dz$ then we get,

$$z_0 = 1,\ n = 2 \text{ and } f(z) = \frac{e^z}{z^2+4}.$$

Then $f'(z) = \frac{(z^2+4)e^z - 2z\,e^z}{(z^2+4)^2}$. Therefore, $f'(1) = \frac{(1+4)\,e^1 - e^1.2}{5^2} = \frac{3e^1}{25}$.

Therefore (1) gives

$$\int_c f(z)\,dz = \frac{6\pi i}{25}e^1.$$

**(l) $f(z) = \frac{e^{2z}}{(z+1)^4}$ where, c: |z| = 4.**

**Solution:** Here, $f(z) = \frac{e^{2z}}{(z+1)^4}$ and c: |z| = 4 (CCW).

Clearly f(z) analytic except at $z = -1$ which lies inside of c. Then, by Cauchy Integral Formula,

$$\oint_c \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{3!} f'''(-1)\ldots(1) \left[\because \oint_c \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{e^{2z}}{(z+1)^4}\,dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}\,dz$ then we get,

$$z_0 = -1,\ n = 4 \text{ and } f(z) = e^{2z}.$$

Then $f'''(z) = 8e^{2z}$. Therefore, $f'''(-1) = 8e^{-2}$.

Therefore (1) gives

$$\int_c f(z)\,dz = \frac{2\pi i}{3!}.8e^{-2} = \frac{8\pi i}{3}e^{-2}.$$

**(m)** $f(z) = \frac{z^3 - z}{(z+2)^3}$, **where c:** $16x^2 + 25y^2 = 400 \Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1.$

**Solution:** Here, $f(z) = \frac{z^3 - z}{(z+2)^3}$ and c: $\frac{x^2}{25} + \frac{y^2}{16} = 1$ (in CCW direction).

Clearly f(z) analytic except at z = −2 which lies inside of c. Then, by Cauchy Integral Formula,

$$\oint_c \frac{z^3 - z}{(z+2)^3}dz = \frac{2\pi i}{2!} f''(-2) \quad \ldots(1) \quad \left[\because \oint_c \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{z^3 - z}{(z+2)^3}dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}dz$ then we get,

$z_0 = -2,\ n = 3$ and $f(z) = z^3 - z.$

Then $f''(z) = 6z$. Therefore, $f''(-2) = -12.$

Therefore (1) gives

$$\int_c f(z)\,dz = \frac{2\pi i}{2!}(-12) = -12\pi i.$$

**(n)** $f(z) = z^{-2} e^{-z}$ **where, c:** $|z| = 1$

**Solution:** Here, $f(z) = z^{-2}e^{-z} = \frac{e^{-z}}{z^2}$ and c: $|z| = 1$ (CCW).

Clearly f(z) analytic except at z = 0 which lies inside of c. Then, by Cauchy Integral Formula,

$$\oint_c \left(\frac{e^{-z}}{z^2}\right)dz = \frac{2\pi i}{1!} f'(0) \quad \ldots(1) \quad \left[\because \oint_c \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \left(\frac{e^{-z}}{z^2}\right)dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}dz$ then we get,

$z_0 = 0,\ n = 2$ and $f(z) = e^{-z}.$

Then $f'(z) = -e^{-z}$. Therefore, $f'(0) = -1.$

Therefore (1) gives

$$\int_c f(z)\,dz = \int_c \frac{e^{-z}}{z^2}dz = -2\pi i.$$

**(o)** $f(z) = \frac{1}{(z^2+4)^3}$ **where c is the circle** $|z - i| = 2.$

**Solution:** Here, $f(z) = \frac{1}{(z^2+4)^3} = \frac{1}{(z+2i)^3(z-2i)^3}$ and c is the circle $|z - i| = 2.$



Clearly f(z) analytic except at $z = \pm 2i$ in which the point z = 2i lies in c but z = −2i does not lie in c. Then, by Cauchy Integral Formula,

$$\oint_c \frac{1}{(z+2i)^3(z-2i)^3}dz = \int_c \frac{(z+2i)^{-3}}{(z-2i)^3}dz = \frac{2\pi i}{2!} f''(2i) \quad \ldots(1)$$

$$\left[\because \oint_c \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{(z+2i)^{-3}}{(z-2i)^3}dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}dz$ then we get,

$z_0 = 2i,\ n = 3$ and $f(z) = (z+2i)^{-3}.$

Then $f''(z) = 12(z+2i)^{-5}$. Therefore, $f''(2i) = 12(4i)^{-5} = \frac{12}{1024\,i} = \frac{-3i}{256}$.

Therefore (1) gives

$$\int_c f(z)\,dz = \frac{2\pi i}{2!} \times \frac{-3i}{256} = \frac{3\pi}{256}.$$

**(p)** $\int_c \left(\frac{2z^3 + z^2 + 4}{z^4 + 4z^2}\right)dz \quad C: |z - 2| = 4$ (CW)

**Solution: Similar to (h).**

**(q)** $\oint_C \left(\frac{z+4}{z^2 + 2z + 5}\right)dz \quad C: |z + 1 - i| = 2.$

**Solution: Similar to (h).**

**(r)** $\oint_C \frac{z+1}{z^3 - 4z}dz$

**Solution: Similar to (h).**

## Miscellaneous Example

**1. Integrate $(1 - z^2)^2$ from −i to i counterclockwise along**
**(a) $|z| = 1$ (b) $x^2 + 4y^2 = 4$. Why are the results equal?**

**Solution:** Here,

$\oint (1 - z^2)^2\,dz \quad C$: from −i to i along

Given function is, $f(z) = (1 - z^2)^2$ ... (i)

We have to integrate if from −i to i

(a) around $|z| = 1$, since the curve is not a closed contour so the Cauchy integral theorem or Cauchy integral formula is not applicable here.

Now,

$$\oint_C f(z)\,dz = \int_{-i}^{i} (1-z^2)^2\,dz$$

$$= \int_{-i}^{i} (1-2z^2+z^4)\,dz$$

$$= \left[z - \frac{2}{3}z^3 + \frac{z^5}{5}\right]_{-i}^{i}$$

$$= i - \frac{2}{3}i^3 + \frac{i^5}{5} + i - \frac{2}{3}i^3 + \frac{i^5}{5}$$

$$= 2\left[i - \frac{2}{3}i^3 + \frac{i^5}{5}\right]$$

$$= 2\left[i + \frac{2}{3}i + \frac{i}{5}\right]$$

$$= \frac{2i}{15}(15+10+3) = \frac{2i}{15}\cdot 28 = \frac{56}{15}i$$

**(b) around $x^2 + 4y^2 = 4$.**

***Note:*** Since the integration is independent to its path. So, (B) has same value as (a).

**2. Integrate $z \cos hz^2$ from 0 to $\pi i$ along any path.**

**Solution:** Given that

$$I = \oint z \cos hz^2\,dz \quad C: 0 \text{ to } \pi i$$

Here,

$$\oint z \cos hz^2\,dz = \frac{1}{2}\int_0^{\pi i} 2z \cos hz^2\,dz$$

$$= \frac{1}{2}\left|\sin hz^2\right|_0^{\pi i} = \frac{1}{2}\sin h(-\pi^2) = -\frac{1}{2}\sin h\pi^2$$

**3. Integrate $\left(\frac{1}{z}+\frac{1}{z-2}\right)$ clockwise around the ellipse $(x-1)^2+4y^2=4$.**

**Solution:** Given that

$$I = \oint_C \left(\frac{1}{z}+\frac{1}{z-2}\right)dz \quad \text{where } C: (x-1)^2+4y^2=4 \quad (CW)$$

$$\Rightarrow C: \frac{(x-1)^2}{4} + \frac{y^2}{1} = 1$$

Clearly, the integrand of I i.e. $f(z) = \left(\frac{1}{z}+\frac{1}{z-2}\right)$ is not analytic at $z = 0, 2$ in which $z = 0$ lies outside C and $z = 2$ lies in C.

So, by Cauchy Integral Theorem,

$$\oint_C \left(\frac{1}{z}\right)dz = 0.$$



Next, $f(z) = \left(\frac{1}{z-2}\right)$. Clearly f(z) analytic except at $z = 2$ which lies in c.

Then, by Cauchy Integral Formula,

$$\oint_c \left(\frac{1}{z-2}\right)dz = 2\pi i\, f(2) \quad \ldots(1) \quad \left[\because \oint_c \frac{f(z)}{(z-z_0)}dz = 2\pi i\, f(z_0)\right]$$

Comparing $\oint_c \left(\frac{1}{z-2}\right)dz$ with $\oint_c \frac{f(z)}{(z-z_0)}dz$ then we get,

$$z_0 = 2 \quad \text{and} \quad f(z) = 1. \text{ Then } f(2) = 1.$$

Therefore (1) gives

$$\oint_c \left(\frac{1}{z-2}\right)dz = 2\pi i$$

Then,

$$\oint_C \left(\frac{1}{z}+\frac{1}{z-2}\right)dz = 0 + 2\pi i = 2\pi i.$$

**4.** $\oint_C \frac{e^z}{z^4}dz \quad$ **C : |z| = $\frac{1}{2}$ (CCW)**

**Solution:** Here, $f(z) = \frac{e^z}{z^4}$ and c: $|z| = \frac{1}{2}$ (CCW).

Clearly f(z) analytic except at $z = 0$ which lies inside of c. Then, by Cauchy Integral Formula,

$$\oint_c \left(\frac{e^z}{z^4}\right)dz = \frac{2\pi i}{3!} f'''(0) \quad \ldots(1) \quad \left[\because \oint_c \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \left(\frac{e^z}{z^4}\right)dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n}dz$ then we get,

$$z_0 = 0,\ n = 4 \quad \text{and} \quad f(z) = e^z.$$

Then $f'''(z) = e^z$. Therefore, $f'''(0) = 1$.

Therefore (1) gives

$$\oint_C \frac{e^z}{z^4}dz = \frac{2\pi i}{3!} \times 1 = \frac{\pi i}{3}$$

**5.** $\oint_C \frac{1}{\log(z+2i)}dz \quad$ **C : |z| = 1 (CW).**

**Solution:** Here, $f(z) = \frac{1}{\log(z+2i)}$ and c: $|z| = 1$ (CW).

Clearly f(z) analytic except at $z \le -2i$ but these points does not lie in c. This means f(z) is analytic in c. Therefore, by Cauchy Integral Theorem,

$$\oint_c \frac{1}{\log(z+2i)}dz = 0.$$

6. $\oint \text{Re}(z)\,dz$ C : 0 to (3 + 27i) along $y = x^3$

**Solution:** Given that $\oint_C \text{Re}(z)\,dz$

Here, $f(z) = \text{Re}(z)$.

Since, $z = x + iy$. Then, $dz = dx + idy$ and given that $y = x^3$. So, $dy = 3x^2\,dx$.

Here, $\text{Re}(z) = x$

Then, $\oint_C \text{Re}(z)\,dz = \oint_C (x)(dx + idy)$.

$$= \int_0^3 (x\,dx + i\,3x^3\,dx)$$

$$= \left[\frac{x^2}{2} + i\frac{3x^4}{4}\right]_0^3$$

$$= \frac{9}{2} + i\frac{(3)^5}{4}.$$

7. $\oint \frac{z\cos hz^2}{(z-2i)^3}$ C : $|z - i| = 2$ (CCW).

**Solution:** Given that $\oint_C \frac{z\cos hz^2}{(z-2i)^3}dz$ and C : $|z - i| = 2$ (CCW).

Clearly, the integrand function is analytic except at $z = 2i$ which lies in C.

So, by the Cauchy's Integral Formula,

$$\oint_C \left(\frac{z\cos hz^2}{(z-2i)^3}\right)dz = \frac{2\pi i}{2!} f''(0) \quad \ldots(1) \qquad \left[\because \oint_C \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_C \left(\frac{z\cos hz^2}{(z-2i)^3}\right)dz$ with $\oint_C \frac{f(z)}{(z-z_0)^n}dz$ then we get,

$z_0 = 2i$, $n = 3$ and $f(z) = z\cosh z^2$.

Then, $f'(z) = \cosh z^2 + 2z^2 \sin hz^2$

And, $f''(z) = 2z\sinh z^2 + 4z\sinh z^2 + 4z^3\cosh z^2$

So, $f''(2i) = 12i\sinh(-4) - 32i\cosh(-4)$

Therefore, (1) becomes

$$\oint \left(\frac{z\cos hz^2}{(z-2i)^3}\right)dz = \pi i\,(12i\sinh(-4) - 32i\cosh(-4))$$

$$= -12\pi\sinh(-4) + 32\pi\cosh(-4).$$

$$= 12\pi\sinh(4) + 32\pi\cosh(4).$$



8. $\oint \frac{\tan\pi z}{(z-1)^2}dz$ C : $|z - 1| = 0.1$ (CCW).

**Solution:** Given that $\oint_C \frac{\tan\pi z}{(z-1)^2}dz$ and C : $|z - 1| = 0.1$ (CCW).

Clearly, the integrand function is analytic except at $z = 1$ which lies in C.

So, by the Cauchy's Integral Formula,

$$\oint_C \frac{\tan\pi z}{(z-1)^2}dz = \frac{2\pi i}{1!} f'(0) \quad \ldots(1) \qquad \left[\because \oint_C \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_C \frac{\tan\pi z}{(z-1)^2}dz$ with $\oint_C \frac{f(z)}{(z-z_0)^n}dz$ then we get,

$z_0 = 1i$, $n = 2$ and $f(z) = \tan\pi z$.

Then, $f'(z) = \pi\sec^2\pi z$. So, $f'(1) = \pi\sec^2\pi = \pi$.

Therefore, (1) becomes

$$\oint_C \frac{\tan\pi z}{(z-1)^2}dz = 2\pi i.\pi = 2\pi^2 i.$$

9. $\oint_C \left(\frac{4}{z+2i} + \frac{2}{4i+z}\right)dz$, c : $|z - 1| = 2.5$ (CW)

**Solution:** Given that

$$I = \oint_C \left(\frac{4}{z+2i} + \frac{2}{4i+z}\right)dz \quad \text{where c : } |z-1| = 2.5 \text{ (CW)}.$$

Here,

$$I_1 = \oint_C \left(\frac{4}{z+2i}\right)dz$$

Clearly, the integrand of $I_1$ i.e. $\left(\frac{4}{z+2i}\right)$ is not analytic at $z = 2i$ which lies in c.

Then, by Cauchy Integral Formula,

$$\oint_C \left(\frac{4}{z+2i}\right)dz = 2\pi i\,f(-2i) \quad \ldots(1) \qquad \left[\because \oint_C \frac{f(z)}{(z-z_0)}dz = 2\pi i\,f(z_0)\right]$$

Comparing $\oint_C \left(\frac{4}{z+2i}\right)dz$ with $\oint_C \frac{f(z)}{(z-z_0)}dz$ then we get,

$z_0 = -2i$ and $f(z) = 4$. Then $f(-2i) = 4$.

Then (1) becomes,

$$\oint_C \left(\frac{4}{z+2i}\right)dz = 2\pi i\,(4) = 8\pi i$$

Next,

$$I_2 = \oint_C \left(\frac{2}{4i+z}\right) dz$$

Clearly, the integrand of $I_2$ i.e. $\left(\frac{2}{4i+z}\right)$ is not analytic at $z = -4i$ which does not lie in c. This means the integrand of $I_2$ is analytic in c. So, by Cauchy Integral Theorem,

$$\oint_C \left(\frac{2}{4i+z}\right) dz = 0.$$

Therefore

$$\oint_C \left(\frac{4}{z+2i} + \frac{2}{4i+z}\right) dz = I_1 + I_2 = 0 + 8\pi i = 8\pi i$$

**10.** $\oint_C \frac{\log z}{(z-2i)^2} dz$ **C : |z − 2i| = 1 (CCW)**

**Solution:** Given that $\oint_C \frac{\log z}{(z-2i)^2} dz$ and C: $|z - 2i| = 1$.

Clearly, the integrand function is analytic except at $z = -2i$ which lies in C.
So, by the Cauchy's Integral Formula,

$$\oint_c \frac{\log z}{(z-2i)^2} dz = \frac{2\pi i}{1!} f'(2i) \quad \ldots(1) \qquad \left[\because \oint_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)\right]$$

Comparing $\oint_c \frac{\log z}{(z-2i)^2} dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n} dz$ then we get,

$$z_0 = 2i,\ n = 2 \text{ and } \quad f(z) = \log(z).$$

Then, $f'(z) = \frac{1}{z}$. So, $f'(2i) = \frac{1}{2i}$.

Therefore, (1) becomes

$$\oint_C \frac{\log z}{(z-2i)^2} dz = 2\pi i \times \frac{1}{2i} = \pi.$$

**11.** $\oint_C \frac{\cos 4z}{z^3(4z-\pi)} dz$ **C : |z − 1| = $\frac{1}{2}$ (CCW)**

**Solution:** Given that $\oint_C \frac{\cos 4z}{z^3(4z-\pi)} dz$ and C : $|z - 1| = \frac{1}{2}$ (CCW).

Let, $f(z) = \frac{\cos 4z}{4z^3(z-\pi/4)}$.

Clearly, the f(z) is analytic except at $z = 0$ and $z = (\pi/4)$ in which only $z = (\pi/4)$ lies in c but $z = 0$ lies outside of c.



So, by the Cauchy's Integral Formula,

$$\oint_c \frac{\cos 4z}{4z^3\left(z-\frac{\pi}{4}\right)} dz = 2\pi i\, f\left(\frac{\pi}{4}\right) \quad \ldots(1) \qquad \left[\because \oint_c \frac{f(z)}{(z-z_0)} dz = 2\pi i\, f(z_0)\right]$$

Comparing $\oint_c \frac{\cos 4z}{4z^3\left(z-\frac{\pi}{4}\right)} dz$ with $\oint_c \frac{f(z)}{(z-z_0)^n} dz$ then we get,

$$z_0 = \pi/4, \quad \text{and} \quad f(z) = \frac{\cos 4z}{4z^3}.$$

Then, $f\left(\frac{\pi}{4}\right) = \frac{16 \cos\pi}{\pi^3} = \frac{-16}{\pi^3}$.

Therefore, (1) becomes

$$\oint_C \frac{\cos 4z}{z^3(4z-\pi)} dz = 2\pi i \times \frac{-16}{\pi^3} = \frac{-32i}{\pi^2}$$

**12.** $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)} dz$ **C : |z| = 3 (CCW).**

**Solution:** Given that $I = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)} dz$ and C : $|z| = 3$ (CCW).

Clearly, the integrand function of I is analytic except at $z = -1$ and $z = -2$ which are lie in C. Therefore,

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)} dz = \oint_C \left(\frac{1}{z+1} - \frac{1}{z+2}\right)(\sin \pi z^2 + \cos \pi z^2)\, dz$$

$$= I_1 - I_2$$

By Cauchy's Integral Formula,

$$I_1 = \oint_c \left(\frac{\sin \pi z^2 + \cos \pi z^2}{z+1}\right) dz = 2\pi i\, f(-1) \quad \ldots(1)$$

Comparing $I_1$ with $\oint_c \frac{f(z)}{(z-z_0)} dz$ then we get,

$$z_0 = -1, \text{ and } \quad f(z) = \sin \pi z^2 + \cos \pi z^2.$$

Then, $f(-1) = \sin \pi + \cos \pi = 0 - 1 = -1$.

Therefore, (1) becomes

$$I_1 = -2\pi i.$$

Also, by Cauchy's Integral Formula,

$$I_1 = \oint_c \left(\frac{\sin \pi z^2 + \cos \pi z^2}{z+2}\right) dz = 2\pi i\, f(-2) \quad \ldots(2)$$

Comparing $I_2$ with $\oint_c \frac{f(z)}{(z - z_0)} dz$ then we get,

$z_0 = -2$, and $f(z) = \sin \pi z^2 + \cos \pi z^2$.

Then, $f(-2) = \sin 4\pi + \cos 4\pi = 0 + 1 = 1$.

Therefore, (2) becomes

$$I_2 = 2\pi i.$$

Thus,

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)} dz = I_1 - I_2 = -2\pi i - 2\pi i = -4\pi i.$$

**13.** $\oint \frac{e^{2z}}{(z-1)(z-2)} dz$ **C : |z| = 3 (CCW)**

**Solution:** Similar to Q.12.

**14.** $\oint_C \frac{z+1}{z^2+2z+4}$ **around the circle |z + 1 + i| = 2.**

**Solution:** Given that $\oint_C \frac{z+1}{z^2+2z+4} dz$ and C: $|z + 1 + i| = 2$.

Here, $\frac{z+1}{z^2+2z+4} = \frac{z+1}{z^2+2z+1+3} = \frac{z+1}{(z+1)^2 - 3i^2}$

$\Rightarrow \frac{z+1}{z^2+2z+4} = \frac{z+1}{(z+1+3i)(z+1-3i)} = \frac{A}{z+1+3i} + \frac{B}{z+1-3i}$

$\Rightarrow z + 1 = A(z + 1 - 3i) + B(z + 1 + 3i)$

Solving we get, $A = B = \frac{1}{2}$

Now,

$$\oint_C \frac{z+1}{z^2+2z+4} dz = \frac{1}{2}\oint_C \frac{dz}{z+1+3i} + \frac{1}{2}\oint_C \frac{dz}{z+1-3i}$$

$$= \frac{1}{2}(I_1 + I_2) \quad \dots \text{(i)}$$

Given circle is C: $|z + 1 + i| = 2$.

Since $I_2$ is analytic except at $3i - 1$ which lies in C but $I_1$ is analytic except at $-1 - 3i$ which does not lie in C. That means $I_1$ is analytic everywhere around C.

So, by Cauchy's Integral Theorem, $I_1 = 0$.

But for $I_2$, by Cauchy's Integral Formula,

$$I_2 = \oint_C \frac{dz}{z+1+3i} dz = 2\pi i\, f(-1-3i). \quad \dots \text{(ii)}$$

$\frac{1}{2}(2\pi i + 0) = \pi i$



Comparing $I_2$ with $\oint_c \frac{f(z)}{(z - z_0)} dz$ then we get,

$z_0 = -1-3i$, and $f(z) = 1$.

Then, $f(-1-3i) = 1$.

Therefore, (ii) becomes

$$I_2 = 2\pi i$$

Thus (i) becomes

$$I = \frac{1}{2}(2\pi i + 0) = \pi i$$

**15.** $\oint_C \frac{\cos \pi z}{z-1} dz$ **C : Square with vertices ± 2, ± 2i**

**Solution:** Given that $\oint_C \frac{\cos \pi z}{z-1} dz$ and C : Square with vertices ± 2, ± 2i.

Clearly, the integrand function is analytic except at z = 1 which lies in C.
So, by the Cauchy's Integral Formula,

$$\oint_c \frac{\cos \pi z}{z-1} dz = 2\pi i\, f(1) \quad \dots(1) \quad \left[\because \oint_c \frac{f(z)}{(z-z_0)} dz = 2\pi i\, f(z_0)\right]$$

Comparing $\oint_c \frac{\cos \pi z}{z-1} dz$ with $\oint_c \frac{f(z)}{(z-z_0)} dz$ then we get,

$z_0 = 1$ and $f(z) = \cos\pi z$.

Then, $f(1) = \cos\pi = -1$.

Therefore, (1) becomes

$$\oint_C \frac{\cos \pi z}{z-1} dz = -2\pi i.$$

## OTHER IMPORTANT QUESTION FROM FINAL EXAM FOR PRACTICE

**2002 Q. No. 1(b)**

**State Cauchy's Integral Formula and use it to evaluate: $\int_c \frac{z^2+1}{z^2-1} dz$ where C is the circle of radius 1 with centre at the point: (i) z = 1 (ii) z = –1.**

**Solution:** **Statement of Cauchy's Integral Formula:**

Let f(z) is analytic in a simply connected domain D. Then, for any point $z_0$ in D and any simple closed path c in D that encloses $z_0$ such that.

$$\oint_c \frac{f(z)}{z - z_0} dz = 2\pi i\, f(z_0)$$

where, the path c is taken in counterclockwise.

**Problem Part:** Given that,

$$\oint_C \frac{z^2+1}{z^2-1}dz = \oint_C \frac{z^2+1}{(z+1)(z-1)}dz$$

Clearly, given function is not analytic at $z = 1$ and $z = -1$.

(i) Given that $\oint_C \frac{(z^2+1)/(z+1)}{(z-1)}dz$ and C : unit circle with 1.

Clearly, the integrand function is analytic except at $z = 1$ which lies in C.

So, by the Cauchy's Integral Formula,

$$\oint_C \frac{(z^2+1)/(z+1)}{(z-1)}dz = 2\pi i\, f(1) \quad \ldots(1) \quad \left[\because \oint_C \frac{f(z)}{(z-z_0)}dz = 2\pi i\, f(z_0)\right]$$

Comparing $\oint_C \frac{(z^2+1)/(z+1)}{(z-1)}dz$ with $\oint_C \frac{f(z)}{(z-z_0)}dz$ then we get,

$$z_0 = 1 \quad \text{and} \quad f(z) = \frac{z^2+1}{z+1}.$$

Then, $f(1) = \frac{2}{2} = 1$.

Therefore, (1) becomes

$$\oint_C \frac{(z^2+1)/(z+1)}{(z-1)}dz = 2\pi i.$$

(ii) Given that $\oint_C \frac{(z^2+1)/(z-1)}{(z+1)}dz$ and C : unit circle with $-1$.

Clearly, the integrand function is analytic except at $z = -1$ which lies in C.

So, by the Cauchy's Integral Formula,

$$\oint_C \frac{(z^2+1)/(z-1)}{(z+1)}dz = 2\pi i\, f(-1) \quad \ldots(1) \quad \left[\because \oint_C \frac{f(z)}{(z-z_0)}dz = 2\pi i\, f(z_0)\right]$$

Comparing $\oint_C \frac{(z^2+1)/(z-1)}{(z+1)}dz$ with $\oint_C \frac{f(z)}{(z-z_0)}dz$ then we get,

$$z_0 = -1 \quad \text{and} \quad f(z) = \frac{z^2+1}{z-1}.$$

Then, $f(-1) = \frac{2}{-2} = -1$.

Therefore, (1) becomes

$$\oint_C \frac{(z^2+1)/(z-1)}{(z+1)}dz = -2\pi i.$$



**2003 (Fall) Q. No. 1(b)**

**State and prove Cauchy Integral Formula. Evaluate $\oint_C \frac{2z^2+3}{(z-2)^2}dz$, $C : |z-1| = 2$**

**2004(Fall) Q. No. 1(b)**

**State Cauchy's Integral Formula. Use it to evaluate the integral: $\int_C \frac{4-\sin z}{z^2-2z}$ dz, where ic is the square with vertices $\pm 1, \pm i$.**

**2004 (Spring) Q. No. 1(b)**

**State Cauchy's theorem. Evaluate the integral $\int_C \frac{1}{z^2+4}dz$, whre c is ellipse $4x^2 + (y-2)^2 = 4$.**

**Hint: See Exercise 3.3 Q. No. 3(a).**

**2005 Fall Q. No. 1(b)**

**State and prove Cauchy-integral formula and hence evaluate $\oint_C \frac{2z^2+4z}{z-2}dz$; $c : |z| = 1$.**

**2005 Spring Q. No. 1(b)**

**State and prove Cauchy Integral Formula. Evaluate $\oint_C \frac{e^{5z}}{(z+i)^3}dz$, where C is the circle $|z| = 3$, counter clockwise.**

**Hint: See Exercise 3.4 Q. No. 2(f).**

**2006 Fall Q. No. 1(b)**

**State and prove Cauchy Integral Theorem. Integrate $f(z) = \frac{z^3}{2z-i}$ around the unit circle clockwise.**

**Hint: See Exercise 3.3 Q. No. 2(a).**

**2006 Spring Q. No. 1(b)**

**Evaluate: $\oint_C \frac{\cot z}{(z-\pi/2)^2}dz$, where C is the ellipse $4x^2 + 9y^2 = 36$.**

**2007 (Fall) Q. No. 1(b)**

**State Cauchy Integral Theorem. Evaluate $\oint_C \frac{\log(z-1)}{z-6}dz$, where c is the circle $|z-6| = 4$.**

**Hint: See Exercise 3.3 Q. No. 3(b).**

**2007 Spring Q. No. 1(b)**

**State Cauchy Integral Theorem. Evaluate the integral $\oint_C \left(\frac{z+1}{(z^3-4z)}\right)dz$, where C is the circle $|z+2| = \frac{3}{2}$ in anticlockwise direction.**

**Hint: See Exercise 3.4 Q. No. 2(r (i)).**

2008 Fall Q. No. 1(b)

State the Cauchy Integral Formula and by using it evaluate the following integral: $\oint_C \frac{z^2}{z^2-1}dz$, where C is a positively oriented circle $|z-1| = 1$.

2008 Spring Q. No. 1(b)

State and prove Cauchy Integral Formula. Use it to integrate $\oint_C \frac{z}{z-2}dz$ where c : $|z| = 2$ counterclockwise.

2009 Fall Q. No. 1(b)

State the Cauchy Integral Formula and by using it evaluate the following integral: $\int_C \frac{e^z}{z-2}dz$.

(i) C is a positively oriented circle $|z| = 1$.
(ii) C is a positively oriented circle $|z-2| = 2$.

2009 (Spring) Q. No. 1(a)

State and prove Cauchy's Integral Formula and use it to find $\int_C \frac{z\sin z}{2z-1}dz$ where C is the unit circle counterclockwise.

2009 Spring Q. No. 2(a) OR

State Cauchy Integral Formula. Evaluate the integral $\oint_C \frac{\cosh 4z}{(z-4)^3}$, where C consists of $|z| = 6$ (counterclockwise) and $|z-3| = 2$ (clockwise).

Hint: See Exercise 3.4 Q. No. 2(e).

2011 Fall Q. No. 1(b)

Evaluate the following integrals using Cauchy's Integral Formula

(i) $\oint_C \frac{z+1}{z^3-4z}dz$, C : $|z-2| = 3/2$ (ii) $\oint_C \frac{z+1}{z^3-2z^2}dz$, C is the unit circle.

2011 Spring Q. No. 1(b)

State and prove Cauchy Integral Theorem. Integrate $f(z) = e^{-z^2}$ around the unit circle counter clockwise.

Hint: See Exercise 3.2 Q. No. 1(a).

2012 Fall Q. No. 1(b) OR

State and prove Cauchy's Integral Theorem.

2016 Fall Q. No. 1(b)

Integrate the followings along the unit circle counterclockwise

(i) $\oint \frac{Z^6}{(2Z-1)^6}$ (ii) $\oint \frac{z+1}{z^3-2z^2}$.



2016 Spring Q. No. 1(b)

State and prove Cauchy integral formula. Integrate $\oint_C \frac{1}{z^2+4}dz$, C . $4x^2 + (y-2)^2 = 4$ counter clockwise.

Hint: Prove the theorem and See Exercise 3.3 Q. No. 3(a).

2017 Fall Q. No. 1(b)

State and prove Cauchy's integral formula. Evaluate the integral $\int \frac{\cos z}{(z-\pi i)^2}dz$ where c is unit circle enclosing the point $\pi i$.

Hint: Prove the theorem and See Exercise 3.4 Q. No. 2(i).

## SHORT QUESTIONS

2002 Q. No. 7(b); 2007 Fall Q. No. 7(b); 2012 Fall Q. No. 7(b)

Evaluate $\int_C \frac{1}{z}dz$ where c is the unit circle.

2006 Fall Q. No. 7(c) OR

Show that $\oint_C \frac{dz}{z} = 2\pi i$, where C is the unit circle, counter–clockwise.

2005 Spring Q. No. 7(i)

Evaluate $\oint_C \frac{dz}{z-3i}$, where c is the circle, $|z| = 4$ counter clockwise direction.

2007 Spring Q. No. 7(e)

Evaluate the integral $\oint_C \frac{1}{z^2}dz$, where C is a unit circle.

2016 Spring Q. No. 7(c)

Evaluate $\oint \frac{z^3 \sin z}{3z-1}$ dz along a unit circle.

2017 Fall Q. No. 7(c)

Evaluate $\oint_C \frac{dz}{z}$, where c is the unit disk $|z| = 1$.

□□□

## Unit 5

# SEQUENCE AND SERIES

### Sequence and Series:

An infinite sequence is obtained by assigning to each positive integer n a number $z_n$, is called a term of sequence and is written as

$$z_1, z_2, z_3, \ldots \text{ or } \{z_n\}.$$

An infinite series has the form $\sum_{n=0}^{\infty} z_n$.

### Series of complex function:

Let, $f_1(z) + f_2(z) + f_3(z) + \ldots$ be an infinite series of complex function of complex number.

### Taylor Series and Maclaurin's Series

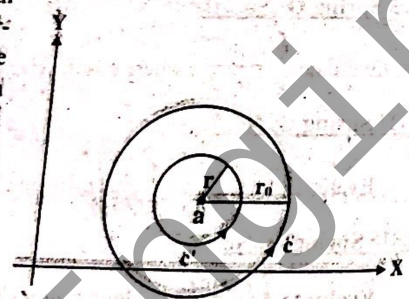
**Statement:** *If f(z) be analytic in c with centre at 'a' and radius $r_0$, then at each point z inside c, the series*

$$f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \ldots \text{ converges to } f(z).$$

*That is*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n;\ \text{where } a_n = \frac{f^{(n)}(a)}{n!}$$

**Proof:** Let c be a closed contour with center at a and radius $r_0$ in counter-clockwise (CCW) direction. Let z be a point inside c. Let c' be the circle in counter-clockwise direction such that $|z - a| = r < r_0$; r being chosen such that, the point z is interior to c'. The function f(z) is analytic inside and on c. Therefore, by Cauchy's Integral Formula,




$$f(z) = \frac{1}{2\pi i}\oint_c \frac{f(w)}{w-z}\,dw \qquad \ldots\ldots(1)$$

Since $1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = \frac{1-\alpha^n}{1-\alpha}$

$$\Rightarrow \frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} + \frac{\alpha^n}{1-\alpha} \qquad \ldots(2)$$

Here,

$$\frac{1}{w-z} = \frac{1}{w-a-(z-a)} = \frac{1}{(w-a)}\left[1 - \frac{z-a}{w-a}\right]^{-1}$$



$$\Rightarrow \frac{1}{w-z} = \frac{1}{(w-a)}\left\{1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \ldots + \left(\frac{z-a}{w-a}\right)^{n-1} + \frac{[(z-a)/(w-a)]^n}{\left[1 - \frac{z-a}{w-a}\right]}\right\}$$

[Applying (2)]

$$\Rightarrow \frac{1}{w-z} = \frac{1}{(w-a)} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \ldots + \frac{(z-a)^{n-1}}{(w-a)^n} + \frac{(z-a)^n}{(w-a)^n(w-z)}$$

Multiplying both sides by $\frac{f(w)}{2\pi i}$ and then integrating around c' we get,

$$\frac{1}{2\pi i}\int_{c'} \frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{c'}\frac{f(w)}{w-a}dw + (z-a)\frac{1}{2\pi i}\int_{c'}\frac{f(w)}{(w-a)^2}dw + \ldots +$$

$$\frac{(z-a)^n}{2\pi i}\int_{c'}\frac{f(w)}{(w-a)^n(w-z)}dz$$

Since the integrand function is analytic in c' except at z and a that lie in c'. So, applying Cauchy Integral Formula to each integral then we get,

$$\Rightarrow f(z) = f(a) + f'(a)(z-a) + (z-a)^2\frac{f''(a)}{2!} + \ldots + \ldots + \frac{(z-a)^{n-1}}{(n-1)!}f^{n-1}(a) + R_n \qquad \ldots(3)$$

where $\quad R_n = \frac{(z-a)^n}{2\pi i}\int_{c'}\frac{f(w)}{(w-a)^n(w-z)}dw$

It can be shown that $|R_n| \to 0$ as $n \to \infty$. Therefore, by taking limit $n \to \infty$ in equation (3) we get,

$$f(z) = f(a) + \sum_{r=1}^{\infty}\frac{(z-a)^r}{r!}f^{(r)}(a)$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad \ldots\ldots(4)$$

where, $\quad a_n = \frac{f^n(a)}{n!}$

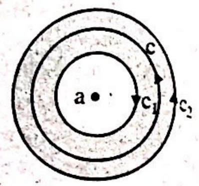
This series in equation represents f(z) for all z interior to c'. Since for any z inside c, corresponding c' and be found, the above representation is valid for any z inside c.

Thus equation (4) is the Taylor's series of f(z) about z = a.

## Laurent Series

**Statement:** *If f(z) is analytic on two concentric circles $c_1$ and $c_2$ with centre at a, and also in the annular region R bounded by $c_1$ and $c_2$, then at any point z in R, f(z) can be expressed as a convergent series of positive and negative power of (z – a) in the form*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$$




*where* $a_n = \dfrac{1}{2\pi i}\displaystyle\int_{c'} \dfrac{f(w)}{(w-a)^{n+1}}\,dw,\ n = 0, 1, 2, 3, \ldots$

*and* $b_n = \dfrac{1}{2\pi i}\displaystyle\int_{c'} \dfrac{f(w)}{(w-a)^{-n+1}}\,dw,\quad n = 1, 2, 3, 4, \ldots$

*where c being any simple closed curve lying within the annular and encircling the inner boundary of the annular region R.*

**Proof:**

Let us defined two circles $c_1$ and $c_2$ such that $|z - a| = r_1$ and $|z - a| = r_2$ respectively, where $r_2 > r_1$.

We have f(z) is analytic in R and also on $c_1$ and $c_2$. If z is any point on R, then we can apply Cauchy Integral Formula, by introducing a cross–cut AB,

$$f(z) = \frac{1}{2\pi i}\oint_{c_2} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\oint_{c_1} \frac{f(w)}{w-z}\,dw.$$

where the integration around $c_1$ and $c_2$ both being counterclockwise direction. Therefore,

$$f(z) = \frac{1}{2\pi i}\oint_{c_2} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\oint_{c_1} \frac{f(w)}{z-w}\,dw \qquad \ldots\ldots\ldots(1)$$

Here,

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)}\left[1 - \frac{z-a}{w-a}\right]$$

$$= \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \frac{(z-a)^2}{(w-a)^2} + \ldots + \frac{(z-a)^{n-1}}{(w-a)^{n-1}} + \frac{\left(\frac{z-a}{w-a}\right)^n}{1-\frac{z-a}{w-a}}\right]$$

Since we have

$$\frac{1}{w-z} = \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \ldots + \frac{(z-a)^{n-1}}{(w-a)^n} + \frac{(z-a)^n}{(w-z)(w-a)} \qquad \ldots..(2)$$

Also, we have



$$\frac{1}{z-w} = \frac{1}{z-a} + \frac{w-a}{(z-a)^2} + \frac{(w-a)^2}{(z-a)^3} + \ldots + \frac{(w-a)^{n-1}}{(z-a)^n} + \frac{(w-a)^n}{(z-a)^n(z-w)}$$

Substituting the values of $\frac{1}{w-z}$ and $\frac{1}{z-w}$ in equation (1) from (2) and (3) we get,

$$f(z) = \sum_{r=0}^{n-1} \frac{(z-a)^r}{2\pi i}\oint_{c_2} \frac{f(w)}{(w-a)^{r+1}}\,dw + R_1 +$$

$$\sum_{r=1}^{n} \frac{1}{(z-a)^r}\frac{1}{2\pi i}\oint_{c_1} \frac{f(w)}{(w-a)^{-r+1}}\,dw + R_2 \qquad \ldots\ldots(4)$$

where

$$R_1 = \frac{(z-a)^n}{2\pi i}\oint_{c_2} \frac{f(w)}{(w-a)^n(w-z)}\,dw$$

$$R_2 = \frac{1}{2\pi i\,(z-a)^n}\oint_{c_1} \frac{(w-a)^n f(w)}{(z-w)}\,dw$$

As $n\to\infty$, then the remainders $R_1$ and $R_2$ tends to zero.

Taking limit $n\to\infty$ in equation (4), we get,

$$f(z) = \sum_{r=0}^{n} a_r(z-a)^r + \sum_{r=1}^{n} b_r(z-a)^{-r}$$

where

$$a_r = \frac{1}{2\pi i}\oint_{c_2} \frac{f(w)}{(w-a)^{r+1}}\,dw,\qquad b_r = \frac{1}{2\pi i}\oint_{c_1} \frac{f(w)}{(w-a)^{r+1}}\,dw.$$

From Cauchy's integral theorem for multiply connected region, it follows the curves $c_2$ and $c_1$ in $a_r$ and $b_r$ be replaced by c, where c be any closed curve lying in the annular region bounded by $c_1$ and $c_2$.

Therefore, we get Laurent series of f(z) is

$$f(z) = \sum_{r=0}^{n} a_n(z-a)^n + \sum_{r=1}^{n} b_n(z-a)^{-n}$$

where,

$$a_n = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{n+1}}\,dz \quad \text{and} \quad b_n = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{-n+1}}\,dz$$

**Some Formulae**

1. Expansion of $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \ldots$ if $|z| < 1$.

2. Expansion of $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\ldots$

3. Expansion of $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \ldots\ldots$

4. Expansion of $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\ldots$

5. Expansion of $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots\ldots$

6. Expansion of $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots..$

7. Expansion of $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots\ldots$

8. Expansion of $\log\left[\frac{1}{1-z}\right] = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots..$

9. Expansion of $\log\left[\frac{1+z}{1-z}\right] = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots..\right)$

## Exercise 5.1

1. **Find the Maclaurin's series of the following functions.**

**(a)** $f(z) = \cos 2z^2$

**Solution:** Since we have,

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{(z)^{2n}}{(2n)!}$$

Then, $f(z) = \cos 2z^2 = \sum_{n=0}^{\infty} (-1)^n \frac{(2z^2)^{2n}}{(2n)!}$

$$= 1 - \frac{(2z^2)^2}{2!} + \frac{(2z^2)^4}{4!} + \ldots$$

$$= 1 - 2z^4 + \frac{2}{3}z^8 \ldots$$

This is required Maclaurin's series.

**(b)** $f(z) = \sin^2 z$

**Solution:** Since we have,

$$f(z) = \frac{\sin^2 z}{2} = \frac{1 - \cos 2z}{2}$$



Then, $f(z) = \frac{1 - \cos 2z}{2} = \frac{1}{2} - \frac{1}{2}\left\{\sum_{n=0}^{\infty} (-1)^n \frac{(2z)^{2n}}{(2n)!}\right\}$

$$= \frac{1}{2} - \frac{1}{2}\left(1 - \frac{4z^2}{2!} + \frac{16z^4}{4!} - \frac{64z^6}{6!} + \ldots\ldots\right)$$

$$= z^2 - \frac{z^4}{3} + \frac{2z^6}{45} - \ldots\ldots$$

which is required Maclaurin's series.

**(c)** $f(z) = \frac{z+2}{1-z^2}$ **[2006 Spring Q. No. 7(a)]**

**Solution:** Since we have,

$$\frac{1}{(1+z)} = \sum_{n=0}^{\infty} (-z)^n \quad \text{for } |z| < 1.$$

Here,

$$f(z) = \frac{z+2}{(1+z)(1-z)} = \frac{3}{2(1-z)} + \frac{1}{2(1+z)}$$

$$= \frac{3}{2}\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty} (-z)^n \quad \text{for } |z| < 1.$$

$$= \frac{3}{2}\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n z^n$$

$$= \frac{1}{2}(3 + 3z + 3z^2 + 3z^3 + \ldots + 1 - z + z^2 - z^3 + \ldots)$$

$$= 2 + z + 2z^2 + z^3 + \ldots\ldots$$

This is required Maclaurin's series.

**(d)** $f(z) = \frac{1}{z+3i}$

**Solution:** Since we have,

$$\frac{1}{(1+z)} = \sum_{n=0}^{\infty} (-z)^n \quad \text{for } |z| < 1.$$

Here, $f(z) = \frac{1}{z+3i} = \frac{1}{3i\left(1 + \frac{z}{3i}\right)}$

$$= \frac{1}{3i}\sum_{n=0}^{\infty}\left(-\frac{z}{3i}\right)^n \quad \text{for } \left|\frac{z}{3i}\right| < 1$$

$$= \frac{1}{3i}\left(1 - \frac{z}{3i} + \frac{z^2}{9i^2} - \frac{z^3}{27i^3} + \frac{z^4}{81i^4} - \ldots\ldots\right)$$

$$= \frac{-i}{3}\left(1 - \frac{z}{3i} + \frac{z^2}{-9} - \frac{z^3}{-27i} + \frac{z^4}{81} - \ldots\ldots\right)$$

$$= \frac{-i}{3} + \frac{z}{9} - \frac{iz^2}{27} - \frac{iz^3}{81} - \frac{iz^4}{243} - \ldots\ldots$$

(e) $f(z) = \frac{1}{z^2+1}$

**Solution:** Since we have,

$$\frac{1}{(1+z)} = \sum_{n=0}^{\infty} (-z)^n \quad \text{for } |z| < 1.$$

Here,

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{A}{z+i} + \frac{B}{z-i} \quad \ldots\ldots\ldots \text{(i)}$$

Here, $A = \lim_{z\to i}\left(\frac{1}{z+i}\right) = \frac{-1}{2i}$

And, $B = \lim_{z\to -i}\left(\frac{1}{z-i}\right) = \frac{1}{2i}$

Therefore (i) becomes,

$$f(z) = \frac{1}{2i}\left\{\frac{-1}{z+i} + \frac{1}{z-i}\right\} = \frac{-1}{2i} \times \frac{1}{i}\left[\frac{1}{1+z/i} + \frac{1}{1-z/i}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1+z/i} + \frac{1}{1-z/i}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(-\frac{z}{i}\right)^n + \sum_{n=0}^{\infty}\left(\frac{z}{i}\right)^n\right] \text{ for } \left|\frac{z}{i}\right| < 1$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}[(-1)^n + 1]\left(\frac{z}{i}\right)^n$$

$$= \frac{1}{2}\left(2 + 2\frac{z^2}{i^2} + 2\frac{z^4}{i^4} + 2\frac{z^6}{i^6} + \ldots\ldots\ldots\right)$$

$$= 1 - z^2 + z^4 - z^6 + \ldots\ldots$$

(f) $f(z) = \frac{z-1}{z+1}$

**Solution:** Since we have,

$$\frac{1}{(1+z)} = \sum_{n=0}^{\infty} (-z)^n \quad \text{for } |z| < 1.$$

Here, $f(z) = \frac{z-1}{z+1} = \frac{z+1-2}{z+1}$

$$= 1 - \frac{2}{z+1}$$

$$= 1 - 2\sum_{n=0}^{\infty}(-1)^n z^n \quad \text{for } |z| < 1.$$

$$= 1 - 2(1 - z + z^2 - z^3 + \ldots\ldots\ldots)$$

$$= -1 + 2(z - z^2 + z^3 - \ldots\ldots\ldots).$$



**2. Find Taylor's expansion of the given function at a specified points.**

**(a)** $f(z) = \frac{1}{z}$ **at** $z = 2$

**Solution:** Since the Taylor's series expansion of f(z) at z = a is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{where, } a_n = \frac{f^n(a)}{n!}$$

$$\text{i.e. } f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n \quad \ldots (1)$$

Given that,

$$f(z) = \frac{1}{z}$$

Differentiate with respect to x we get,

$$f'(z) = \frac{(-1)}{z^2}, \quad f''(z) = \frac{2}{z^3}, \ldots, f^n(z) = (-1)^n \frac{(n)!}{z^{n+1}}$$

So, at z = 2, $f^n(2) = (-1)^n \frac{n!}{2^{n+1}}$ and therefore $a_n = \frac{(-1)^n n!}{2^{n+1}} \times \frac{1}{n!} = \frac{(-1)^n}{2^{n+1}}$

Then, the equation (1) at z = 2 becomes,

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(z-2)^n = \frac{1}{2} - \frac{(z-2)}{4} + \frac{(z-2)^2}{8} - \frac{(z-2)^3}{16} + \ldots$$

**(b)** $f(z) = e^z$ **at** $z = a$

**Solution:** Since the Taylor's series expansion of f(z) at z = a is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{where, } a_n = \frac{f^n(a)}{n!}$$

$$\text{i.e. } f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n \quad \ldots (1)$$

Given that, $f(z) = e^z$.

Differentiate with respect to x we get,

$$f'(z) = e^z, \quad f''(z) = e^z, \ldots, f^n(z) = e^z.$$

At z = a, $f^n(a) = e^a$ and so, $a_n = e^a \times \frac{1}{n!}$

Then, the equation (1) at z = a becomes,

$$f(z) = \sum_{n=0}^{\infty} \frac{e^a}{n!}(z-a)^n = e^a\left(1 + (z-a) + \frac{(z-a)^2}{2!} + \frac{(z-a)^3}{3!} + \ldots\right)$$

**(c)** $f(z) = \log z$ **at** $z = 1$. ***(note that the book use log for natural logarithm i.e. ln)***

**Solution:** Since the Taylor's series expansion of f(z) at z = a is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad \text{where, } a_n = \frac{f^n(a)}{n!}$$

$$\text{i.e. } f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n \qquad \dots (1)$$

Given that. $f(z) = \log(z)$.

Differentiate with respect to x then we get.

$$f(z) = \log z. \quad f'(z) = \frac{1}{z}, \quad f''(z) = -\frac{1}{z^2}, \dots \; f^n(z) = (-1)^{n+1}\frac{(n-1)!}{z^n}$$

$$\text{At } z = 1 \text{ Then } a_n = \frac{f^n(1)}{n!} = \frac{(-1)^{n+1}(n-1)!}{(n-1)!\, n\, 1^n} = \frac{(-1)^{n+1}}{n}$$

Then, the equation (1) at z = 1 becomes.

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n = \left((z-1) - \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} - \dots\dots\right)$$

**(d)** $f(z) = \sin z$ at $z = \frac{\pi}{2}$

**Solution:** Since the Taylor's series expansion of f(z) at z = a is.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad \text{where, } a_n = \frac{f^n(a)}{n!}$$

$$\text{i.e. } f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n \qquad \dots (1)$$

Given that, $f(z) = \sin z$.

Differentiate with respect to x then we get.

$$f(z) = \sin z, \quad f'(z) = \cos z = \sin\left(z + \frac{\pi}{2}\right), \dots, f^n(z) = \sin\left(z + \frac{n\pi}{2}\right)$$

$$\text{At } z = \frac{\pi}{2}, \; f^n\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} + n\frac{\pi}{2}\right) = \sin\frac{\pi}{2}\cos\frac{n\pi}{2} + \cos\frac{\pi}{2}\sin\frac{n\pi}{2} = \cos\frac{n\pi}{2}$$

$$\text{So, } a_n = \frac{f^n\left(\frac{\pi}{2}\right)}{n!} = \frac{\cos\frac{n\pi}{2}}{n!}$$

Then, the equation (1) at $z = \frac{\pi}{2}$ becomes.

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{\cos\frac{n\pi}{2}}{n!}\right)\left(z - \frac{\pi}{2}\right)^n$$

$$= \left(1 - \frac{(z-\pi/2)^2}{2!} + \frac{(z-\pi/4)^2}{4!} - \frac{(z-\pi/2)^3}{6!} + \dots\dots\right)$$



**(e)** $f(z) = \sinh(2z - i)$ at $z = \frac{i}{2}$

**Solution:** Since the Taylor's series expansion of f(z) at z = a is.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad \text{where, } a_n = \frac{f^n(a)}{n!}$$

$$\text{i.e. } f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n \qquad \dots (1)$$

Here, we have to find series at $z = \frac{i}{2}$ i.e. $2z - i = 0$. Given that

$$f(z) = \sinh(2z - i)$$

Differentiate with respect to x we get,

$$f'(z) = \cosh(2z-i) \cdot 2 = 2\sinh\left(2z - i + \frac{\pi}{2}\right),$$

$$f''(z) = 2\cosh\left(2z - i + \frac{\pi}{2}\right) \cdot 2 = 2^2\sinh\left(2z - i + \frac{2\pi}{2}\right), \dots\dots$$

$$f^n(z) = 2^{n-1}\cosh\left(2z - i + \frac{(n-1)\pi}{2}\right) \cdot 2 = 2^n\sinh\left(2z - i + \frac{n\pi}{2}\right).$$

$$\text{At } z = \frac{i}{2}, \quad f^n\left(\frac{i}{2}\right) = 2^n\sinh\left(2\left(\frac{i}{2}\right) - i + \frac{n\pi}{2}\right) = 2^n\sinh\left(\frac{n\pi}{2}\right).$$

$$\text{So, } a_n = \frac{f^n\left(\frac{i}{2}\right)}{n!} = \frac{2^n\sinh\frac{n\pi}{2}}{n!}$$

Then, the equation (1) at $z = \frac{i}{2}$ becomes,

$$f(z) = \sum_{n=0}^{\infty} \frac{2^n\sinh\frac{n\pi}{2}}{n!}\left(z - \frac{i}{2}\right)^n$$

$$= \left(2(z - i/2) + \frac{2^3(z-i/2)^3}{3!} + \frac{2^5(z-i/4)^5}{5!} + \dots\dots\right)$$

**(f)** $f(z) = e^{z^2 - 2z}$ **at z = 1.**

**Solution:** Since the Taylor's series expansion of f(z) at z = a is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad \text{where, } a_n = \frac{f^n(a)}{n!}$$

$$\text{i.e. } f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n \qquad \dots (1)$$

Given that,

$$f(z) = e^{z^2 - 2z} = e^{(z-1)^2 + 2z - 1 - 2z} = \frac{1}{e}e^{(z-1)^2}$$

Since we have the Maclaurin's series of $e^z$ is, $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Therefore (1) becomes at $z = 1$ is,

$$f(z) = \frac{1}{e} \sum_{n=0}^{\infty} \frac{(z-1)^{2n}}{n!}.$$

**(g) $f(z) = \cosh(z - \pi i)$ at $z = \pi i$**

**Solution:** Since the Taylor's series expansion of f(z) at z = a is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad \text{where, } a_n = \frac{f^n(a)}{n!}$$

$$\text{i.e. } f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (z-a)^n \qquad \dots (1)$$

Given that,

$$f(z) = \cosh(z - \pi i)$$

Differentiate with respect to x we get,

$$f'(z) = \sinh(z - \pi i) = \cosh\left(z - \pi i + \frac{\pi}{2}\right).$$

$$f''(z) = \sinh\left(z - \pi i + \frac{\pi}{2}\right) = \cosh\left(z - \pi i + \frac{2\pi}{2}\right), \dots\dots\dots$$

$$f^n(z) = \cosh\left(z - \pi i + \frac{n\pi}{2}\right).$$

At $z = \pi i$, $\quad f^n(\pi i) = \cosh\left(\frac{n\pi}{2}\right)$. Then $a_n = \frac{f^n(\pi i)}{n!} = \frac{\cosh\frac{n\pi}{2}}{n!}$

Then, the equation (1) at $z = \pi i$ becomes,

$$f(z) = \sum_{n=0}^{\infty} \frac{\cosh\frac{n\pi}{2}}{n!} (z - \pi i)^n.$$

$$= \sum_{n=0}^{\infty} \frac{(z - \pi i)^n}{n!}$$

$$= \left(1 + \frac{(z-\pi i)^2}{2!} + \frac{(z-\pi i)^4}{4!} + \frac{(z-\pi i)^6}{6!} + \dots\dots\dots\right)$$

3. **Write Taylor's expansion for the functions given below in the specified region.**

**(a) $f(z) = \frac{1}{1-z}$ in $|z| < \frac{1}{2}$**

**Solution:** Since we have,



$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{for } |z| < 1.$$

Therefore,

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{is valid for } |z| < \frac{1}{2} < 1.$$

**(b) $f(z) = \frac{1}{z^2 + 2z + 3}$ in $|z| < 1$.**

**Solution:** Since we have,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{for } |z| < 1.$$

Here,

$$f(z) = \frac{1}{z^2 + 2z + 3}$$

$$= \frac{1}{(z+1)^2 + (\sqrt{2})^2} = \frac{1}{(z + 1 + \sqrt{2}\,i)(z + 1 - \sqrt{2}\,i)}$$

$$\Rightarrow \quad f(z) = \frac{A}{z + 1 + \sqrt{2}\,i} + \frac{B}{2 + 1 - \sqrt{2}\,i}$$

Since,

$$A = \lim_{z \to (-1-\sqrt{2}\,i)} \frac{1}{z + 1 - \sqrt{2}\,i} = \frac{1}{-1 - \sqrt{2}\,i + 1 - \sqrt{2}\,i} = \frac{1}{-2\sqrt{2}\,i}$$

$$B = \lim_{z \to (-1-\sqrt{2}\,i)} \frac{1}{z + 1 + \sqrt{2}\,i} = \frac{1}{-1 + \sqrt{2}\,i + 1 + \sqrt{2}\,i} = \frac{1}{2\sqrt{2}\,i}$$

Then, $\quad f(z) = \frac{1}{2\sqrt{2}i}\left[\frac{-1}{z + 1 + \sqrt{2}i} + \frac{1}{2 + 1 - \sqrt{2}i}\right]$

$$= \frac{1}{2\sqrt{2}i}\left[\left\{-\frac{1}{1+\sqrt{2}i}\right\}\left\{\frac{1}{1 + \frac{z}{1+\sqrt{2}i}}\right\} + \left\{\frac{1}{1-\sqrt{2}i}\right\}\left\{\frac{1}{1 + \frac{z}{1-\sqrt{2}i}}\right\}\right]$$

$$= \frac{1}{2\sqrt{2}i}\left\{\left\{-\frac{1}{1+\sqrt{2}i}\right\} \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(1+\sqrt{2}i)^n} + \left\{\frac{1}{1-\sqrt{2}i}\right\} \sum_{n=0}^{\infty} \frac{z^n}{(1-\sqrt{2}i)}\right\}$$

$$= \frac{1}{2\sqrt{2}i}\left\{\sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^n}{(1+\sqrt{2}i)^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{(1-\sqrt{2}i)^{n+1}}\right\}. \qquad \dots\dots(i)$$

This expansion of $\left(1 + \frac{z}{1+\sqrt{2}i}\right)^{-1}$ is valid for

$$\left\{\frac{z}{1+\sqrt{2}i}\right\} < 1 \Rightarrow |z| < \sqrt{3}$$

And the expansion of $\left(1+\dfrac{z}{1-\sqrt{2}i}\right)^{-1}$ is valid for

$$\left\{\frac{z}{1-\sqrt{2}i}\right\} < 1 \Rightarrow |z| < \sqrt{3}$$

Thus the Taylor expansion of f(z) is (i) that is valid for $|z| < \sqrt{3}$.

**(c)** $f(z) = \dfrac{z^3+2z^2}{z^2+2z+3}$ in $|z| < \dfrac{1}{2}$

**Solution:** Since we have,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{for } |z| < 1.$$

Here,

$$f(z) = z - \frac{3z}{z^2+2z+3}$$

$$= z - \frac{3z}{(z+1-i\sqrt{2})(z+1+i\sqrt{2})}$$

$$= z - 3\left[\frac{A}{z+1-i\sqrt{2}} + \frac{B}{2+1+i\sqrt{2}}\right] \qquad \ldots\ldots\ldots \text{(i)}$$

Since, $A = \lim_{z\to(-1+i\sqrt{2})} \dfrac{z}{z+1+i\sqrt{2}}$

$$= \frac{-1+i\sqrt{2}}{2\sqrt{2}\,i}$$

And, $B = \lim_{z\to(-1-i\sqrt{2})} \dfrac{z}{z+1-i\sqrt{2}} = \dfrac{-(1+i\sqrt{2})}{-2\sqrt{2}\,i} = \dfrac{1+i\sqrt{2}}{2\sqrt{2}\,i}$

Therefore,

$$f(z) = z - \frac{3}{2i\sqrt{2}}\left[\frac{-1+i\sqrt{2}}{z+1-i\sqrt{2}} + \frac{1+i\sqrt{2}}{z+1+i\sqrt{2}}\right]$$

$$= z - \frac{3}{2i\sqrt{2}}\left[(-1+i\sqrt{2})\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{(1-i\sqrt{2})^{n+1}} + \sum_{n=0}^{\infty}\frac{(1+i\sqrt{2})}{(1+i\sqrt{2})}(-1)\frac{z^n}{(1+i\sqrt{2})^n}\right]$$

$$= z - \frac{3}{2i\sqrt{2}}\left[\sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^n}{(1-i\sqrt{2}^n)} + \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{(1+i\sqrt{2}^n)}\right]$$

$$= z - \frac{3}{2i\sqrt{2}}\sum_{n=0}^{\infty}\left[\frac{1}{(1+i\sqrt{2})^n} - \frac{1}{(1-i\sqrt{2})^n}\right](-1)^n z^n.$$



This expansion is valid only when,

$$\left|\frac{z}{1-i\sqrt{2}}\right| < 1 \quad \text{or} \quad \left|\frac{z}{1+i\sqrt{2}}\right| < 1$$

$$\Rightarrow |z| < 3 \qquad\qquad \Rightarrow |z| < 3.$$

Thus the Taylor series expansion of given function is valid for $|z| < 3$.

**4. Obtain Taylor's expansion for the functions given below and specify the region of validity in each case.**

**(a)** $f(z) = \dfrac{z^3+1}{z(z+1)}$ **about the point z = i**

**Solution:** Since we have,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{for } |z| < 1.$$

Here,

$$f(z) = \frac{2z^3+1}{z(z+1)} = 2z-2+\frac{2z+1}{z(z+1)} = 2z-2+\frac{1}{z}+\frac{1}{z+1}$$

Now for $|z-i| < 1$,

$$f(z) = 2z-2+\frac{1}{z-i+i}+\frac{1}{z-i+i+1}$$

$$= 2z-2+\frac{1}{i\left(1+\frac{z-i}{i}\right)}+\frac{1}{(i+1)\left(1+\frac{z-i}{i+1}\right)}$$

$$= 2z-2+\frac{1}{i}\sum_{n=0}^{\infty}(-1)^n\frac{(z-i)^n}{i^n}+\frac{1}{i+1}\sum_{n=0}^{\infty}(-1)^n\frac{(z-i)^n}{(i+1)^n}$$

$$= 2z-2+\sum_{n=0}^{\infty}\left[\frac{1}{i^{n+1}}+\frac{1}{(i+1)^{n+1}}\right](-1)^n(z-i)^n \qquad \ldots \text{(i)}$$

This expansion is valid only when

$$\left|\frac{z-i}{i}\right| < 1 \Rightarrow \quad |z-i| < 1 \text{ (valid)}$$

and $\quad \left|\dfrac{z-i}{i+1}\right| < 1 \Rightarrow \quad |z-i| < \sqrt{2}$ (not valid).

Thus, the Taylor's expansion of f(x) is (i) with validity $|z-i| < 1$.

**(b)** $f(z) = \dfrac{2z^3+1}{z(z+1)}$ **about the point z = –i**

**Solution:** Since we have,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{for } |z| < 1.$$

Here for $|z + i| < 1$,

$$f(z) = 2z - 2 + \frac{2z+1}{z(z+1)}$$

$$= 2z - 2 + \frac{1}{z} + \frac{1}{z+1}$$

$$= 2z - 2 + \frac{1}{z+i-i} + \frac{1}{z+i-i+1}$$

$$= 2z - 2 + \frac{1}{-i\left(1 - \frac{(z+i)}{i}\right)} + \frac{1}{-(i-1)} \frac{1}{\left(1 - \frac{z+i}{i-1}\right)}$$

$$= 2z - 2 + \left(-\frac{1}{i}\right) \sum_{n=0}^{\infty} \left(\frac{z+i}{i}\right)^n + \left(\frac{-1}{i-1}\right) \sum_{n=0}^{\infty} \left(\frac{z+i}{i-1}\right)$$

$$= 2z - 2 - \sum_{n=0}^{\infty} \frac{(z+i)^n}{i^{n+1}} - \sum_{n=0}^{\infty} \frac{(z+i)^n}{(i-1)^{n+1}} \quad \dots (i)$$

This expansion is valid only when

$$\left|\frac{z+i}{i}\right| < 1 \qquad \text{or} \qquad \left|\frac{z+i}{i-1}\right| < 1$$

$\Rightarrow$ $|z + i| < 1$ (valid) $\qquad |z + i| < 2$ (not valid).

Thus, the Taylor's expansion of f(x) is given in (i) and is valid for $|z + i| < 1$.

**(c)** $f(z) = \cos z$ **about** $z = \frac{\pi}{4}$

**Solution:** Here for $|z - \pi/4| < 1$,

$$f(z) = \cos\left(z - \frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$= \cos\left(z - \frac{\pi}{4}\right) . \cos\frac{\pi}{4} - \sin\left(z - \frac{\pi}{4}\right) \sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}\left[\cos\left(z - \frac{\pi}{4}\right) - \sin\left(z - \frac{\pi}{4}\right)\right]$$

Since,

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\therefore \quad f(z) = \frac{1}{\sqrt{2}}\left(\sum_{n=0}^{\infty} (-1)^n \frac{(z - \pi/4)^{2n}}{(2n)!} - \sum_{n=0}^{\infty} (-1)^n \frac{(z - \pi/4)^{2n+1}}{(2n+1)!}\right)$$

This is required Taylor's expansion which is valid for each region being the function cosine and sine are valid for region R..



## Exercise 5.2

1. **Find Laurent series that converges for $0 < |z| < R$ of the functions:**

**(a)** $\frac{\cos z}{z^4}$ **(b)** $\frac{e^{z^2}}{z^3}$ **(c)** $z^3 \cosh\frac{1}{z}$ **(d)** $\frac{e^{-1/z^2}}{z^2}$.

**Solution:** Let $|z| < R$.

Since we have by Maclaurin's series expansion,

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \qquad \cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \qquad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

(a) $$f(z) = \frac{\cos z}{z^4} = \frac{1}{z^4} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$$= \frac{1}{z^4}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\dots\dots\right)$$

$$= \left(z^{-4} - \frac{z^{-2}}{2} + \frac{1}{24} - \frac{z^2}{720} + \dots\dots\dots\right)$$

This is required Laurent series expansion of f(z).

(b) $$f(z) = \frac{e^{z^2}}{z^3}$$

$$= \frac{1}{z^3} \sum_{n=0}^{\infty} \frac{(z^2)^n}{n!} = \frac{1}{z^3}\left(1 + \frac{z^2}{1!} + \frac{z^4}{2!} + \frac{z^6}{3!} + \frac{z^8}{4!} + \dots\dots\dots\right)$$

$$= \left(z^{-3} + z^{-1} + \frac{z}{2} + \frac{z^3}{6} + \frac{z^5}{24} + \dots\dots\dots\right).$$

This is required Laurent series expansion of f(z).

(c) $$f(z) = z^3 \cosh\frac{1}{z} = z^3 \sum_{n=0}^{\infty} \frac{(1/z)^{2n}}{(2n)!}$$

$$= z^3\left(1 + \frac{z^{-2}}{2!} + \frac{z^{-4}}{4!} + \frac{z^{-6}}{6!} + \dots\dots\dots\right)$$

$$= \left(z^3 + \frac{z}{2} + \frac{z^{-1}}{24} + \frac{z^{-3}}{720} + \dots\dots\dots\right)$$

This is required Laurent series expansion of f(z).

(d) $$f(z) = \frac{e^{-1/z^2}}{z^2} = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{(-1/z^2)^n}{n!} = \frac{1}{z^2} \sum_{n=0}^{\infty} (-1)^n \frac{z^{-2n}}{n!}$$

$$= \frac{1}{z^2}\left(1 - \frac{z^{-2}}{1!} + \frac{z^{-4}}{2!} - \frac{z^{-6}}{3!} + \dots\dots\dots\right)$$

$$= \left(z^{-2} - z^{-4} + \frac{z^{-6}}{2} - \frac{z^{-8}}{6} + \dots\dots\dots\right).$$

This is required Laurent series expansion of f(z).

**2. Find Laurent's series that converges for $0 < |z - z_0| < R$ of the function:**

(a) $f(z) = \frac{1}{z^2+1}$ at $z_0 = i$ (b) $f(z) = \frac{\cos z}{(z-\pi)^2}$ at $z_0 = \pi$

(c) $f(x) =$ (d) $f(z) = \frac{\cos hz}{(z+\pi i)^2}$ at $z_0 = -\pi i$

**Solution:** Let $|z - z_0| < R$.

Since we have by Maclaurin's series expansion,

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad \cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \quad \frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n$$

(a) For $z_0 = i$, we observe the series for $|z - i| < 1$.

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

$$= \frac{1}{(z-i)(z-i+2i)}$$

$$= \frac{1}{(z-i)\times 2i}\left[\frac{1}{1+\left(\frac{z-i}{2i}\right)}\right]$$

$$= \frac{i}{2i(z-i)} \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(2i)^n}$$

$$= \frac{i}{2i(z-i)}\left(1 - \frac{(z-i)}{2i} + \frac{(z-i)^2}{4i^2} - \frac{(z-i)^3}{8i^3} + \dots\dots\right)$$

$$= \left(-\frac{i}{2}(z-i)^{-1} + \frac{1}{4} + \frac{i}{8}(z-i) - \frac{1}{16}(z-i)^2 - \frac{i}{32}(z-i)^3 + \dots\dots\right)$$

(b) For $z_0 = \pi$, we observe the series for $|z - \pi| < 1$.

$$f(z) = \frac{\cos z}{(z-\pi)^2} = \frac{\cos(z - \pi + \pi)}{(z-\pi)^2}$$

$$= \frac{\cos(z-\pi)\times -1}{(z-\pi)^2} \quad [\text{Since } \sin\pi = 0.]$$

$$= \frac{-1}{(z-\pi)^2} \sum_{n=0}^{\infty} (-1)^n \frac{(z-\pi)^{2n}}{(2n)!}$$

$$= \frac{-1}{(z-\pi)^2}\left(1 - \frac{(z-\pi)^2}{2!} + \frac{(z-\pi)^4}{4!} - \frac{(z-\pi)^6}{6!} + \dots\dots\right)$$

$$= \left(-(z-\pi)^{-2} + \frac{1}{2} - \frac{(z-\pi)^2}{24} + \frac{(z-\pi)^4}{720} - \dots\dots\right)$$

(d) For $z_0 = -\pi i$, we observe the series for $|z + \pi i| < 1$.

$$f(z) = \frac{\cosh z}{(z+\pi i)^2} = \frac{e^z + e^{-z}}{2(z+\pi i)^2}$$



$$= -\frac{e^{z+\pi i - \pi i} + e^{-(z+\pi i)+\pi i}}{2(z+\pi i)^2}$$

$$= -\frac{e^{z+\pi i} - e^{-(z+\pi i)}}{2(z+\pi i)^2}$$

$$= -\frac{\cos h(z+\pi i)}{(z+\pi i)^2}$$

$$= -\sum_{n=0}^{\infty} \frac{(z+\pi i)^{2n-2}}{(2n)!}$$

**3. Find Taylor and Laurent's series of $f(z) = \frac{2z-3i}{z^2-3iz-2}$ in the following regions:**

**(a) $0 < |z| < 1$. (b) $|z| > 2$.**

**Solution:** Since we have by Maclaurin's series expansion,

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n \quad \text{for } |z| < 1$$

Here,

$$f(z) = \frac{2z-3i}{(z-2i)(z-i)} = \frac{1}{z-2i} + \frac{1}{z-i}$$

We have, $0 < |z| < 1$, so we can expand $(1+z)^{-1}$ if $|z| < 1$. For this we have to arrange for making valid for given region.

$$f(z) = -\frac{1}{2i}\left(\frac{1}{1-\frac{z}{2i}}\right) - \frac{1}{i}\left(\frac{1}{1-\frac{z}{i}}\right)$$

$$= -\frac{1}{2i}\sum_{n=0}^{\infty}\left(\frac{z}{2i}\right)^n - \frac{1}{i}\sum_{n=0}^{\infty}\left(\frac{z}{i}\right)^n$$

This is required Taylor's series.

(b) Here,

$$f(z) = \frac{1}{z-2i} + \frac{1}{z-i}$$

We have, $|z| > 2 \Rightarrow \frac{2}{|z|} < 1$, so we arrange suitably for making expansion valid, we get

$$f(z) = \frac{1}{z}\left(\frac{1}{1-\frac{2i}{z}}\right) + \frac{1}{z}\left(\frac{1}{1-\frac{i}{z}}\right)$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2i}{z}\right)^n + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(2i)^n}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{i^n}{z^{n+1}}$$

This is required Laurent series.

**4. Find the Laurent's expansion for $f(z) = \frac{z^2-1}{z^2+5z+6}$ in region $2 < |z| < 3$.**

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**Solution:** Since we have by Maclaurin's series expansion,

$$\frac{1}{1+z} = \sum_{n=0}^{\infty}(-z)^n \quad \text{for } |z| < 1$$

Here,

$$f(z) = \frac{z^2-1}{z^2+5z+6} = 1 - \frac{5z+7}{z^2+5z+6} = 1 - \left[\frac{8}{z+3} - \frac{3}{z+2}\right]$$

We have, $2 < |z| < 3 \Rightarrow \frac{2}{|z|} < 1$ and $\frac{|z|}{3} < 1$, so we can expand the term, we have to arrange to make valid for given region,

$$f(z) = 1 - \frac{8}{3}\left(\frac{1}{1+\frac{z}{3}}\right) + \frac{3}{z}\left(\frac{1}{1+\frac{2}{z}}\right)$$

$$= 1 - \frac{8}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n(-1)^n + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n$$

$$= 1 + 8\sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^n}{3^{n+1}} + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^{n+1}}$$

This is the required Laurent's series.

**5. Find the Laurent's expansion for $f(z) = \frac{1}{z^2(1-z)}$ in the region.**

**(a) $0 < |z| < 1$ (b) $1 < |z| < 4$.**

**Solution:** Since we have by Maclaurin's series expansion,

$$\frac{1}{1+z} = \sum_{n=0}^{\infty}(-z)^n \quad \text{for } |z| < 1$$

(a) Here,

$$f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z^2}\left(\frac{1}{1-z}\right) = \frac{1}{z^2}\sum_{n=0}^{\infty}z^n = \sum_{n=0}^{\infty}z^{n-2}$$

This is required Laurent's series.



(b) We have,

$$f(z) = \frac{1}{z^2(1-z)}$$

Since we have the region is $1 < |z| < 4 \Rightarrow \frac{1}{|z|} < 1$ and $\frac{|z|}{4} < 1$. For this we arrange the term(s) to make the expansion valid for given region.

$$f(z) = -\frac{1}{z^3}\left(\frac{1}{1-\frac{1}{z}}\right) = -\frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=0}^{\infty}\frac{1}{z^{n+3}}$$

This is required Laurent's series.

**6. Find the expansion of $f(z) = \frac{1}{z-z^3}$ in the region $1 < |z-1| < 2$.**

**Solution:** Since we have by Maclaurin's series expansion,

$$\frac{1}{1+z} = \sum_{n=0}^{\infty}(-z)^n \quad \text{for } |z| < 1$$

Here,

$$f(z) = \frac{1}{z-z^3} = \frac{1}{z(1-z)(1+z)} = \frac{A}{z} + \frac{B}{1-z} + \frac{C}{1+z} \quad \ldots\ldots (i)$$

Since we have the region is $1 < |z-1| < 2 \Rightarrow \frac{1}{|z-1|} < 1$ and $\frac{|z-1|}{2} < 1$. So,

$$A = \lim_{z\to 0}\frac{1}{1-z^2} = 1, \qquad B = \lim_{z\to 1}\frac{1}{z(1+z)} = \frac{1}{2}$$

$$C = \lim_{z\to -1}\frac{1}{z(1-z)} = -\frac{1}{2}$$

Then (i) becomes,

$$f(z) = \frac{1}{z} + \frac{1}{2}\left(\frac{1}{1-z}\right) - \frac{1}{2}\left(\frac{1}{1+z}\right)$$

$$= \frac{1}{(z-1+1)} + \frac{1}{2}\left(\frac{1}{1-z}\right) - \frac{1}{2}\frac{1}{(z-1+2)}$$

$$= \frac{1}{(z-1)\left(1+\frac{1}{z-1}\right)} - \frac{1}{2}\left(\frac{1}{z-1}\right) - \frac{1}{4}\left(\frac{1}{1+\frac{z-1}{2}}\right)$$

$$= \frac{1}{(z-1)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z-1}\right)^n - \frac{1}{2}\left(\frac{1}{z-1}\right) - \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{2}\right)^n$$

$$= -\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+2}}(z-1)^n - \frac{1}{2}\left(\frac{1}{z-1}\right) + \sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^{n+1}}$$

This is required Laurent series of the given function f(z).

**7. Obtain the expansion of $f(z) = \left(\frac{z-1}{z^2}\right)$ in power of $(z-1)$. Indicate the region of validity.**

**Solution:** Here,

$$f(z) = \frac{(z-1)}{z^2}$$

We have to find the expansion of f(z) in power of $(z-1)$ i.e. at $z = 1$.

If $|z-1| < 1$ then Laurent's series of $\frac{1}{z^2}$ is,

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} a_n (z-1)^n \quad \text{with} \quad a_n = \frac{f^n(1)}{n!}$$

Here, $f^n\left(\frac{1}{z^2}\right) = (-1)^n (n+1)! \times \frac{1}{z^{n+2}}$

Then $a_n = \frac{f^n(1)}{n!} = (-1)^n (n+1)$

Therefore, $\frac{1}{z^2} = \sum_{n=0}^{\infty} (-1)^n (n+1)(z-1)^n$

So, $f(z) = (z-1)\sum_{n=0}^{\infty} (-1)^n (n+1)(z-1)^n$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1)(z-1)^{n+1}$$

which is valid for $|z-1| < 1$.

If $|z-1| > 1 \Rightarrow \frac{1}{|z-1|} < 1$ then Laurent's series of $\frac{1}{z^2}$ is,

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} a_n \frac{1}{(z-1)^n} \quad \text{with} \quad a_n = \frac{f^n(1)}{n!}$$

Here, $f^n\left(\frac{1}{z^2}\right) = (-1)^n (n+1)! \times \frac{1}{z^{n+2}}$

Then $a_n = \frac{f^n(1)}{n!} = (-1)^n (n+1)$

Therefore, $\frac{1}{z^2} = \sum_{n=0}^{\infty} (-1)^n (n+1)\frac{1}{(z-1)^n}$



So, $f(z) = (z-1)\sum_{n=0}^{\infty} (-1)^n (n+1)\frac{1}{(z-1)^n}$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1)\frac{1}{(z-1)^{n-1}}$$

which is valid for $|z-1| > 1$.

Thus, $f(z) = \sum_{n=0}^{\infty} (-1)^n (n+1)(z-1)^{n+1}$ for $|z-1| < 1$.

and $f(z) = \sum_{n=0}^{\infty} (-1)^n (n+1)\frac{1}{(z-1)^{n-1}}$ for $|z-1| > 1$.

**8. Find the expansion for $\frac{7z-2}{(z+1)z(z-2)}$ in the regions given by**

**(a) In region $0 < |z+1| < 1$ (b) $1 < |z+1| < 3$ (c) $|z+1| > 3$**

**Solution:** Since we have by MacIaurin's series expansion,

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n \quad \text{for } |z| < 1$$

Here,

$$f(z) = \frac{7z-2}{(z+1)z(z-2)} = \frac{A}{z+1} + \frac{B}{z} + \frac{C}{z-2} = \frac{-3}{z+1} + \frac{1}{z} + \frac{2}{z-2}$$

**(a) In region $0 < |z+1| < 1$,**

$$f(z) = -\frac{3}{z+1} + \frac{1}{z+1-1} + \frac{2}{z+1-3}$$

$$= -\frac{3}{z+1} - \frac{1}{[1-(z+1)]} + \frac{2}{3}\left(\frac{1}{1-\left(\frac{z+1}{3}\right)}\right)$$

$$= -\frac{3}{z+1} - \sum_{n=0}^{\infty} (z+1)^n + \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

$$= \left[\sum_{n=0}^{\infty}\left(-1 + \frac{2}{3^{n+1}}\right)(z+1)^n\right] - \frac{3}{z+1}$$

**(b) For region $1 < |z+1| < 3 \Rightarrow \frac{1}{|z+1|} < 1$ and $\frac{|z+1|}{3} < 1$.**

By arranging the terms for making expansion valid for given region

$$f(z) = -\frac{3}{z+1} + \frac{1}{z+1-1} + \frac{2}{z+1-3}$$

$$= -\frac{3}{z+1} + \frac{1}{(z+1)}\frac{1}{\left(1-\frac{1}{z+1}\right)} - \frac{2}{3}\frac{1}{\left(1-\frac{z+1}{3}\right)}$$

$$= -\frac{3}{z+1} + \sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^{n+1} - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^{n}$$

**(c) For region $|z+1| > 3 \Rightarrow \left|\frac{3}{z+1}\right| < 1.$**

$$f(z) = -\frac{3}{z+1} + \frac{1}{z+1-1} + \frac{2}{z+1-3}$$

$$= -\frac{3}{z+1} + \frac{1}{(z+1)}\frac{1}{\left(1-\frac{1}{z+1}\right)} + \frac{2}{(z+1)}\frac{1}{\left(1-\frac{3}{z+1}\right)}$$

$$= -\frac{3}{z+1} + \sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^{n+1} + 2.\sum_{n=0}^{\infty}\frac{3^n}{(z+1)^{n+1}}$$

**9. Give the Laurent's series expansion for $f(z) = \frac{1}{(z+1)(z+3)}$ for the region $0 < |z+1| < 2.$**

**Solution:** Since we have by Maclaurin's series expansion,

$$\frac{1}{1+z} = \sum_{n=0}^{\infty}(-z)^n \text{ for } |z| < 1$$

Given region is $0 < |z+1| < 2 \Rightarrow \frac{|z+1|}{2} < 1.$

Here,

$$f(z) = \frac{1}{(z+1)(z+3)}$$

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+1+2}\right]$$

$$= \frac{1}{2(z+1)} - \frac{1}{4}\left(\frac{1}{1+\left(\frac{z+1}{2}\right)}\right)$$

$$= \frac{1}{2(z+1)} - \frac{1}{2^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z+1}{2}\right)^n$$



$$= \frac{1}{2(z+1)} + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{(z+1)^n}{2^{n+2}}$$

This is required Laurent series for f(z) for given region.

**10. Find the Laurent's series expansion for the function f(z) given below in the specified regions:**

**(a) $f(z) = \frac{1}{z(1+z)^2}$ in $0 < |z| < 1$**

**(b) $f(z) = \frac{z+3}{z(z^2-z-2)}$ in (i) $0 < |z| < 1$ (ii) $1 < |z| < 2$**

**Solution: (a)** Since we have by Maclaurin's series expansion,

$$\frac{1}{1+z} = \sum_{n=0}^{\infty}(-z)^n \text{ for } |z| < 1$$

Given region is $0 < |z| < 1.$

Here,

$$f(z) = \frac{1}{z(1+z)^2} = \frac{1}{z}\left(\frac{1}{(1+z)^2}\right)$$

$$= \frac{1}{z}\left(\sum_{n=0}^{\infty}(-z)^n\right)^2$$

$$= \frac{1}{z}(1 - z + z^2 - \ldots)^2$$

$$= \frac{1}{z}(1 + 2z + 3z^2 + \ldots)$$

$$= \frac{1}{z} + 2 + 3z + \ldots\ldots$$

**(b)** Since we have by Maclaurin's series expansion,

$$\frac{1}{1+z} = \sum_{n=0}^{\infty}(-z)^n \text{ for } |z| < 1$$

(i) Given region is $0 < |z| < 1.$

Here,

$$f(z) = \frac{z+3}{z(z^2-z-2)}$$

By partial fraction

$$= -\frac{3}{2z} + \frac{2}{3(z+1)} + \frac{5}{6(z-2)}$$

$$= -\frac{3}{2z} + \frac{2}{3}\left(\frac{1}{1+z}\right) - \frac{5}{12}\left(\frac{1}{1-\frac{z}{2}}\right)$$

$$= -\frac{3}{2z} + \frac{2}{3}\sum_{n=0}^{\infty}(-1)^n z^n - \frac{5}{12}\sum_{n=0}^{\infty}\frac{z^n}{2^n}$$

$$= -\frac{3}{2z} + \frac{2}{3}\sum_{n=0}^{\infty}(-1)^n z^n - \frac{5}{3}\sum_{n=0}^{\infty}\frac{z^n}{2^{n+2}}$$

(ii) Given region is $1 < |z| < 2 \Rightarrow \frac{1}{|z|} < 1$ and $\frac{|z|}{2} < 1$

Here,

$$f(z) = -\frac{3}{2z} + \frac{2}{3(z+1)} + \frac{5}{6(z-2)}$$

$$= -\frac{3}{2z} + \frac{2}{3z}\frac{1}{\left(1+\frac{1}{z}\right)} - \frac{5}{12}\frac{1}{\left(1-\frac{z}{2}\right)}$$

$$= -\frac{3}{2z} + \frac{2}{3z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n - \frac{5}{3(2^2)}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= -\frac{3}{2z} + \frac{2}{3}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n+1}} - \frac{5}{3}\sum_{n=0}^{\infty}\frac{z^n}{2^{n+2}}$$

**11. Find the Laurent's series expansion for f(z) in the region specified:**

**(a)** $\frac{z}{(z-1)(z-3)}$ **in** $0 < |z-1| < 2$

**Solution:** Since we have by Maclaurin's series expansion,

$$\frac{1}{1+z} = \sum_{n=0}^{\infty}(-z)^n \quad \text{for } |z| < 1$$

(a) Given region is $0 < |z-1| < 2 \Rightarrow \frac{|z-1|}{2} < 1$.

Let,

$$f(z) = \frac{z}{(z-1)(z-3)}$$

$$= \left[\frac{-1}{2(z-1)} + \frac{3}{2(z-3)}\right]$$

$$= \frac{1}{2}\left[\frac{-1}{(z-1)} + \frac{3}{(z-3)}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{(z-1)} + \frac{3}{(z-1-2)}\right]$$



$$= \frac{1}{2}\left[\frac{-1}{(z-1)} - \frac{3}{2}\frac{1}{\left[1-\left(\frac{z-1}{2}\right)\right]}\right]$$

$$= \frac{1}{2}\left[\frac{-1}{(z-1)} - \frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n\right]$$

**(b)** $\frac{e^{2z}}{(z-1)^3}$ **in** $|z-1| > 1$

**Solution:** Given region is $|z-1| > 1 \Rightarrow \frac{1}{|z-1|} < 1$.

Let,

$$f(z) = \frac{e^{2z}}{(z-1)^3}$$

$$= \frac{e^{2(z-1)}\cdot e^2}{(z-1)^3}$$

$$= \frac{e^2}{(z-1)^3}\sum_{n=0}^{\infty}\frac{[2(z-1)]^n}{n!} = e^2\sum_{n=0}^{\infty}\frac{2^n}{n!}(z-1)^{n-3}$$

**(c)** $\frac{1}{z-z^3}$ **in** $1 < |z+1| < 2$.

**Solution:** Given region is $1 < |z+1| < 2 \Rightarrow \frac{1}{|z+1|} < 1$ and $\frac{|z+1|}{2} < 1$.

Let,

$$f(z) = \frac{1}{z-z^3}$$

$$= \frac{1}{z(1-z)(1+z)}$$

$$= \frac{1}{z} + \frac{1}{2(1-z)} - \frac{1}{2(1+z)}$$

$$= \frac{1}{z+1-1} + \frac{1}{2(1-z-1+1)} - \frac{1}{2(z+1)}$$

$$= \frac{1}{(z+1)\left(1-\frac{1}{z+1}\right)} + \frac{1}{2.2}\left(\frac{1}{1-\left(\frac{z+1}{2}\right)}\right) - \frac{1}{2(z+1)}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(z+1)^{n+1}} + \sum_{n=0}^{\infty}\frac{(z+1)^n}{2^{n+2}} - \frac{1}{2(z+1)}$$

## OTHER IMPORTANT QUESTION FROM FINAL EXAM FOR PRACTICE

**2002 Q. No. 1(b) OR**

State Laurent's theorem. Find all Laurent's series of the function $f(z) = \frac{1}{1-z^2}$ with centre at z = 1.

**2003 Fall Q. No. 1(b) OR**

(i) Explain $e^z$ at $z_0 = \frac{\pi i}{2}$ as Taylor's series.

(ii) Find the Laurent series of $\frac{e^z}{z(1-z)}$ at $z_0 = 1$ that converges for $0 < |z - 1| < R$.

**2004 Spring Q. No. 1(b) OR**

State Taylor' theorem. Find the Laurent series of the function $f(z) = \frac{z^2-4}{(z-1)^2}$ about z = 1.

**2005 Spring Q. No. 1(b) OR**

Find Laurent series of the function $f(z) = \frac{z^2-1}{z^2+5z+6}$ in the region $2 < |z| < 3$.

**2005 Fall Q. No. 1(b) OR**

(i) Integrate: $\oint_c \frac{dz}{z^2+4}$; $c: 4x^2 + (y-2)^2 = 4$.

(Hint: Use Cauchy's integral formula)

(ii) Expand $\frac{1}{z}$ at z = 2 as Taylor's series.

**2007 Spring Q. No. 2(a)**

State Laurent's theorem. Find the Lauren't series for $f(z) = \frac{z^2-1}{(z^2+5z+6)}$ in the region $1 < |z| < 2$.

**2007 Fall Q. No. 1(b) OR**

State Lauren't theorem. Find all Taylor series and Laurent series of the function $f(z) = \frac{3-2z}{z^2-3z+2}$ with centre 0.

**2008 Spring Q. No. 2(a) OR**

Find Laurent series of the function $f(z) = \frac{e^z}{z(1-z)}$, which converges to $0 < |z - 1| < R$.

**2008 Fall Q. No. 2(a); 2009 Fall Q. No. 2(a)**

Find the Laurent series expansion of the function $f(z) = \frac{1}{z^2+1}$ for the region $0 < |z - 1| < R$ an determine the precise region of convergence.

Hint: See Exercise 4.2 Q. No. 2(a).



**2011 Fall Q. No.2(a); 2016 Fall 2(b)**

State Laurent's series. Expand the function $f(z) = \frac{1}{z-z^2}$ in the region $1 < |z - 1| < 2$.

Hint: See Exercise 4.2 Q. No. 6.

**2016 Spring 2(b)**

**Obtain the Taylor series and Laurent series of the function $f(z) = \frac{1}{(z+2)(z^2+1)}$ when $1 < |z| < 2$.**

## SHORT QUESTIONS

**2005 Spring Q. No. 7(ii)**

Find Maclaurin's expansion of the function $f(z) = \frac{2-z}{(1-z)^2}$.

❑❑❑

## Unit 6
# ZEROS AND SINGULARITIES OF A FUNCTION

### Singularity:

A function $f(z)$ is said to have a singularity at a point $z = z_0$ if $f(z)$ is not analytic at that point but $f(z)$ is analytic at all other points in the neighborhood of the point $z = z_0$.

### Types of Singularity:

We have three types of singularities which are

**(a) Isolated Singularity:**

A function $f(z)$ is said to have isolated singularity at $z = z_0$ if $f(z)$ has exactly one singular point $z_0$ in the neighborhood of $f(z)$.

**(b) Removable Singularity:**

Let, $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ is analytic in $|z - a| < R$ but is not analytic at $z = a$ then we call $f(z)$ has removable singularity at $z = a$.

**(c) Essential Singularity:**

Let, $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{(z-a)^n}$ then we call $f(z)$ has essential singular point at $z = a$.

### Pole of a Complex Function f(z):

If a complex function $f(z)$ is defined as $f(z) = \sum_{n=0}^{m} \frac{a_n}{(z-a)^n}$ then $z = a$ is known as pole of $f(z)$.

*Note*: If $m = 1$ then we call the pole is simple pole.

### Zeros of a Complex Function f(z):

A complex function $f(z)$ is said to have zeros at $z = a$ if $f(a) = 0$.

*Note*: If $z = a$ is zeros of $f(z)$ and $f(a) = 0$ but $f'(a) \neq 0$ then $z = a$ is simple zeros of $f(z)$.

If $z = a$ is zeros of $f(z)$ and $f(a) = 0 = f'(a) = f''(a) = \dots\dots\dots = f^{n-1}(a)$ but $f^n(a) \neq 0$ then $z = a$ is zeros of $f(z)$ of order n.

***NOTE:** If $f(a) = 0$ then we called $f(z)$ has zeros at $z = a$ and if $f(a) = \infty$ then we called $f(z)$ has singularity at $z = a$.*

### Residue:

If a complex function $f(z)$ has singular point $z = z_0$ in C then the coefficient of $\left(\frac{1}{z - z_0}\right)$ is called residue of $f(z)$.



### Process to Evaluation of Residues

1. If $f(z)$ has a simple pole at $z = z_0$, then

$$f(z) = \sum_{r=0}^{n} a_n (z - z_0)^n + \frac{b_1}{z - z_0}$$

$$\Rightarrow (z - z_0) f(z) = \sum_{r=0}^{n} a_n (z - z_0)^{n+1} + b_1$$

Taking $\lim_{z \to z_0}$ on both sides,

$$\lim_{z \to z_0} (z - z_0) f(z) = b_1$$

Thus the residues of $f(z)$ at a simple pole at $z = z_0$ is $\lim_{z \to z_0} (z - z_0) f(z)$.

2. Suppose $f(z)$ has a simple pole at $z = z_0$ and $f(z) = \frac{P(z)}{Q(z)}$, where

$$Q(z) = (z - z_0)\,\phi(z) \text{ and } \phi(z_0) \neq 0$$

Thus residue of $f(z) = \lim_{z \to z_0} (z - z_0) \frac{P(z)}{(z - z_0) f(z)}$

$$= \lim_{z \to z_0} \frac{P(z)}{f(z)} \text{ at } z = z_0$$

$$= \frac{P(z_0)}{\phi(z_0)} \qquad \dots\dots\dots(1)$$

Here,

$$Q'(z) = (z - z_0)\,\phi'(z) + \phi(z)$$

$$Q'(z_0) = \phi(z_0)$$

Therefore from (1), we get,

Residue of $f(z)$ at $z = z_0$ is, $= \frac{P(z_0)}{Q'(z_0)}$

3. If $f(z)$ has a pole of order m at $z = z_0$, then

$$f(z) = \sum_{r=0}^{n} a_n (z - z_0)^n + \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \dots + \frac{b_m}{(z - z_0)^m}$$

$$\Rightarrow (z - z_0)^m f(z) = \sum_{r=0}^{n} a_n (z - z_0)^{m+n} + b_1 (z - z_0)^{m-1} + \dots + b_m$$

Differentiating w.r.to z up to (m – 1) times and then taking as $z \to z_0$ we get,

$$\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] = (m - 1)!\, b_1$$

The other terms vanish as the remaining terms contain $(z - z_0)$ as a factor.

Therefore, residue of $f(z)$,

$$b_1 = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z)\right] \text{ at } z = z_0.$$

## Cauchy's Residue Theorem

**Let f(z) be analytic in a closed contour c except at finitely many points $z_1, z_2, \ldots, z_k$ where each $z_j$ for j = 1, 2, …, k lie in c. Then,**

$$\oint_c f(z)\,dz = 2\pi i \sum_{j=1}^{k} \underset{z=z_j}{\text{Res}} [f(z)].$$

**Proof:**

We have f(z) is analytic in a closed contour except at $z_1, z_2, \ldots, z_k$ where each $z_j$ for j = 1, 2, …, k lie in c. Let us enclose each singular points $z_j$ by a circle $c_j^*$ with as small as possible radius so that no one circle intersects to another where c has counter clockwise direction and $c_j^*$ has clockwise.

Then f(z) is analytic in the multiply connected domain D bounded by c and $c_1^*$, $c_2^*$, ……, $c_k^*$ and on the entire boundary of D.

Then by Cauchy's integral theorem,

$$\oint_c f(z)\,dz + \oint_{c_1^*} f(z)\,dz + \oint_{c_2^*} f(z)\,dz + \ldots + \oint_{c_k^*} f(z)\,dz = 0.$$

$$\Rightarrow \oint_c f(z)\,dz + \sum_{j=1}^{k} \oint_{c_j^*} f(z)\,dz = 0.$$

where the integral c being taken in counter clockwise and the other integrals $c_j^*$ has clockwise direction.

Now, reversing the path of integration in each integral $c_j^*$. Then

$$\oint_c f(z)\,dz = \sum_{j=1}^{k} \oint_{c_j} f(z)\,dz. \quad \ldots (1)$$

where all the integrals are now taken in counterclockwise direction.

Since the function has singular point $z = z_j$ in $c_j$ for j = 1, 2, …, k. So, by definition of residue,

$$\oint_{c_j} f(z)\,dz = \underset{z=z_j}{\text{Res}} f(z) \quad \text{for } j = 1, 2, \ldots, k.$$

Then (1) becomes,

$$\oint_c f(z)\,dz = 2\pi i \sum_{j=1}^{k} \underset{z=z_j}{\text{Res}} f(z).$$

This completes the proof.



## Exercise 6.1

1. **Determine the location and order of the zeros of the following functions:**

(i) $\tan\pi z$ (ii) $\cos^2\frac{z}{2}$ (iii) $(z^2+1)(e^z-1)$ (iv) $(z^4-z^2-6)^3$

(v) $\frac{(z^2+4)}{e^z}$ (vi) $\sin\frac{1}{1-z}$ (vii) $\sin\frac{1}{z}$

**Solution:** Since we know that f(z) has zeros at z = a if f(a) = 0.

(i) Here, $\tan\pi z = 0 = \tan n\pi$ **[2004 Spring Q. No. 7(c)]**

$\Rightarrow \pi z = n\pi$ for $n = 0 \pm 1, \pm 2, \pm 3, \ldots$

$\Rightarrow z = \pm n$ for $n = 0 \pm 1, \pm 2, \pm 3, \ldots$

Clearly for each z, the function is of simple degree. So, the zeros are simple.

(ii) Here, $\cos^2\left(\frac{z}{2}\right) = 0$

$\Rightarrow \frac{1+\cos z}{2} = 0$

$\Rightarrow \cos z = 1 = \cos n\pi$ for n is odd.

$\Rightarrow z = n\pi$ for $n = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$

Clearly for each z, the function is of second degree. So, the zeros are of second order.

(iii) Here, $(z^2+1)(e^z-1) = 0$

$\Rightarrow (z+i)(z-i)(e^z-1) = 0$

$\Rightarrow z = i, -i, 0$

Clearly for each z, the function is of simple degree. So, the zeros are simple.

(iv) Here, $(z^4-z^2-6)^3 = 0$

$\Rightarrow z^4-z^2-6 = 0$

$\Rightarrow (z^2-3)(z^2+2) = 0$

$\Rightarrow z = 3, -3, i\sqrt{2}, -i\sqrt{2}$

Clearly for each z, the function is of third degree. So, the zeros are of third order.

(v) Here, $\frac{(z^2+4)}{e^z} = 0.$

$\Rightarrow e^{-z}(z^2+4) = 0$

$\Rightarrow e^{-z}(z+2i)(z-2i) = 0$

$\Rightarrow z = \pm 2i, \infty.$

Clearly for each z, the function is of simple degree. So, the zeros are simple.

(vi) Here, $\sin\left(\frac{1}{1-z}\right) = 0 = \sin n\pi$

$$\Rightarrow 1 - z = \frac{1}{n\pi} \quad \text{for } n = \pm 1, \pm 2, \ldots$$

$$\Rightarrow z = 1 - \frac{1}{n\pi} \quad \text{for } n = \pm 1, \pm 2, \ldots$$

Clearly for each z, the function is of simple degree. So, the zeros are simple.

(vii) Here, $\sin\frac{1}{z} = 0 = \sin n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$

$$\Rightarrow \frac{1}{z} = n\pi \quad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

$$\Rightarrow z = \frac{1}{n\pi} \quad \text{for } n = \pm 1, \pm 2, \ldots$$

Clearly for each z, the function is of simple degree. So, the zeros are simple.

**2. Determine the location and type of singularities of the following functions, including those at infinity. Also, find the poles and its order.**

**(i)** $z^2 - \frac{1}{z^2}$ **(ii)** $\cot 2z$ **(iii)** $2z^{-3} - z^{-1}$ **(iv)** $(z^2 + a^2)^{-2}$

**(v)** $z^{-2}\sin^2 z$ **(vi)** $(z - \pi i)^{-2}\sinh z$

**Solution:**

(i) Given that, $f(z) = z^2 - \frac{1}{z^2}$

Clearly, the function f(z) has pole of second order.

For singularity, set

$$f(z_0) = \infty \Rightarrow z_0^2 + \frac{1}{z_0^2} = \infty$$

$$\Rightarrow \frac{z_0^4 - 1}{z_0^2} = \infty$$

$$\Rightarrow z_0 = 0, \infty.$$

Thus f(z) has pole of second order.

(ii) Given that, $f(z) = \cot 2z$

At $z = 0$ and $z = n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$ the function f(z) does not exist. Therefore, $z = 0, \pm\pi, \pm 2\pi, \ldots$ are poles of f(z).

And, $z = \infty$ is essential singularity.

(iii) Given that $f(z) = 2z^{-3} - z^{-1} = \frac{2 - z^2}{z^3}$

Clearly, $z = 0$ is the pole of f(z) of order 3.

Also, $f(\pm\sqrt{2}) = 0$. So, f(z) has zeros at $z = \pm\sqrt{2}$.



(iv) Given that $f(z) = (z^2 + a^2)^{-2} = \frac{1}{z^2 + a^2}$.

This shows f(z) has poles at $z = \pm ia$ of second order.

(v) Given that,

$$f(z) = z^{-2}\sin^2 z = \frac{\sin^2 z}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!\, z^2}$$

This shows, $z_0 = \infty$ is an essential singularity of f(z).

(vi) Given that,

$$f(z) = (z - \pi i)^{-2}\sinh z$$ **[2008 Spring Q. No. 7(c)]**

At $z = \pi i$, f(z) is undefined. So, $z = \pi i$ is the pole of f(z).

And, $\sinh z_0 = \infty \Rightarrow z_0 = \infty$ is essential singularity.

## Exercise 6.2

**1. Find residue at singular points from the Laurent's series or by definition of the following functions:**

(a) $\frac{4z}{1+z^2}$ (b) $\frac{\sin 2z}{z^6}$ (c) $\frac{1}{1-e^z}$ (d) $\frac{1}{(z^2-1)^2}$ (e) $\cos\pi z$

**Solution:**

(a) Here, $f(z) = \frac{4z}{1+z^2} = \frac{4z}{(z-i)(z+i)}$

Clearly f(z) has simple poles are at $z = i, -i$.

Now, residue of f(z) at the poles are

$$\underset{z=i}{\text{Res}}\, f(z) = \lim_{z\to i} (z-i)\frac{4z}{(z-i)(z+i)} = \frac{4i}{2i} = 2$$

$$\underset{z=-i}{\text{Res}}\, f(z) = \lim_{z\to -i} (z+i)\frac{4z}{(z-i)(z+i)} = 2.$$

Thus the residue of f(z) are 2, 2 at $z = \pm i$.

(b) Here, $f(z) = \frac{\sin 2z}{z^6}$

Clearly, the function f(z) has poles at $z = 0$ of order 6.

Now, residue of f(z) at the poles are

$$\underset{z=0}{\text{Res}}\, f(z) = \lim_{z\to 0} \frac{1}{5!}\frac{d^5}{dz^5}\left(z^6 \frac{\sin 2z}{z^6}\right)$$

$$= \lim_{z\to 0} \frac{1}{5!}[2\times(-2)\times(-2)\times(2)\times(2)\cos 2z] = \frac{32}{120}\times 1 = \frac{4}{15}$$

Thus the residue of f(z) are $\frac{4}{15}$ at $z = 0$.

(c) Here, $f(z) = \frac{1}{1-e^z}$

Here, $1 - e^z = 0 \Rightarrow e^z = 1 \Rightarrow z = 0$.

Clearly, the function f(z) has simple pole at z = 0.

Now, residue of f(z) at the poles are

$$\underset{z=0}{\text{Res}}\ f(z) = \lim_{z\to 0}\left(\frac{1}{1-e^z}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{z\to 0}\left(\frac{1}{-e^z}\right) = -1.$$

Thus the residue of f(z) are −1 at z = 0.

(d) Here, $f(z) = \frac{1}{(z^2-1)^2} = \frac{1}{(z-1)^2(z+1)^2}$

Since,

$$z^2 - 1 = 0 \Rightarrow z = \pm 1.$$

Clearly, the function f(z) has poles at $z = \pm 1$ of order 2.

Now, residue of f(z) at the poles are

$$\underset{z=1}{\text{Res}}\ f(z) = \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{1}{(z-1)^2(z+1)^2}\right] = \lim_{z\to 1}\left[\frac{-2}{(z+1)^3}\right] = -\frac{1}{4}$$

$$\underset{z=-1}{\text{Res}}\ f(z) = \lim_{z\to -1}\frac{d}{dz}\left[\frac{1}{(z-1)^2}\right] = \frac{-2}{(z-1)^3} = \frac{-2}{(-2)^3} = \frac{2}{8} = \frac{1}{4}$$

Thus the residue of f(z) are $-\frac{1}{4}, \frac{1}{4}$ at $z = \pm 1$.

(e) Here, $f(z) = \cot\pi z = \frac{\cos\pi z}{\sin\pi z}$

Now,

$$\sin z\pi = 0 = \sin n\pi \qquad \Rightarrow z = 0, \pm 1, \pm 2, \ldots$$

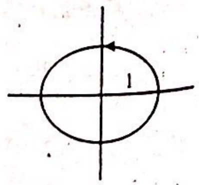
Clearly, the function f(z) has simple pole at z = 0.

$$\underset{z=0}{\text{Res}}\ f(z) = \lim_{z\to 0}\frac{\cos\pi z}{\pi\cos\pi z} = \frac{1}{\pi} \qquad z = 0, \pm 1, \pm 2, \ldots$$

Thus the residue of f(z) are $\frac{1}{\pi}$ at $z = 0, \pm 1, \pm 2, \ldots$

**2. Evaluate the following integrals (counter–clockwise):**

(a) $\oint_C \tan\pi z\, dz \qquad C: |z| = 1$




**Solution:** We have,

$$f(z) = \tan\pi z = \frac{\sin\pi z}{\cos\pi z}$$



For poles of f(z),

$$\cos\pi z = 0 = \cos\frac{n\pi}{2}$$

$$\Rightarrow z = \pm\frac{n}{2} = \pm\frac{1}{2}, \pm\frac{3}{2}\ldots$$

Given that the region is C: |z| = 1 that includes only $z = \pm\frac{1}{2}$ which are simple poles. So, the residue of f(z) at $z = \pm\frac{1}{2}$ is,

$$\underset{z=(1/2)}{\text{Res}}\ f(z) = \lim_{z\to(1/2)}\left(\frac{\sin\pi z}{-\pi\sin z}\right) = \frac{-1}{\pi}$$

$$\underset{z=(-1/2)}{\text{Res}}\ f(z) = \lim_{z\to(-1/2)}\left(\frac{\sin\pi z}{-\pi\sin z}\right) = \frac{-1}{\pi}$$

Now by Cauchy Residue Theorem,

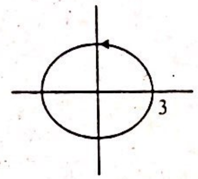
$$\oint_C \tan\pi z dz = 2\pi i \text{ (sum of residues of f(z) at z = ½ and z = −1/2}$$

$$= 2\pi i\left(-\frac{1}{\pi} - \frac{1}{\pi}\right) = -4i.$$

(b) $\oint_C \frac{e^z}{\cos z}dz \qquad$ **C : |z| = 3**

**Solution:** Here,

$$f(z) = \frac{e^z}{\cos z}$$




For poles of f(z), set

$$\cos z = 0 = \cos\frac{n\pi}{2} \Rightarrow z = \pm\frac{n\pi}{2} = \pm\frac{\pi}{2} \pm \frac{3\pi}{2}, \ldots$$

Given that the region is C: |z| = 3 that include only $z = \pm\frac{\pi}{2}$ which are simple poles. So, the residue of f(z) at $z = \pm\frac{\pi}{2}$ is,

$$\underset{z=\pi/2}{\text{Res}}\ f(z) = \lim_{z\to\pi/2}\frac{e^z}{-\sin z} = -e^{\pi/2}$$

$$\underset{z=-\pi/2}{\text{Res}}\ f(z) = \lim_{z\to-\pi/2}\frac{e^z}{-\sin z} = e^{-\pi/2}$$

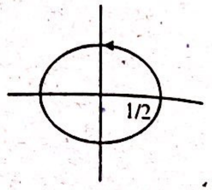
Now by Cauchy Residue Theorem,

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of Res of f(z) in C)}$$

$$\Rightarrow \oint_C \frac{e^z}{\cos z}dz = 2\pi i \times (-e^{\pi/2} + e^{-\pi/2}) = -4\pi i \sinh\frac{\pi}{2}$$

(c) $\oint_C \frac{z+1}{z^4-2z^3}dz$ $C: |z| = \frac{1}{2}$

**Solution:** We have,




$$f(z) = \frac{z+1}{z^4-2z^3}$$

For poles of f(z), set

$$z^3(z-2) = 0 \Rightarrow z = 0, 2$$

Given that the region is C: $|z| = \frac{1}{2}$ that includes only z = 0 which is a pole of order three. So, the residue of f(z) at z = 0 be,

$$\underset{z=0}{\text{Res}} f(z) = \lim_{z\to 0} \frac{1}{2!}\frac{d^2}{dz^2}\left[z^3 \frac{(z+1)}{z^3(z-2)}\right]$$

$$= \lim_{z\to 0} \frac{1}{2!}\frac{d}{dz}\left[\frac{z-2-z-1}{(z-2)^2}\right]$$

$$= -\frac{3}{2}\lim_{z\to 0}\frac{-2}{(z-2)^3} = -\frac{3}{2}\times\frac{1}{4} = -\frac{3}{8}$$

By Cauchy's Residue Theorem,

$$\oint_C f(z)\,dz = 2\pi i\ (\text{sum of Res. of f(z) in C})$$

$$\Rightarrow \oint_C \frac{z+1}{z^4-2z^3}dz = 2\pi i \times -\frac{3}{8} = -\frac{3\pi i}{4}$$

(d) $\oint_C \frac{z\cosh \pi z}{z^4+13z^2+36}dz$ $C: |z| = \pi$

**Solution:** We have,

$$f(z) = \frac{z\cosh\pi z}{z^4+13z^2+36}$$

For poles of f(z) set,

$$z^4 + 13z^2 + 30 = 0 \Rightarrow (z^2+9)(z^2+4) = 0$$
$$\Rightarrow z = \pm 3i, \pm 2i$$

Given that the region is C: $|z| = \pi$ that includes $z = \pm 3i, \pm 2i$ which are simple poles. Now, the residue of f(z) at the points is,

$$\underset{z=3i}{\text{Res}} f(z) = \lim_{z\to 3i}\frac{z\cosh\pi z}{z^4+13z^2+36}$$

$$= \frac{3i\cosh\pi 3i}{4(3i)^3+26\times 3i} = \frac{3i\cosh 3\pi}{-4\times 27i+26\times 3i} = \frac{\cosh 3\pi}{-36+26} = \frac{\cosh 3\pi}{-10}$$

$$\underset{z=-3i}{\text{Res}} f(z) = \lim_{z\to -3i}\frac{z\cosh\pi z}{z^4+13z^2+36}$$



$$= \frac{-3i\cosh 3\pi}{4(-3i)^3-26\times 3i} = \frac{-3i\cosh 3\pi}{4\times 27i-26\times 3i} = \frac{\cosh 3\pi}{-10}$$

$$\underset{z=2i}{\text{Res}} f(z) = \lim_{z\to 2i}\frac{z\cosh\pi z}{z^4+13z^2+36} = \frac{2i\cosh 2\pi}{-4\times 2^3 i+26\times 2i} = \frac{\cosh 2\pi}{10}$$

Similarly,

$$\underset{z=-2}{\text{Res}} f(z) = \frac{\cos 2\pi}{10}$$

By Cauchy's Residue Theorem,

$$\oint_C f(z)\,dz = 2\pi i\ (\text{sum of Res. of f(z) in C})$$

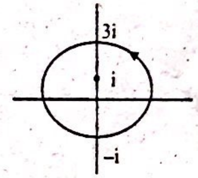
$$\Rightarrow \oint_C \frac{z\cosh\pi z}{z^4+13z^2+36}dz = 2\pi i\left(\frac{2\cosh 3\pi}{-10}+\frac{2\cosh 2\pi}{10}\right)$$

$$= \frac{2\pi i}{5}(\cosh 2\pi - \cosh 3\pi).$$

**Note:** *To obtain the book's answer please process the above question with*

$$f(z) = \frac{z\cos\pi z}{z^4+13z^2+36}.$$

3. $\oint_C \frac{dz}{(z^2+4)^3}$ $C: |z - i| = 2$




**Solution:** We have,

$$f(z) = \frac{1}{(z^2+4)^3}$$

Clearly, f(z) has poles at $z = \pm 2i$ of order 3.

And, given region is C: $|z - i| = 2$ that include only the point z = 2i which is simple pole.

Now, the residue of f(z) at the point is,

$$\underset{z=2i}{\text{Res}} f(z) = \frac{1}{2i}\underset{z=2i}{\text{Res}}\frac{d^2}{dz^2}\left((z-2i)^3\frac{1}{(z-2i)^3(z+2i)^3}\right)$$

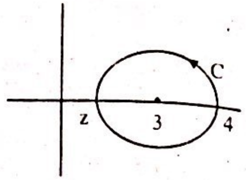
$$= \frac{1}{2}\lim_{z\to 2i}\frac{12}{(z+2i)^5} = 6\times\frac{1}{(4i)^5} = \frac{3}{2\times 256i}$$

By Cauchy's Residue Theorem,

$$\oint_C f(z)\,dz = 2\pi i\ (\text{sum of Residues in C})$$

i.e. $\oint_C \frac{dz}{(z^2+4)^3} = 2\pi i\times\frac{3}{2\times 256\,i} = \frac{3\pi}{256}$

4. $\oint_C \frac{1-\cos 2(z-3)}{(z-3)^3}\,dz \qquad C: |z-3| = 1$




**Solution:** We have,

$$f(z) = \frac{1-\cos 2(z-3)}{(z-3)^3}$$

Clearly, f(z) has poles are at z = 3 of order 3.

Given that the region is C: $|z-3| = 1$ that includes the point z = 3 of order 3.

Now, the residue of f(z) at the points be,

$$\underset{z\to 3}{\text{Res}} f(z) = \lim_{z\to 3} \frac{1}{2!}\frac{d^2}{dz^2}[1-\cos 2(z-3)] = \frac{1}{2}\lim_{z\to 3} 4\cos 2(z-3) = 2$$

By Cauchy's Residue Theorem,

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of residue of f(z) in C)}$$

$$\Rightarrow \oint_C \left(\frac{1-\cos 2(z-3)}{(z-3)^3}\right) dz = 2\pi i \times 2 = 4\pi i$$

5. $\oint_C \frac{dz}{z^8(z+4)}$ **where C is the circle**

(a) $|z| = 2$ (b) $|z+2| = 3$

**Solution:** We have,

$$f(z) = \frac{1}{z^8(z+4)}$$

Clearly, f(z) has poles are at z = 0 of order 8 and at z = –4 of simple order.

(a) Given region is C: $|z| = 2$ that include only the point z = 0 of order 8.

Now, the residue of f(z) at the point is,

$$\underset{z=0}{\text{Res}} f(z) = \lim_{z\to 0}\frac{1}{7!}\frac{d^7}{dz^7}\left(\frac{1}{z+4}\right) = \lim_{z\to 0}\frac{1}{7!}\times\frac{(-1)^7\times 7!}{(z+4)^8} = \frac{-1}{4^8} = \frac{-1}{2^{16}}$$

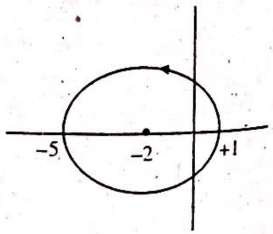
By Cauchy's Residue Theorem,

$$\oint_C \frac{dz}{z^8(z+4)} = 2\pi i \times \frac{-1}{2^{16}} = \frac{-\pi i}{2^{15}}$$

(b) Given that the region is C: $|z+2| = 3$ that includes the point z = 0 of order 8 and z = –4 of simple pole.

Now, the residue of f(z) at the point are,

$$\underset{z=0}{\text{Res}} f(z) = \frac{-1}{2^{16}} \qquad \text{[by (a)]}$$




And,



$$\underset{z=-4}{\text{Res}} f(z) = \lim_{z\to -4}\frac{1}{z^8} = \frac{1}{(-4)^8} = \frac{1}{2^{16}}$$

By Cauchy's Residue Theorem,

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of residue of f(z) in C)}$$

$$\Rightarrow \oint_C \frac{dz}{z^8(z+4)} = 2\pi i\left(-\frac{1}{2^{16}}+\frac{1}{2^{16}}\right) = 0$$

6. $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$, **where C is $|z| = 4$**

**Solution:** We have,

$$f(z) = \frac{e^z}{(z^2+\pi^2)^2}$$

Clearly, f(z) has poles at $z = \pm\pi i$ of order 2.

And, given region is C: $|z| = 4$ that include the point $z = \pm\pi i$ of order 2.

Now, the residue of f(z) at the points are,

$$\underset{z=\pi i}{\text{Res}} f(z) = \lim_{z\to\pi i}\frac{d}{dz}\left[\frac{e^z}{(z+\pi i)^2}\right]$$

$$= \lim_{z\to\pi i}\left[\frac{e^z(z+\pi i)^2 - 2e^z(z+\pi i)}{(z+\pi i)^4}\right]$$

$$= \lim_{z\to\pi i}\left[\frac{e^z[(z+\pi i)-2]}{(z+\pi i)^3}\right]$$

$$= \frac{e^{i\pi}[(2\pi i)-2]}{(2\pi i)^3}$$

$$= \frac{2-2\pi i}{-8\pi^3 i} \qquad [\text{Since } e^{i\pi} = \cos\pi + i\sin\pi = -1]$$

$$= \frac{i+\pi}{4\pi^3}$$

And,

$$\underset{z=-\pi i}{\text{Res}} f(z) = \lim_{z\to-\pi i}\frac{d}{dz}\left[\frac{e^z}{(z-\pi i)^2}\right]$$

$$= \lim_{z\to-\pi i}\frac{e^z(z-\pi i)-2e^z}{(z-\pi i)^3}$$

$$= \frac{e^{-\pi i}[-2\pi i-2]}{(-2\pi i)^3}$$

$$= \frac{2\pi i+2}{8\pi^3 i} \qquad [\text{Since } e^{-i\pi} = \cos\pi - i\sin\pi = -1]$$

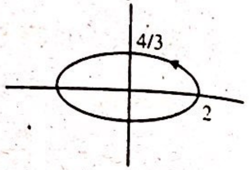
$$= \frac{\pi - i}{4\pi^3}$$

By Cauchy's Residue Theorem,

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of residue of } f(z) \text{ in C)}$$

$$\Rightarrow \oint_C \frac{e^z}{(z^2+\pi^2)^2}dz = 2\pi i\left[\frac{\pi+i}{4\pi^3}+\frac{\pi-i}{4\pi^3}\right] = \frac{4\pi^2}{4\pi^3} = \frac{i}{\pi}$$

**7.** $\oint_C \frac{e^{2z}}{(z+1)^3}dz$ C : $4x^2+9y^2=16 \Rightarrow \frac{x^2}{4}+\frac{y^2}{16/9}=1$




**Solution:** We have,

$$f(z) = \frac{e^{2z}}{(z+1)^3}$$

Clearly, f(z) has poles at $z = -1$ of order 3.

Given that the region is the ellipse $\frac{x^2}{4}+\frac{y^2}{16/9} = 1$ that include the points $z = -1$ of order 3.

Now, the residue of f(z) at the point is,

$$\underset{z=-1}{\text{Res}} f(z) = \lim_{z\to -1}\frac{1}{2!}\frac{d^2}{dz^2}[e^{2z}] = \frac{1}{2}\cdot 4e^{-2} = \frac{2}{e^2}$$

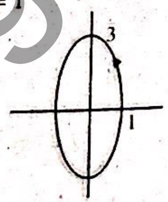
By Cauchy's Residue Theorem,

$$\oint_C \frac{e^{2z}}{(z+1)^3}dz = 2\pi i \times \frac{2}{e^2} = \frac{4\pi i}{e^2}$$

**8. Evaluate** $\oint_C \left[\frac{z\,e^{\pi z}}{z^4-16}+ze^{\pi/z}\right]dz$ **where C is the ellipse** $9x^2+y^2=9$.

**Solution:** Here,

$$\oint_C \left[\frac{z\,e^{\pi z}}{z^4-16}+ze^{\pi/z}\right]dz \qquad C: 9x^2+y^2=9 \Rightarrow \frac{x^2}{1}+\frac{y^2}{9}=1$$




$$= \oint_C \frac{ze^{\pi z}}{z^4-16}dz + \oint_C ze^{\pi/z}dz$$

$$= \oint_C f_1(z)dz + \oint_C f_2(z)\,dz$$

$$= I_1 + I_2 \qquad f_1(z) = \frac{ze^{\pi z}}{z^4-16} = \frac{P(z)}{Q(z)}$$

To evaluate $I_1$: We have,

$$f_1(z) = \frac{ze^{\pi z}}{z^4-16} = \frac{ze^{\pi z}}{(z^2-4)(z^2+4)} = \frac{ze^{\pi z}}{(z-2)(z+2)(z-2i)(z+2i)}$$

Clearly, $f_1(z)$ has poles at $z = \pm 2, \pm 2i$ of order 1.



Given that the region is C: $9x^2+y^2=9$ that include only the points $z = \pm 2i$ of order 1.

Now, the residue of $f_1(z)$ at the poles is,

$$\underset{z=2i}{\text{Res}} f_1(z) = \lim_{z\to 2i}(z-2i)\frac{ze^{\pi z}/(z^2-4)}{(z-2i)(z+2i)}$$

$$= \lim_{z\to 2i}\frac{ze^{\pi z}/(z^2-4)}{(z+2i)}$$

$$= \frac{2i\,e^{2\pi i}/(-4-4)}{(4i)}$$

$$= \frac{-1}{16} \qquad [\text{Since } e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1]$$

And,

$$\underset{z=-2i}{\text{Res}} f_1(z) = \lim_{z\to(-2i)}\frac{ze^{\pi z}/(z^2-4)}{(z-2i)}$$

$$= \frac{-2i\,e^{-2\pi i}/(-4-4)}{(-4i)}$$

$$= \frac{-1}{16} \qquad [\text{Since } e^{-2\pi i} = \cos 2\pi - i\sin 2\pi = 1]$$

By Cauchy's Residue Theorem,

$$\oint_C f_1(z)\,dz = I_1 = 2\pi i\left(-\frac{1}{16}-\frac{1}{16}\right) = \frac{-\pi i}{4}$$

Again Residues of $ze^{\pi/z}$ is [i.e. $f_2(z)$]

$$ze^{\pi/z} = z\left(1+\frac{\pi}{2}+\frac{\pi^2}{2!\,z^2}\cdots\right) = \left(z+\pi+\frac{\pi^2}{2!z}+\ldots\right) \text{ Res.} = \frac{\pi^2}{2}$$

Therefore, $I_2 = 2\pi i \times \frac{\pi^2}{2} = \pi^3 i$

Hence, $I = I_1 + I_2 = -\frac{\pi i}{4} + \pi^3 i = \pi\left(\pi^2 - \frac{1}{4}\right)i$

**9. Integrate** $\oint_C \frac{\tan z}{z^2-1}dz$ **where C:** $|z| = \frac{3}{2}$ **in a counter clockwise direction.**

**Solution:** We have,

$$f(z) = \frac{\tan z}{z^2-1}$$

Clearly, f(z) has poles are at $z = \pm 1$ of simple order.

Given that the region is C: $|z| = \frac{3}{2}$ that include the points $z = \pm 1$ of order 1.

Now, the residue of f(z) at the points be,

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{\tan z}{z+1} = \frac{\tan 1}{2}$$

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z\to -1} \frac{\tan z}{z-1} = \frac{-\tan 1}{-2} = \frac{\tan 1}{2}$$

By Cauchy's Residue Theorem,

$$\oint_C \frac{\tan z}{z^2-1} dz = 2\pi i \left(\frac{\tan 1}{2} + \frac{\tan 1}{2}\right) = 2\pi i \tan 1.$$

**10. Evaluate $\oint_C \frac{4-3z}{z^2-z} dz$ where C is the curve, which is counterclockwise simple closed path such that**

**(a) If 0 and 1 i.e. poles lies in C.** **(b) If 0 is inside and 1 outside C**

**(c) If 1 inside and 0 outside** **(d) If both outside C**

**Solution:** We have,

$$f(z) = \frac{4-3z}{z^2-z} = \frac{4-3z}{z(z-1)}$$

Clearly, f(z) has simple poles at z = 0, 1.

(a) Given that 0 and 1 i.e. poles lies in C.

Now, the residue of f(z) at the point is,

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{4-3z}{z-1} = \frac{4}{-1} = -4$$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{4-3z}{z} = \frac{4-3}{1} = 1$$

By Cauchy's Residue Theorem,

$$\oint_C \frac{4-3z}{z^2-z} dz = 2\pi i(-4+1) = -6\pi i$$

(b) Given that 0 is inside and 1 outside C.

Now, the residue of f(z) at the point is,

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{4-3z}{z-1} = \frac{4}{-1} = -4.$$

By Cauchy's Residue Theorem,

$$\oint_C \frac{4-3z}{z^2-z} dz = 2\pi i \times (-4) = -8\pi i$$

(c) Given that 1 inside and 0 outside.

Now, the residue of f(z) at the points be,

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{4-3z}{z} = \frac{4-3}{1} = 1$$



By Cauchy's Residue Theorem,

$$\oint_C \frac{4-3z}{z^2-z} dz = 2\pi i$$

(d) Given that both outside C then the function is analytic everywhere. So, by Cauchy's Integral Theorem,

$$\oint_C \frac{4-3z}{z^2-z} dz = 0.$$

## OTHER IMPORTANT QUESTION FROM FINAL EXAM FOR PRACTICE

**2002 Q. No. 2(a)**

State and prove Residue theorem.

Hint: See theorem.

**2003 Fall Q. No. 2(a)**

Define singularities and zeros of function. Evaluate

(i) $\oint_c \frac{4dz}{1+z^2}$ c : |z| = 2.

(ii) $\oint_c \frac{z^2 \sin z}{4z^2-1} dz$, c : |z| = 1.

**2004 Spring Q. No. 2(a); 2011 Fall Q. No. 2(a) OR**

State Residue theorem. Use it to find the value of the integral $\int_c \frac{z-23}{z^2-4z-5} dz$, where c is the circle |z − 2| = 4.

**2004 Fall Q. No. 2(a)**

Define a singular point and the residue of the function f(z) at $z = z_0$. Find the sum of the residues of the function $f(z) = \frac{z+1}{(z-3)(z-5)^2}$.

**2005 Spring Q. No. 2(a)**

State Cauchy Residue Theorem. Evaluate

(i) $\oint_c \frac{\tan z}{z^2-1} dz$, where c is the circle |z| = 3/2 in counter clockwise direction.

(ii) $\oint_c \left(\frac{z-23}{z^2-4z-5}\right) dz$, where C is the circle |z − 2| = 4, counter clockwise.

Hint: (i) See Exercise 5.2 Q. No. 9. (ii) See 2004 Spring.

**2005 Fall Q. No. 2 (a)**

Define singularity, zeros and pole of a function. State Cauchy Residue theorem.

Evaluate: $\oint_C \left(\frac{z^2 \sin z}{4z^2 - 1}\right) dz$ where c : |z| = 2, counter–clockwise.

Hint: See 2003 Fall.

**2006 Spring Q. No. 2(a)**

Define pole and zeroes of a function. State Cauchy's residue theorem and evaluate $\oint_C \frac{e^z}{\cos z} dz$ where C: |z| = 3.

**2006 Fall Q. No. 2(a)**

State Taylors' series of a function f(z). Evaluate $\oint_C \frac{z^2 \sin z}{4z^2 - 1} dz$ where C : |z| = 2 counter–clockwise, by using Residue theorem.

**2007 Fall Q. No. 2(a)**

Define singularity of a function. Evaluate the following integrals:

(i) $\int_C \frac{e^z}{\cos z} dz$, C : |x| = 3 (ii) $\int_C \frac{z + 1}{z^4 - 2z^3} dz$, C: $|z| = \frac{1}{2}$

Hint: (i) See 2006 Spring. (ii) See exercise 5.2 Q. No. 2(c).

**2008 Spring Q. No. 2(a)**

Define zeros and pole of a function f(z). Find the Residues of $f(z) = \frac{z + 2}{(z + 1)(z^2 + 1)^2}$.

**2008 Fall Q. No. 2(a) OR**

State Cauchy residue theorem and then evaluate $\oint_C f(z)\, dz$, where $f(z) = \frac{50z}{(z + 4)(z - 1)^2}$ and C is a positively oriented circle |z| = 5.

**2009 Fall Q. No. 2(a) OR**

State Cauchy residue theorem and then evaluate $\oint_C f(z)\, dz$, where $f(z) = \frac{9z + 1}{z(z + i)}$ and C is a positively oriented circle |z| = 2.



**2009 Spring Q. No. 2(a)**

Define residue of the complex valued function at their poles. State and prove Cauchy residue theorem. Evaluate $\oint_C \frac{e^z}{\cos z} dz$, where C : the circle |z| = 3, in counterclockwise direction.

Hint: For problem see 2006 Spring.

**2011 Spring Q. No. 2(a)**

Define Pole and Zeroes of a function. State Cauchy's residue theorem and evaluate $\oint_C \frac{z - 23}{z^2 - 4z - 5} dz$ where C : |z – 2| = 4.

Hint: For problem see 2006 Spring.

**2015 Spring Q. No. 2(a)**

**State and Prove Cauchy Reside theorem. Evaluate $\oint_C \frac{e^{5z}}{(z + i)^4} dz$, where C is a circle |z| = 3 along anticlockwise direction.**

**2016 Fall Q. No. 2(a)**

**Find the singular points and residues of the function $f(z) = \frac{z + 2}{(z - 2)(z^2 + 1)^2}$.**

**2016 Spring Q. No. 2(b)**

**State Cauchy Residue Theorem and hence evaluate $\oint_C \frac{z - 23}{z^2 - 4z - 5} dz$ where C: |z – 2| = 4.**

**2017 Fall Q. No. 2(a)**

**Define singularity of a function. Evaluate the following integrals:**

**(i) $\int \frac{e^2}{\cos z} dz$, C: |z| = 3 (ii) $\int \frac{z + 1}{z^4 - 2z^3} dz$, C: $|z| = \frac{1}{2}$**

## SHORT QUESTIONS

**2003 Fall Q. No. 7(e)**

Find the residue of $f(z) = \frac{1}{z^2 - 1}$ at z = 1.

**2004 Fall Q. No. 7(b)**

Find the type of singularity of the function sin z at z = ∞.

**2004 Fall Q. No. 7(d)**

Determine the location and the order of zero of the function. $f(z) = (z-1)^3$.

**2005 Fall Q. No. 7(c)**

Find the residue at the poles of $\frac{z^4}{z^2 - iz + 2}$

**2017 Fall Q. No. 7(c)**

**Find poles with their order of function $f(z) = \frac{1}{(z^2 + a^2)^2}$.**

❑❑❑

## Unit 7
# FOURIER INTEGRAL AND FOURIER TRANSFORM

**Periodic function**

Let f(z) be a function of x > 0 then a number p is called a ***period*** of f(x) if $f(x \pm p) = f(x)$.

***Fourier series:***

Let f(x) be a function of x of period $2\pi$. Then, the ***Fourier series*** of f(x) is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt$ for $n = 0, 1, 2, \ldots$

and, $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt$ for $n = 1, 2, \ldots$

**Note:** If f(x) is of period $2l$ then,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right]$$

where, $a_n = \frac{1}{l} \int_{-l}^{l} f(t) \cos \frac{n\pi t}{l} \, dt$ for $n = 0, 1, 2, \ldots$

and, $b_n = \frac{1}{l} \int_{-l}^{l} f(t) \sin \frac{n\pi t}{l} \, dt$ for $n = 1, 2, 3, \ldots$

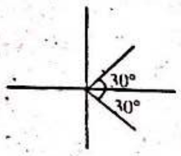
***Even and Odd function:***

A function f(x) is said to be odd if $f(-x) = -f(x)$ and is called even if $f(-x) = f(x)$.

***Example:*** sinx is odd and cosx is even.

So, $\sin(-30°) = \sin(2\pi - 30°) = \sin 330° = -\frac{1}{2} = -\sin 30°$

$\cos(-30°) = \cos(2\pi - 30°) = \cos 330° = \frac{\sqrt{3}}{2} = \cos 30°$




***Properties of odd and even function***

(i) **If f(x) is even then $f(-x) = f(x)$**

Now,

$$\int_{-l}^{l} f(x)\,dx = \int_{-l}^{0} f(x)\,dx + \int_{0}^{l} f(x)\,dx$$

Put x = –t then,

$$= \int_{l}^{0} f(-t)(-dt) + \int_{0}^{l} f(x)\,dx$$

$$= -\int_{l}^{0} f(t)\,dt + \int_{0}^{l} f(x)\,dx = \int_{0}^{l} f(t)\,dt + \int_{0}^{l} f(x)\,dx$$

Put t = x then,

$$= \int_{0}^{l} f(x)\,dx + \int_{0}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$$

2. **If f(x) is odd then f(–x) = –f(x)**

Now,

$$\int_{-l}^{l} f(x)\,dx = \int_{-l}^{0} f(x)\,dx + \int_{0}^{l} f(x)\,dx$$

Put x = –t then,

$$= \int_{0}^{l} f(-t)\,dt + \int_{0}^{l} f(x)\,dx$$

$$= -\int_{0}^{l} f(t)\,dt + \int_{0}^{l} f(x)\,dx$$

Put x = t then,

$$= -\int_{0}^{l} f(x)\,dx + \int_{0}^{l} f(x)\,dx = 0.$$

3. The product of even and even function is even
4. The product of odd and odd function is even
5. The product of even and odd function is odd.
6. The product of odd and even function is odd.

***Fourier sine and cosine series***

The Fourier series of odd function is Fourier sine series.

***NOTE:*** **(i)** Let f(x) is odd function then f(x) cosnx is odd and **f(x) sinnx is even.**

Then, f(x) $\cos\frac{n\pi x}{l}$ is odd and f(x) $\sin\frac{n\pi x}{l}$ is even. So that,

$$\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = 0 \quad \text{and} \quad \int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = 2\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Therefore, $a_n = 0$. $b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$



Hence, the Fourier series becomes,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \text{where, } b_n = \frac{2}{l}\int_{0}^{l} f(t)\sin\frac{n\pi x}{l}\,dt$$

This is called a Fourier Sine series.

**(ii) The Fourier Series of even function is Fourier cosine series.**

Let f(x) is even then f(x) $\cos\frac{n\pi x}{l}$ is even and f(x) $\sin\frac{n\pi x}{l}$ is odd. So that,

$$a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l} \quad \text{and } b_n = 0.$$

Hence, the Fourier series becomes,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}\,dx$$

with $a_0 = \frac{2}{l}\int_{0}^{l} f(t)\,dt$ and $a_n = \frac{2}{l}\int_{0}^{l} f(t)\cos\frac{n\pi t}{l}\,dt$

This is called Fourier cosine series.

**(iii) Complex form of Fourier series**

The complex form of Fourier series of a $2\pi$ periodic function f(x) is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \text{ where, } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx, \quad n = 0, \pm1, \pm2, \ldots$$

**Justification:**

Since, the Fourier series of a $2\pi$ periodic function f(x) is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

Since, $e^{inx} = \cos nx + i\sin nx$ and $e^{-inx} = \cos nx - i\sin nx$

Then,

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}; \qquad \sin nx = \frac{e^{inx} - e^{-inx}}{2i}$$

So that,

$$a_n \cos nx + b_n \sin nx = \frac{a_n - ib_n}{2}e^{inx} + \frac{a_n + ib_n}{2}e^{-inx}$$
$$= c_n e^{inx} + c_{-n} e^{-inx}$$

Hence,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[c_n e^{inx} + c_{-n} e^{-inx}] = \frac{a_0}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n e^{inx} = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2}(a_n - i\,b_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,(\cos nx - i\sin nx)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx$$

## *Fourier Integral*

**2003 Fall Q. No. 6(a) OR; 2007 Fall Q. No. 6(a); 2012 Fall Q. No. 2(a); 2016 Spring Q. No. 3(a)**

Obtain the Fourier integral formula from the Fourier series assuming the required conditions.

**2006 Fall Q. No. 6(b) OR; 2011 Fall Q. No. 6(b)**

Derive an expression for the Fourier integral from a Fourier series.

**Solution:**

The Fourier integral of f(x) for x>0 is

$$f(x) = \int_0^{\infty}[A(w)\cos wx + B(w)\sin wx]\,dw$$

where, $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt\,dt,\qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin wt\,dt$

Consider a Fourier series of f(x) for x>0 of period $2l$ is,

$$f_l(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos w_n x + b_n\sin w_n x] \quad \text{.......... (i)}$$

with,

$$w_n = \frac{n\pi}{l},\qquad a_n = \int_{-l}^{l} f_l(t)\cos w_n t\,dt,\qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f_l(t)\sin w_n(t)\,dt \quad n = 1, 2, \ldots$$

Then, (i) becomes,

$$f_l(x) = \frac{1}{2l}\int_{-l}^{l} f_l(t)\,dt + \sum_{n=1}^{\infty}$$

$$\left[\cos \ldots \frac{1}{l}\int f_l(t)\sin w_n(t)\,dt + \sin w_n x\,\frac{1}{l}\int f(t) \ldots\right]$$



Set. $\Delta w = w_{n+1} - w_n = \frac{n+1}{l}\pi - \frac{n\pi}{l} = \frac{\pi}{l} \qquad \Rightarrow \frac{1}{l} = \frac{\Delta w}{\pi}$

So that,

$$f_l(x) = \frac{1}{2l - l}\int_{-l}^{l} f_l(t)\,dt +$$

$$\sum_{n=1}^{\infty}\left[\cos w_n x\,\frac{\Delta w}{\pi}\int_{-l}^{l} f_l(t)\cos w_n(t)dt + \sin w_n x\,\frac{\Delta w}{\pi}\int_{-l}^{l} f_l(t)\cos w_n(t)dt\right]$$

Suppose that f(x) is absolutely integrable and the function is valid for any $l$.

So,

$$f(x) = \lim_{l\to\infty} f_l(x)$$

Also, if $l \to \infty$ then, $\Delta w = \frac{\pi}{l} \to 0$. So, we may replace $\Delta w$ by dw. Also, changing $\Sigma$ by $\int$. Then, as $l \to \infty$,

$$f(x) = \frac{1}{\pi}\int_1^{\infty}\left(\cos wx\int_{-\infty}^{\infty} f(t)\cos wt\,dt + \sin wx\int_{-\infty}^{\infty} f(t)\cos wt\,dt\right)dw$$

$$= \frac{1}{\pi}\int_0^{\infty}\left(\cos wx\int_{-\infty}^{\infty} f(t)\cos wt\,dt + \sin wx\int_{-\infty}^{\infty} f(t)\cos wt\,dt\right)dw$$

Then,

$$f(x) = \int_0^{\infty}(A(w)\cos wx + B(w)\sin wx)\,dw \quad \ldots\ldots\ldots \text{(ii)}$$

Where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt\,dt \quad \text{and} \quad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt\,dt$$

Thus, (ii) is the Fourier integral with the value of A(w) and B(w).

### Fourier cosine and sine integral

A function f(x) is

$$f(x) = \int_0^{\infty} A(w)\cos wx\,dw \quad \ldots\ldots\ldots \text{(iii)}$$

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt\,dt$

Then, (iii) is the Fourier cosine integral with the value of A(w).

A function f(x) ...

$$f(x)=\int_0^\infty B(w)\sin wx\,dw \quad \ldots\ldots\ldots (iv)$$

where $\quad B(w)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt\,dt$

Thus, (iv) is the Fourier sine integral with the value of B(w).

## Fourier cosine transform:

Since, the Fourier cosine integral of f(x) is,

$$f(x)=\int_0^\infty A(w)\cos wx\,dw$$

Where $\quad A(w)=\frac{2}{\pi}\int_0^\infty f(t)\cos wt\,dt$

$$=\sqrt{\frac{2}{\pi}}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos wt\,dt\right\}$$

$$=\sqrt{\frac{2}{\pi}}\,\hat{f}_c(w) \qquad \text{where, } \hat{f}_c(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos wt\,dt$$

Thus,

$$\hat{f}_c(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos wt\,dt \quad \ldots\ldots\ldots\ldots (i)$$

and, $$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(w)\cos wx\,dw \quad \ldots\ldots\ldots\ldots (ii)$$

Here, (i) is called Fourier cosine transform of f(x) and (ii) is called inverse Fourier sine transform.
Then, (iii) is the Fourier cosine integral with the value of A(w).

## Fourier sine transform:

Since, the Fourier sine integral of f(x) is,

$$f(x)=\int_0^\infty B_n(w)\sin wx\,dx$$

where, $\quad B(w)=\frac{2}{\pi}\int_0^\infty f(t)\sin wt\,dt$



$$=\sqrt{\frac{2}{\pi}}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin wt\,dt\right\}$$

$$=\sqrt{\frac{2}{\pi}}\,\hat{f}_s(w) \qquad \text{where, } \hat{f}_s(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin wt\,dt$$

Thus.

$$\hat{f}_s(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin wt\,dt \quad \ldots\ldots\ldots\ldots (i)$$

and. $$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(w)\sin wx\,dw \quad \ldots\ldots\ldots\ldots (ii)$$

Here, (i) is called Fourier sine transform of f(x) and (ii) is called inverse Fourier sine transform.

### Linearity Theorem of Fourier Cosine Transform:

Let. $\hat{f}_c=F_c(f)$ and $\hat{g}_c=F_c(g)$ are Fourier cosine transform of f(x) and g(x). Also, let a and b are constants then, $F_c(af+bg)=a\,F_c(f)+b\,F_c(g)$.

**[2005 Fall Q. No. 7(e); 2009 Fall Q. No. 7(e); 2011 Fall Q. No. 7(d)]**

**Proof:**
Let $F_c(f)$ and $F_c(g)$ are Fourier cosine transform of f(x) and g(x). So,

$$F_c(f)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx \quad \text{and} \quad F_c(g)=\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\sin wx\,dx$$

Now, by definition of Fourier cosine transform

$$F_c(af+bg)=\sqrt{\frac{2}{\pi}}\int_0^\infty (af+bg)(x)\cos wx\,dx$$

$$=a\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx+b\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx$$

$$=a\,F_c(f)+b\,F_c(g)$$

### Linearity Theorem of Fourier Sine Transform:

Let. $\hat{f}_s=F_s(f)$ and $\hat{g}_s=F_s(g)$ are Fourier cosine transform of f(x) and g(x). Also, let a and b are constants then.

$$F_s(af+bg)=a\,F_s(f)+b\,F_s(g).$$

**Proof:** Process as Linearity Theorem of Fourier Cosine Transform:

**Theorem:** Let f(x) is continuous and absolutely integrable on the x–axis, let f'(x) be piecewise continuous on each finite integral and let $f(x)\to 0$ as $x\to\infty$. Then,

$$F_c[f']=w\,F_s(f)-\sqrt{\frac{2}{\pi}}\,f(0) \quad \text{and} \quad F_s[f']=-w\,F_c(f)$$

**2005 Spring Q.No.6(a)OR; 2008 Fall Q.No.4(b)OR; 2008 Spring Q. No. 6(b)OR**

**Proof:**

Let f(x) is continuous and absolutely integrable on x–axis and also f '(x) is piecewise continuous. So, the integration by parts exist. Also, let f(x)→0 as x→∞.

Now,

$$F_c[f'] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f'(x)\cos wx\,dx$$

$$= \sqrt{\frac{2}{\pi}}[\cos wx\, f'(x)]_0^{\infty} + w\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin wx\,dx$$

$$= -\sqrt{\frac{2}{\pi}} f(0) + wf_s(w) \quad [\because f(x)\to 0 \text{ as } x\to\infty]$$

And,

$$F_s[f'] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f'(x)\sin wx\,dx$$

$$= \sqrt{\frac{2}{\pi}}[f(x)\sin wx]_0^{\infty} - w\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos wx\,dx$$

$$= -w F_c[f]$$

## Complex form of Fourier Integral

Since, the Fourier integral of f(x) is,

$$f(x) = \int_0^{\infty}[A(w)\cos wx + B(w)\sin wx]\,dw$$

where, $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt\,dt$ and $B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin wt\,dt$

Then,

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)[\cos wt\cos wx + \sin wt\sin wx]\,dt\,dw$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos w(t-x)\,dt\right]dw$$

Since, f(t) is even otherwise the integral value is zero. So, we may write,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos w(t-x)\,dt\,dw$$



Also, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\sin w(t-x)\,dt\,dw = 0$, being f(t) is even.

Then,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)[\cos w(t-x) + i\sin w(t-x)]\,dt\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{iw(t-x)}\,dt\,dw$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-iwt}\,dt\right]e^{iwx}\,dw$$

$[\because \cos w(t-x) = \cos w(x-t)]$

Put,

$$\hat{f}(w) = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-iwt}\,dt \quad \ldots\ldots\ldots\ldots (i)$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{iwx}\,dw \quad \ldots\ldots\ldots\ldots (ii)$$

Equation (i) is called the Fourier transform of f(t) is complex form and (ii) is called inverse Fourier transform of f(t).

### Theorem [Linearity of Fourier Transform]

Let F(f) and F(g) are Fourier transform of f(x) and g(x). Also, let a and b are constants then,

$$F[af + bg] = a\,F[f] + b\,F[g]$$

**Proof:**

Let, F[f] and F[g] are Fourier transform of f(x) and g(x).

Now,

$$F(af + bg) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(af + bg)(t)\,e^{-iwt}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-iwt}dt + b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)\,e^{-iwt}dt$$

$$= a\,F[f] + b\,F[g].$$

### Theorem [Fourier transform of the derivative of f(x)]

Let f(x) is continuous on x–axis and f(x) → 0 as |x| → ∞ and f '(x) is absolutely integrable on x–axis then,

$F[f'(x)] = iw\,F[f]$ and $F[f''(x)] = i^2w^2\,F[f] = -w^2F[f]$.

**Proof:**

Let f(x) is continuous on x–axis and $f(x) \to 0$ as $|x| \to \infty$. Also, $f'(x)$ is absolutely integrable on x–axis. So, the integrating by parts exists.

Now,

$$F[f'(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(t)\, e^{-iwt}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ [f(t)\, e^{-iwt}]_{-\infty}^{\infty} + iw \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt \right\}$$

$$= \frac{1}{\sqrt{2\pi}}\, iw \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt \qquad [\because f(x) \to 0 \text{ as } |z| \to \infty]$$

$$= iw\, F[f]$$

Also,

$$F[f''(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f''(t)\, e^{-iwt}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ [f'(t)\, e^{-iwt} + iw\, f(t)\, e^{-iwt}]_{0}^{\infty} + (iw)^2 \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt \right\}$$

$$= \frac{1}{\sqrt{2\pi}} (iw)^2 \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt$$

$$= i^2 w^2 F[f]$$

$$= -w^2 F[f]$$

## Parseval's Identity for Fourier Transform

If f(x) and g(x) are piecewise continuous, bounded and absolutely integrable on x–axis with Fourier transform F(w) and G(w) respectively then

$$\int_{-\infty}^{\infty} F(w)\bar{G}(w)\, dw = \int_{-\infty}^{\infty} f(x)\bar{g}(x)\, dx$$

and $$\int_{-\infty}^{\infty} |F(w)|^2\, dw = \int_{-\infty}^{\infty} |f(x)|^2\, dx.$$

where, $(\bar{\ })$ indicates the complex conjugate.

**Proof:**

Let f(x) and g(x) are piecewise continuous, bounded and absolutely integrable on x–axis. So, the Fourier transform and its complex conjugate exist.



Now,

$$\int_{-\infty}^{\infty} F(w)\,\bar{G}(w)\, dw = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\,\bar{G}(w)\, e^{-iwx}\, dx\, dw$$

$$= \int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{G}(w)\, e^{-iwx}\, dw\, dx$$

$$= \int_{-\infty}^{\infty} f(x)\,\bar{g}(x)\, dx \qquad \ldots (1)$$

Also, $$\int_{-\infty}^{\infty} |F(w)|^2\, dw = \int_{-\infty}^{\infty} F(w)\,\bar{F}(w)\, dw = \int_{-\infty}^{\infty} f(x)\,\bar{f}(x)\, dx \qquad [\text{as } (1)]$$

$$= \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

## Parseval's identity for Fourier cosine and sine transform

i. $\int_0^{\infty} F_c(w)\, G_c(w)\, dw = \int_0^{\infty} f(x)\, g(x)\, dx$ ii. $\int_0^{\infty} |F_c(w)|^2\, dw = \int_0^{\infty} |f(x)|^2\, dx$

iii. $\int_0^{\infty} F_s(w)\, G_s(w)\, dw = \int_0^{\infty} f(x)\, g(x)\, dx$ iv. $\int_0^{\infty} |F_s(w)|^2\, dw = \int_0^{\infty} |f(x)|^2\, dx$

## Convolution of Two Functions

The convolution of given two functions f(x) and g(x) is denoted by f*g and is defined by

$$(f*g)(x) = \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp = \int_{-\infty}^{\infty} f(x-p)\, g(p)\, dp$$

## Fourier Transform of Convolution of Two Functions

Let F(f) and F(G) are Fourier transform of given functions f(x) and g(x), exist. Then, the Fourier transform of the convolution of f(x) and g(x) is define as,

$$F[(f*g)(x)] = \sqrt{2\pi}\, F[f]\, F[g]$$

**Proof:** Since,

$$(f*g)(x) = \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp$$

Now, $$F[(f*g)(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp\, e^{-iwx}\, dx$$

Put, $x - p = q$ so, $dx = dq$. Also, $x \to (-\infty) \Rightarrow q \to (-\infty)$, $x \to \infty \Rightarrow q \to \infty$.
Then,

$$F[(f*g)(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p)\, g(q)\, dp\, e^{-iw(p+q)}\, dq$$

$$= \sqrt{2\pi}\,.\,\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)\, e^{-iwp}\, dp.\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(q)\, e^{-iwq}\, dq.$$

$$= \sqrt{2\pi}\; F[f]\, F[g].$$

**2005 Fall Q. No. 6(b)**

Define convolution of two functions. State and prove convolution theorem on Fourier transform.

**2006 Spring Q. No. 6(b)**

Define convolution of the two functions. If f(x) and g(x) are piecewise continuous, bounded and absolutely integrable on the x–axis then $F(f*g) = \sqrt{2\pi}\, F(f)\, F(g)$.

## Exercise – 7.1

1. **Show that,**

a. $$\int_0^{\infty} \left(\frac{\cos xw + w \sin wx}{1 + w^2}\right) dw = \begin{cases} 0 & \text{for } x < 0 \\ \frac{\pi}{2} & \text{for } x = 0, \\ \pi e^{-x} & \text{for } x > 0 \end{cases}$$

**[2016 Fall Q. No. 4(b)]**
**[2017 Fall Q. No. 5(b)] [2011 Spring Q. No. 4(a)] [2004 Fall Q. No. 6(a)]**

**Solution:** Let,

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{\pi}{2} & \text{for } x = 0 \\ \pi e^{-x} & \text{for } x > 0 \end{cases} \qquad \text{.......... (i)}$$

Then, we wish to show

$$f(x) = \int_0^{\infty} \left(\frac{\cos xw + w \sin wx}{1 + w^2}\right) dw \qquad \text{........... (ii)}$$

Since, on the right part of (ii), the integral includes sine and cosine function. So, we should determine the Fourier integral for f(x).
Since, the Fourier integral for f(x) is,

$$f(x) = \int_0^{\infty} [A(w) \cos wx + B(w) \sin wx]\, dw \qquad \text{… … … (iii)}$$



with $A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos wt\, dt$ & $B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin wt\, dt$

Here,

$$A(w) = \frac{1}{\pi}\left[\int_{-\infty}^{0} 0.\cos wt\, dt + \int_0^0 \frac{\pi}{2} \cos wt\, dt + \int_0^{\infty} \pi e^{-x} \cos wt\, dt\right]$$

[∵ using (i)]

$$= \frac{\pi}{\pi} \int_0^{\infty} e^{-t} \cos wt\, dt$$

$$= \left[\frac{e^{-t}}{(-1)^2 + w^2}((-1)\cos wt + w \sin wt)\right]_0^{\infty}$$

$$\left[\because \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + c\right]$$

$$= \frac{1}{1 + w^2} \qquad [\because e^{-\infty} \sim 0]$$

And,

$$B(w) = \frac{1}{\pi}\left[\int_{-\infty}^{0} 0.\sin wt\, dt + \int_0^0 \frac{\pi}{2} \sin wt\, dt + \int_0^{\infty} \pi e^{-t} \sin wt\, dt\right] \quad [\because \text{using (i)}]$$

$$= \frac{\pi}{\pi} \int_0^{\infty} e^{-t} \sin wt\, dt$$

$$= \left[\frac{e^{-t}}{1 + w^2}(-\sin wt - w \cos wt)\right]_0^{\infty}$$

$$\left[\because \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx + b \cos bx) + c\right]$$

$$= \frac{w}{1 + w^2} \quad [\because e^{-\infty} \sim 0]$$

Then (iii) becomes,

$$f(x) = \int_0^{\infty} \left(\frac{\cos wx + w \sin wx}{1 + w^2}\right) dw \qquad \text{… … … (iv)}$$

Thus, from (i) and (iv) we get,

$$\int_0^{\infty} \left(\frac{\cos xw + w \sin wx}{1 + w^2}\right) dw = \begin{cases} 0 & \text{for } x < 0 \\ \frac{\pi}{2} & \text{for } x = 0 \\ \pi e^{-x} & \text{for } x > 0 \end{cases}$$

b. $\displaystyle\int_0^\infty \left(\frac{1-\cos\pi w}{w}\right)\sin\pi w\, dw = \begin{cases}\frac{\pi}{2} & \text{for } 0<x<\pi \\ 0 & \text{for } x>\pi\end{cases}$ **[2016 Spring 3(a) OR]**

**[2012 Fall 2(a) OR] [2007 Fall Q. No. 6(a) OR] [2004 Spring Q. No. 6(a)]**

**Solution:** Let,

$$f(x) = \begin{cases}\frac{\pi}{2} & \text{for } 0<x<\pi \\ 0 & \text{for } x>\pi\end{cases} \quad \ldots\ldots\ldots (i)$$

Then, we wish to show,

$$f(x) = \int_0^\infty \left(\frac{1-\cos\pi w}{w}\right)\sin\pi w\, dw \quad \ldots\ldots\ldots (ii)$$

The integral part in (ii) has only sine function that included parametric value x. So, we wish to determine the Fourier sine integrand for f(x).
Since, the Fourier sine integral for f(x) is

$$f(x) = \int_0^\infty B(w)\sin wx\, dw \quad \ldots\ldots\ldots (iii)$$

with $B(w) = \frac{2}{\pi}\int_0^\infty f(t)\sin wt\, dt$

$$= \frac{2}{\pi}\left[\int_0^\pi \frac{\pi}{2}\sin wt\, dt + \int_\pi^\infty 0.\sin wt\, dt\right] \quad [\because \text{using (i)}]$$

$$= \int_0^\pi \sin wt\, dt = \left[\frac{-\cos wt}{w}\right]_0^\pi = \frac{1-\cos w\pi}{w} \quad \ldots\ldots\ldots (iv)$$

Thus, from (i), (iii) and (iv) we get,

$$\int_0^\infty \left(\frac{1-\cos w\pi}{w}\right)\sin wx\, dw = \int_0^\infty \left(\frac{1-\cos\pi w}{w}\right)\sin\pi w\, dw$$

c. $\displaystyle\int_0^\infty \left(\frac{\cos wx}{1+w^2}\right)dw = \frac{\pi e^{-x}}{2}$ for $x>0$. **[2006 Spring Q. No. 7(e)]**

**Solution:** Let,

$$f(x) = \frac{\pi e^{-x}}{2} \quad \text{for } x>0 \quad \ldots\ldots\ldots (i)$$

and we wish to show,

$$f(x) = \int_0^\infty \left(\frac{\cos wx}{1+w^2}\right)dw \quad \ldots\ldots\ldots (ii)$$



The integral part in (ii) has only cosine function that included the parametric value x. So, we wish to determine the Fourier cosine integral for f(x).
Since, the Fourier cosine integral for f(x) is,

$$f(x) = \int_0^\infty A(w)\cos wx\, dw \quad \ldots\ldots\ldots (iii)$$

with, $A(w) = \frac{2}{\pi}\int_0^\infty f(t)\cos wt\, dt$

Here,

$$A(w) = \frac{2}{\pi}\int_0^\infty \frac{\pi e^{-t}}{2}\cos wt\, dt$$

$$= \int_0^\infty e^{-t}\cos wt\, dt$$

$$= \left[\frac{e^{-t}}{1+w^2}(-\cos wt + w\sin wt)\right]_0^\infty$$

$$\left[\because \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c\right]$$

$$= \frac{1}{1+w^2} \quad [\because e^{-\infty} \sim 0]$$

Therefore, (iii) becomes

$$f(x) = \int_0^\infty \frac{\cos wx}{1+w^2}dw \quad \ldots\ldots\ldots (iv)$$

Thus, from (i) and (iv) we get,

$$\int_0^\infty \frac{\cos wx}{1+w^2}dw = \frac{\pi e^{-x}}{2} \quad \text{for } x>0.$$

d. $\displaystyle\int_0^\infty \frac{\sin\pi w\sin xw}{1-w^2}dw = \begin{cases}\frac{\pi\sin x}{2} & \text{for } 0\le x\le\pi \\ 0 & \text{for } x>\pi\end{cases}$

**Solution:** Let,

$$f(x) = \begin{cases}\frac{\pi\sin x}{2} & \text{for } 0\le x\le\pi \\ 0 & \text{for } x>\pi\end{cases} \quad \ldots\ldots\ldots (i)$$

and wish to show

$$f(x) = \int_0^\infty \frac{\sin\pi w\sin xw}{1-w^2}dw \quad \ldots\ldots\ldots (ii)$$

Since, the integral part of (ii) has only sine function that includes the parametric variable x. So, we wish to determine the Fourier sine integral of f(x).
Since, the Fourier sine integral of f(x) is,

$$f(x) = \int_0^\infty B(w)\sin wx\,dw \quad \text{.................. (iii)}$$

with $B(w) = \frac{2}{\pi}\int_0^\infty f(t)\sin wt\,dt$

Here,

$$B(w) = \frac{2}{\pi}\left[\int_0^\pi \frac{\pi \sin t}{2}\sin wt\,dt + \int_\pi^\infty 0.\sin wt\,dt\right] \quad [\because \text{ using (i)}]$$

$$= \int_0^\pi \sin wt \sin t\,dt$$

$$= \frac{1}{2}\int_0^\pi [\cos(w-1)t - \cos(w+1)t]dt$$

$$[\because 2\sin A\sin B = \cos(A-B) - \cos(A+B)]$$

$$= \frac{1}{2}\left[\frac{\sin(w-1)t}{w-1} - \frac{\sin(w+1)t}{w+1}\right]_0^\pi$$

$$= \frac{1}{2}\left[\frac{\sin(w-1)\pi}{w-1} - \frac{\sin(w+1)\pi}{w+1}\right] \quad [\because \sin 0 = 0]$$

$$= \frac{1}{2}\left[\frac{\sin w\pi\cos w\pi}{w-1} - \frac{\sin w\pi\cos\pi}{w+1}\right] \quad [\because \sin\pi = 0]$$

$$= \frac{\sin w\pi}{2}\left[\frac{-1}{w-1} + \frac{1}{w+1}\right] \quad [\because \cos\pi = -1]$$

$$= \frac{\sin w\pi}{2}\left(\frac{-2}{w^2-1}\right)$$

$$= \frac{\sin w\pi}{1-w^2}$$

Now, (iii) becomes,

$$f(x) = \int_0^\infty \frac{\sin\pi w\,\sin xw}{1-w^2}dw \quad \text{......... (iv)}$$

Thus, from (i) and (iv) we get,

$$\int_0^\infty \frac{\sin\pi w\,\sin xw}{1-w^2}dw = \begin{cases}\frac{\pi\sin x}{2} & \text{for } 0 \le x \le \pi\\ 0 & \text{for } x > \pi\end{cases}$$



e. $\int_0^\infty \frac{w\sin wx}{a^2+w^2}dw = \frac{\pi e^{-ax}}{2}$ **for x > 0, a < 0.**

**Solution:** Let,

$$f(x) = \frac{\pi e^{-ax}}{2} \quad \text{for } x > 0, a < 0 \quad \text{............ (i)}$$

Then, we wish to show,

$$f(x) = \int_0^\infty \frac{w\sin wx}{a^2+w^2}dw \quad \text{............ (ii)}$$

Clearly, the integral part of (ii) has only sine function that includes than parametric variable x. So, the Fourier sine integral for f(x) is applicable. Since, the Fourier sine integral of f(x) is,

$$f(x) = \int_0^\infty B(w)\sin wx\,dw \quad \text{............ (iii)}$$

with, $B(w) = \frac{2}{\pi}\int_0^\infty f(t)\sin wt\,dt$

Here,

$$\int_0^\infty \frac{\pi}{2}e^{-at}\sin wt\,dt = \int_0^\infty e^{-at}\sin wt\,dt$$

$$= \left[\frac{e^{-at}}{a^2+w^2}(a\sin wt - w\cos wt)\right]_0^\infty$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c\right]$$

$$= \frac{w}{a^2+w^2} \quad [\because e^{-\infty} \sim 0]$$

Now, (iii) becomes,

$$f(x) = \int_0^\infty \frac{w\sin wx}{a^2+w^2}dw = \frac{\pi e^{-ax}}{2} \quad [\because \text{ using (i)}] \quad \text{......... (iv)}$$

Thus, from (i) and (iv), the requirement holds.

f. $\int_0^\infty \left(\frac{k\cos wx}{k^2+w^2}\right)dw = \frac{\pi e^{-kx}}{2}$ **for x > 0.**

**<u>OR 2002 Q. No. 6(a)</u>**

**Find the Fourier cosine integral of the function $f(x) = e^{-kx}$, for x > 0, k > 0 and hence show that $\int_0^\infty \frac{k\cos wx}{k^2+w^2}dw = \frac{\pi}{2}e^{-kx}$.**

**Solution:** Let,

$$f(x) = \frac{\pi e^{-kx}}{2} \quad \text{for } x > 0 \qquad \text{......... (i)}$$

and we wish to show,

$$f(x) = \int_0^\infty \left(\frac{k \cos wx}{k^2 + w^2}\right) dw \qquad \text{......... (ii)}$$

The integral part in (ii) has only cosine function that included the parametric value x. So, we wish to determine the Fourier cosine integral for f(x).

Since, the Fourier cosine integral for f(x) is,

$$f(x) = \int_0^\infty A(w) \cos wx \, dw \qquad \text{.................. (iii)}$$

with, $A(w) = \frac{2}{\pi}\int_0^\infty f(t) \cos wt \, dt$

Here,

$$A(w) = \frac{2}{\pi}\int_0^\infty \frac{\pi e^{-kt}}{2} \cos wt \, dt$$

$$= \int_0^\infty e^{-kt} \cos wt \, dt$$

$$= \left[\frac{e^{-kt}}{k^2 + w^2}(-k \cos wt + w \sin wt)\right]_0^\infty$$

$$\left[\because \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + c\right]$$

$$= \frac{k}{k^2 + w^2} \qquad [\because e^{-\infty} \sim 0]$$

Therefore, (iii) becomes

$$f(x) = \int_0^\infty \frac{k \cos wx}{k^2 + w^2} dw \qquad \text{......... (iv)}$$

Thus, from (i) and (iv) we get,

$$\int_0^\infty \left(\frac{k \cos wx}{k^2 + w^2}\right) dw = \frac{\pi e^{-kx}}{2} \quad \text{for } x > 0.$$

2. **Find Fourier cosine integral of the following functions:**

a. $f(x) = \begin{cases} 1 & ,0 < x < 1 \\ 0 & ,x > 1 \end{cases}$



**Solution:** Here,

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases} \qquad \text{.............. (i)}$$

Now, the Fourier cosine integral of f(x) is,

$$F_c(x) = \int_0^\infty A(w) \cos wx \, dw \qquad \text{............. (ii)}$$

where, $A(w) = \frac{2}{\pi}\int_0^\infty f(t) \cos wt \, dt$

$$= \frac{2}{\pi}\int_0^\infty \cos wt \, dt = \frac{2}{\pi}\left[\frac{\sin wt}{w}\right]_0^1 = \frac{2 \sin w}{\pi w}$$

Hence, the Fourier cosine transform of f(x) is,

$$F_c(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos wx \sin w}{w} dw$$

b. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

**Solution:** Here,

$$f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$$

Now, the Fourier cosine integral of f(x) is,

$$F_c(x) = \int_0^\infty A(w) \cos wx \, dw$$

where, $A(w) = \frac{2}{\pi}\int_0^\infty f(t) \cos wt \, dt$

$$= \frac{2}{\pi}\int_0^\infty t \cos wt \, dt$$

$$= \frac{2}{\pi}\int_0^\infty t \cos wt \, dt$$

$$= \frac{2}{\pi}\left\{\left[t \frac{\sin wt}{w}\right]_0^a - \frac{1}{w}\int_0^a \sin wt \, dt\right\}$$

$$= \frac{2a}{\pi w} \sin wa - \frac{2}{\pi w}\left[\frac{-\cos wt}{w}\right]_0^a$$

$$= \frac{2a}{\pi w} \sin wa + \frac{2}{\pi w^2} \cos wa - \frac{2}{\pi w^2}$$

Hence,

$$F_c(x) = \frac{2}{\pi}\int_0^\infty \left[\frac{a\sin wa}{w} + \frac{\cos wa - 1}{w^2}\right]\cos wx\, dw$$

**c.** $f(x) = \frac{1}{1+x^2}$ $x > 0$

**Solution:** Here,

$$f(x) = \frac{1}{1+x^2} \quad x > 0 \quad \text{.............. (i)}$$

Now, Fourier cosine series of f(x) is,

$$F_c(x) = \int_0^\infty A(w)\cos wx\, dw \quad \text{.............. (ii)}$$

where, $A(w) = \frac{2}{\pi}\int_0^\infty f(t)\cos wt\, dt$

$$= \frac{2}{\pi}\int_0^\infty (1+t^2)^{-1}\cos wt\, dt = e^{-w}$$

Therefore, (ii) becomes,

$$F_c(x) = \int_0^\infty e^{-w}\cos wx\, dw$$

**3. Find Fourier sine integral of the following functions:**

**(a)** $f(x) = \begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

**Solution:** Let,

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases} \quad \text{.............. (i)}$$

Now, Fourier sine integral of f(x) is,

$$F_s(x) = \int_0^\infty B(w)\sin wx\, dw \quad \text{.............. (ii)}$$

where, $B(w) = \frac{2}{\pi}\int_0^\infty f(t)\sin wt\, dt.$

Here, $B(w) = \frac{2}{\pi}\left[\int_0^a 1.\sin wt\, dt + \int_a^\infty 0.\sin wt\, dt\right]$

$$= \frac{2}{\pi}\left[\frac{-\cos wt}{w}\right]_0^a = \frac{2}{\pi}\left[\frac{1-\cos aw}{w}\right]$$



Then (ii) becomes,

$$F_s(x) = \frac{2}{\pi}\int_0^\infty \left[\frac{1-\cos aw}{w}\right]\sin wx\, dw.$$

**(b)** $f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

**Solution:** Let,

$$f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases} \quad \text{.............. (i)}$$

Now, Fourier sine integral of f(x) is,

$$F_s(x) = \int_0^\infty B(w)\sin wx\, dw \quad \text{.............. (ii)}$$

where, $B(w) = \frac{2}{\pi}\int_0^\infty f(t)\sin wt\, dt.$

Here, $B(w) = \frac{2}{\pi}\left[\int_0^\pi \sin t\, \sin wt\, dt + \int_\pi^\infty 0.\sin wt\, dt\right]$

$$= \frac{2}{\pi}\int_0^\pi \sin t \sin wt\, dt$$

$$= \frac{2}{2\pi}\int_0^\pi [\cos(w-1)t - \cos(w+1)t]\, dt$$

$$= \frac{1}{\pi}\left[\frac{\sin(w-1)t}{w-1} - \frac{\sin(w+1)t}{w+1}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\frac{-\sin w\pi}{w-1} - \frac{\sin w\pi}{w+1}\right]$$

$$= \frac{2\sin w\pi}{\pi(1-w^2)}$$

Then (ii) becomes,

$$F_s(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin w\pi \sin wx\, dw}{(1-w^2)}$$

**c.** $f(x) = \begin{cases} e^x & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

**Solution:** Let,

$$f(x) = \begin{cases} e^x & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases} \quad \text{.............. (i)}$$

Since, the Fourier since integral of f(x) is,

$$F_s(x) = \int_0^\infty B(w)\sin wx\, dw \qquad \text{.................. (ii)}$$

with, $B(w) = \frac{2}{\pi}\int_0^\infty f(t)\sin wt\, dt$

Here, $B(w) = \frac{2}{\pi}\left[\int_0^1 e^t \sin wt\, dt + \int_1^\infty 0.\sin wt\, dt\right]$

$$= \frac{2}{\pi}\int_0^1 e^t \sin wt\, dt$$

$$= \frac{2}{\pi}\left[\frac{e^t}{1+w^2}(1.\sin wt - w\cos wt)\right]_0^1$$

$$\left[\because \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c\right]$$

$$= \frac{2}{\pi}\left[\frac{e(\sin w - w\cos w) + w}{1+w^2}\right]$$

Then (ii) becomes,

$$F_s(x) = \frac{2}{\pi}\int_0^\infty \left[\frac{e(\sin w - w\cos w) + w}{1+w^2}\right]\sin wx\, dw$$

This is required Fourier sine integral for f(x).

## Exercise – 7.2

1. **Find Fourier cosine transform of the following functions:**

(a) $f(x) = \begin{cases} 1 & 0 < x < 1 \\ -1 & 1 < x < 1 \\ 0 & x > 2 \end{cases}$

**Solution:** Here, $f(x) = \begin{cases} 1 & 0 < x < 1 \\ -1 & 1 < x < 1 \\ 0 & x > 2 \end{cases}$

Then, the Fourier cosine transform of f(x) is,

$$F_c(f) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^1 \cos wx\, dx - \sqrt{\frac{2}{\pi}}\int_0^2 \cos wx\, dx$$



$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin wx}{w}\right]_0^1 - \sqrt{\frac{2}{\pi}}\left[\frac{\sin wx}{w}\right]_0^2$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin w}{w} - \frac{\sin 2w}{w} + \frac{\sin w}{w}\right]$$

**(b)** $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

**Solution:** Here, $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$ ........ (i)

Now, the Fourier cosine transform of (i) is

$$F_c(f) = \sqrt{\frac{2}{\pi}}\int_0^a x\cos wx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[(x)\left(\frac{\sin wx}{w}\right) - (1)\left(\frac{-\cos wx}{w^2}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(\frac{a\sin wa}{w}\right) - \left(\frac{1-\cos wa}{w^2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{aw\sin wa + \cos wa - 1}{w^2}\right]$$

**(c)** $f(x) = \frac{1}{1+x^2}$

**Solution:** Here, $f(x) = \frac{1}{1+x^2}$

Now, the Fourier cosine transform of f(x) is,

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+x^2}\cos\omega t\, dt$$

We have, $\int_0^\infty \frac{1}{1+t^2}\cos\omega t\, dt = \frac{\pi}{2}e^{-\omega}$

Therefore, $F_c(f(x)) = \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2}e^{-\omega} = \sqrt{\frac{\pi}{2}}e^{-\omega}$

**(d)** $f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

**Solution:** Here,

$$f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases} \qquad \text{............... (i)}$$

Now, the Fourier cosine transform of (i) is,

$$F_c(f) = \sqrt{\frac{2}{\pi}}\int_0^\infty x^2\cos wx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[x^2\frac{\sin wx}{w} + 2x\frac{\cos wx}{w^2} - 2\frac{\sin wx}{w^3}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin w}{w} + \frac{2\cos w}{w^2} - \frac{2\sin w}{w^3}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2w\cos w + (w^2 - 2)\sin w}{w^3}\right]$$

**(e)** $f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 < x < 2 \\ 0 & x > 2 \end{cases}$

**Solution:** Here,

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 < x < 2 \\ 0 & x > 2 \end{cases} \quad \text{............. (i)}$$

Now, the Fourier cosine transform of (i) is,

$$F_c(f) = \sqrt{\frac{2}{\pi}}\int_0^1 x\cos wx\,dx + \sqrt{\frac{2}{\pi}}\,2\int_1^2 \cos wx\,dx - \sqrt{\frac{2}{\pi}}\int_1^2 x\cos wx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[x\frac{\sin wx}{w} + \frac{\cos wx}{w^2}\right]_0^1 + 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin wx}{w}\right]_1^2 - \sqrt{\frac{2}{\pi}}\left[x\frac{\sin wx}{w} + \frac{\cos wx}{w^2}\right]_1^2$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin w}{w} + \frac{\cos w - 1}{w^2}\right] + 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin 2w - \sin w}{w}\right] -$$

$$\sqrt{\frac{2}{\pi}}\left[\frac{2\sin 2w - \sin w}{w} + \frac{\cos 2w - \cos w}{w^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin w}{w} + \frac{\cos w}{w^2} - \frac{1}{w^2} + \frac{2\sin 2w}{w} - \frac{2\sin w}{w} - \frac{2\sin 2w}{w} + \frac{\sin w}{w} - \frac{\cos 2w}{w^2} + \frac{\cos w}{w^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2\cos w + \cos 2w - 1}{w^2}\right]$$

**(f)** $f(x) = \begin{cases} \cos x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

**Solution:** Here, $f(x) = \begin{cases} \cos x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$ ............. (i)

Now, Fourier cosine transform of (i) is,

$$F_c(f) = \sqrt{\frac{2}{\pi}}\int_0^a \cos x\cos wx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int_0^a [\cos(w+1)x + \cos(w-1)x]\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(w+1)x}{w+1} + \frac{\sin(w-1)x}{w-1}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(w+1)a}{w+1} + \frac{\sin(w-1)a}{w-1}\right]$$



**2. Find Fourier sine transform of the following functions:**

**a.** $f(x) = e^{-ax}$

**Solution:** Here, $f(x) = e^{-ax}$ $\quad a > 0$ ................ (i)

Now, Fourier sine transform of (i) is,

$$F_S(f) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin wx\,dx \quad \text{............... (ii)}$$

Since,

$$\int_0^\infty e^{-ax}\sin wx\,dx = \left[\frac{e^{-ax}}{a^2 + w^2}(-a\sin wx - w\cos wx)\right]_0^\infty = \frac{w}{a^2 + w^2}$$

Hence, (i) becomes,

$$F_S[f] = \sqrt{\frac{2}{\pi}}\frac{w}{a^2 + w^2}$$

**b.** $f(x) = \begin{cases} x^2 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$

**Solution:** Here, $f(x) = \begin{cases} x^2 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$

Now, the Fourier sine transform of f(x) is,

$$F_s[f] = \sqrt{\frac{2}{\pi}}\int_0^1 x^2\sin wx\,dx = \sqrt{\frac{2}{\pi}}\left[-x^2\frac{\cos wx}{w} + 2x\frac{\sin wx}{w^2} + 2\frac{\cos wx}{w^3}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-\cos w}{w} + \frac{2\sin w}{w^2} + \frac{2\cos w}{w^3} - \frac{2}{w^3}\right]$$

**c.** $f(x) = e^{-x}$

**Solution:** Here, $f(x) = e^{-x}$ ................ (i)

Now, Fourier sine transform of (i) is,

$$F_S(f) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin wx\,dx \quad \text{............... (ii)}$$

Since,

$$\int_0^\infty e^{-x}\sin wx\,dx = \left[\frac{e^{-x}}{1 + w^2}(-\sin wx - w\cos wx)\right]_0^\infty = \frac{w}{1 + w^2}$$

Hence, (i) becomes,

$$F_S[f] = \sqrt{\frac{2}{\pi}}\left(\frac{w}{a^2 + w^2}\right)$$

**(d)** $f(x) = \left(\frac{x^3}{x^4 + 4}\right)$

**Solution:** Here, $f(x) = \left(\frac{x^3}{x^4 + 4}\right)$ .................. (i)

Now, Fourier sine transform of f(x) is,

$$F_S(f) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left(\frac{t^3}{t^2+4}\right)\sin\omega t\,dt$$

$$= \sqrt{\frac{2}{\pi}}\left|\frac{\pi}{2}e^{-\omega}\right|$$

$$= \sqrt{\frac{\pi}{2}}e^{-\omega}$$

**3. Find the Fourier sine and cosine transform of the following functions:**

**(a) $f(x) = 2e^{-5x} + 5e^{-2x}$** **[2015 Fall Q. No. 4(b)]**

**Solution:** Let $f(x) = 2e^{-5x} + 5e^{-2x}$

Then the Fourier sine transform of f(x) is,

$$F_S\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[2.\int_0^\infty e^{-5x}\sin wx\,dx + 5\int_0^\infty e^{-2x}\sin wx\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[2\frac{e^{-5x}}{25+w^2}(-5\sin wx - w\cos wx)\right] + 5\left[\frac{e^{-2x}}{4+w^2}(-2\sin wx - w\cos wx)\right]\right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left(\left[\frac{2w}{25+w^2}\right] + \left[\frac{5w}{4+w^2}\right]\right)$$

And the Fourier cosine transform of f(x) is,

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[2.\int_0^\infty e^{-5x}\cos wx\,dx + 5\int_0^\infty e^{-2x}\cos wx\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[2\frac{e^{-5x}}{25+w^2}(-5\cos wx + w\sin wx)\right] + 5\left[\frac{e^{-2x}}{4+w^2}(-2\cos wx + w\sin wx)\right]\right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{10}{25+w^2} + \frac{10}{4+w^2}\right]$$

**b.** $f(x) = \begin{cases} 1 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$

**Solution:** Let, $f(x) = \begin{cases} 1 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$

Then, the Fourier sine transform of f(x) is,



$$F_S\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^a \sin wx\,dx \qquad [\because f(x) = 0 \text{ for } x > a]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-\cos wx}{w}\right]_0^a = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos wa}{w}\right)$$

And the Fourier cosine transform of f(x) is,

$$F_S\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^a \cos wx\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{\sin wx}{w}\right]_0^a = \sqrt{\frac{2}{\pi}}\left(\frac{\sin wa}{w}\right)$$

## Exercise – 7.3

**1. Find the Fourier transform of the following function:**

(i) $f(x) = \begin{cases} 1 & ,a < x < b \\ 0 & ,\text{otherwise} \end{cases}$

**Solution:** Here,

$$f(x) = \begin{cases} 1 & ,a < x < b \\ 0 & ,\text{otherwise} \end{cases} \quad \text{.......... (i)}$$

Now, the Fourier transform of (i) is,

$$F(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_a^b e^{-iwx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-iwx}}{-iw}\right]_a^b$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-iwb} - e^{-iwa}}{-iw}\right] = \frac{-i}{w\sqrt{2\pi}}(e^{iwb} - e^{-iwa}) \qquad [\because i^2 = -1]$$

**2.** $f(x) = \begin{cases} e^x & ,-a < x < a \\ 0 & ,\text{otherwise} \end{cases}$

**Solution:** Here, $f(x) = \begin{cases} e^x & ,-a < x < a \\ 0 & ,\text{otherwise} \end{cases}$ .................. (i)

Now the Fourier transform of (i) is

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{x} e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{(1-iw)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \frac{e^{(1-iw)x}}{1-iw} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \frac{e^{(1-iw)a} - e^{-(1-iw)a}}{1-iw} \right]$$

**3.** $f(x) = \begin{cases} x & ,0 < x < a \\ 0 & ,\text{otherwise} \end{cases}$

**Solution:** Here,

$$f(x) = \begin{cases} e^{x} & ,0 < x < a \\ 0 & ,\text{otherwise} \end{cases} \quad \text{.................. (i)}$$

Now, the Fourier transform of (i) is,

$$F(w) = \int_{0}^{a} x\, e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \left[ \frac{xe^{-iwx}}{-iw} - \frac{e^{-iwx}}{(-iw)^2} \right]_{0}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \frac{ae^{-iwa}}{-iw} - \frac{e^{-iwa}}{-w^2} - \frac{1}{w^2} \right] \quad [\because i^2 = -1]$$

$$= \frac{e^{iwa}(iaw + 1) - 1}{w^2 \sqrt{2\pi}}$$

**4.** $f(x) = \begin{cases} xe^{-x} & ,x > 0 \\ 0 & ,x < 0 \end{cases}$

**Solution:** Here,

$$f(x) = \begin{cases} xe^{-x} & ,x > 0 \\ 0 & ,x < 0 \end{cases} \quad \text{.................. (i)}$$

Now, Fourier transform of (i) is,

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x\, e^{-x} e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x\, e^{-(1+iw)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[ x \frac{e^{-(1+iw)x}}{-(1+iw)} - \frac{e^{-(1+iw)x}}{[-(1+iw)]^2} \right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left( \frac{1}{(1+iw)^2} \right)$$

**5.** $f(x) = \begin{cases} -1 & ,-a < x < 0 \\ 1 & ,0 < x < a \\ 0 & \text{Otherwise} \end{cases}$



**Solution:** Here,

$$f(x) = \begin{cases} -1 & ,-a < x < 0 \\ 1 & ,0 < x < a \\ 0 & \text{Otherwise} \end{cases} \quad \text{.................. (i)}$$

Now, Fourier transform of (i) is,

$$F(w) = \frac{1}{\sqrt{2\pi}} \left[ -\int_{-a}^{0} e^{-iwx} dx + \int_{0}^{a} e^{-iwx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[ -\frac{e^{-iwx}}{-iw} \right]_{-a}^{0} + \left[ \frac{e^{-iwx}}{-iw} \right]_{0}^{a} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{iw} - \frac{e^{iwa}}{iw} - \frac{e^{-iwa}}{iw} + \frac{1}{iw} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \frac{2}{iw} - \frac{2}{iw} \cos wa \right]$$

$$= \sqrt{\frac{2}{\pi}} \left( \frac{1}{iw} [-\cos wa + 1] \right)$$

$$= \sqrt{\frac{2}{\pi}} \left( \frac{i}{w} (\cos aw - 1) \right) \quad [\because i^2 = -1]$$

**6.** $f(x) = \begin{cases} e^{-2x} & ,x > 0 \\ 0 & , x = 0, x < 0 \end{cases}$

**Solution:** Here,

$$f(x) = \begin{cases} e^{-2x} & ,x > 0 \\ 0 & , x = 0, x < 0 \end{cases} \quad \text{.................. (i)}$$

Now, Fourier transform of (i) is,

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-2x} e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(2+iw)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \frac{e^{-(2+iw)x}}{-(2+iw)} \right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}\,(2+iw)}$$

**7.** $f(x) = e^{\frac{-x^2}{2}}$ **[2006 Spring Q. No. 6(b) OR]**

**Solution:** Here,

$$f(x) = e^{\frac{-x^2}{2}} \quad \text{.................. (i)}$$

Now, the Fourier transform of (i) is,

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2} + iwx\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\{x^2 + 2iwx + (iw)^2\}} e^{\frac{(iw)^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} \int_{-\infty}^{\infty} e^{-\left(\frac{x+iw}{\sqrt{2}}\right)^2} dx$$

Put, $\left(\frac{x+iw}{\sqrt{2}}\right) = p$ then, $dx = \sqrt{2}\, dp$. Then,

$$F(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} \sqrt{2} \int_{-\infty}^{\infty} e^{-p^2} dp = \frac{1}{\sqrt{\pi}} e^{-\frac{w^2}{2}} \sqrt{\pi} \qquad \left[\because \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\right]$$

$$= e^{-\frac{w^2}{2}}.$$

**Note:** $\int_{-\infty}^{\infty} e^{(-x^2)} dx = 2\int_{0}^{\infty} e^{(-x^2)} dx$ **[Being the given integrand is even]**

$$= 2\int_0^{\infty} e^{-t} \frac{dt}{2\sqrt{t}} \quad \text{[Setting } t = x^2\text{]}$$

$$= \frac{2}{2}\int_0^{\infty} e^{-t} (t)^{-1/2} dt$$

$$= \int_0^{\infty} e^{-t} (t)^{(1/2)-1} dt$$

$$= \Gamma(1/2)$$

$= \sqrt{\pi}$ [Using properties of beta function]

8. $f(x) = \begin{cases} 1 & ,|x| < a \\ 0 & ,|x| > a \end{cases}$

**Solution:** Here,

$$f(x) = \begin{cases} 1 & ,|x| < a \\ 0 & ,|x| > a \end{cases} \quad \text{.................. (i)}$$

Now, Fourier transform of (i) is,



$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-iwx}}{-iw}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-iwa} - e^{iwa}}{-iw}\right] = \sqrt{\frac{2}{\pi}} \left(\frac{\sin wa}{w}\right).$$

9. $f(x) = \begin{cases} x & ,|x| < a \\ 0 & ,|x| > a \end{cases}$

**Solution:** Here,

$$f(x) = \begin{cases} x & ,|x| < a \\ 0 & ,|x| > a \end{cases} \quad \text{.............. (i)}$$

Now, the Fourier transform of (i) is,

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x\, e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[x \frac{e^{-iwx}}{-iw} - \frac{e^{-iwx}}{(-iw)^2}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-iw}(a\, e^{-iwa} + a\, e^{iwa}) - \frac{1}{i^2 w^2}(e^{-iwa} - e^{iwa})\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2ai}{w}\left(\frac{e^{iwa} + e^{-iwa}}{2}\right) - \frac{2i}{w^2}\left(\frac{e^{iwa} - e^{-iwa}}{2i}\right)\right] \qquad [\because i^2 = -1]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2i}{w^2}(aw.\cos wa - \sin wa)\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{i(aw.\cos wa - \sin wa)}{w^2}\right]$$

**10. Find the Fourier transform of $f(x) = \begin{cases} 1 - x^2 & \text{for } |x| < 1 \\ 0 & |x| > 1 \end{cases}$ and then show that**

$$\int_0^{\infty} \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\frac{x}{2}\, dx = \frac{-3\pi}{16}.$$

**Solution:** Let,

$$f(x) = \begin{cases} 1 - x^2 & \text{for } |x| < 1 \Rightarrow -1 < x < 1 \\ 0 & \text{for } |x| > 1 \Rightarrow x < -1,\ x > 1 \end{cases}$$

Then, the Fourier transform of f(x) is,

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2)\, e^{-iwx}\, dx \qquad [\because f(x) = 0 \text{ for } |x| > 1]$$

$$= \frac{1}{\sqrt{2\pi}} \left[ (1 - x^2) \frac{e^{-iwx}}{iw} - (-2x) \frac{e^{-iwx}}{(-iw)^2} + (-2) \frac{e^{-iwx}}{(-iw)^3} \right]_{-1}^{1}$$

[∵ Applying by parts]

$$= \frac{1}{\sqrt{2\pi}} \left[ 0 + \frac{2e^{-iw} + 2e^{iw}}{i^2 w^2} + \frac{2(e^{-iw} - e^{iw})}{i^3 w^3} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \frac{4\cos w}{-w^2} - \frac{4\sin w}{-w^3} \right] \qquad [\because i^2 = -1]$$

$$= \frac{-4}{\sqrt{2\pi}} \left( \frac{w\cos w - \sin w}{w^3} \right) \qquad \text{.............. (i)}$$

This is the required Fourier transform for f(x).

Now, the inverse Fourier transform for f(x) is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F\{f(x)\}\, e^{iwx}\, dw$$

$$\Rightarrow \sqrt{2\pi}\, f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F\{f(x)\}\, e^{iwx}\, dw$$

$$\Rightarrow \int_{-\infty}^{\infty} F\{f(x)\} (\cos wx + i \sin wx)\, dw = \sqrt{2\pi}\, f(x)$$

$$\Rightarrow \frac{-4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{w\cos w - \sin w}{w^3} \right] [\cos wx + i \sin wx]\, dw = \sqrt{2\pi} \begin{cases} 1 - x^2 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

... (ii) [∵ using (i)]

Clearly $\frac{w\cos w - \sin w}{w^3} \sin wx$ is an odd function. Therefore,

$$\int_{-\infty}^{\infty} \frac{w\cos w - \sin w}{w^3} \sin wx\, dx = 0$$

Also, $\frac{w\cos w - \sin w}{w^3} \cos wx$ is an even function. Therefore,

$$\int_{-\infty}^{\infty} \frac{w\cos w - \sin w}{w^3} \cos wx\, dw = 2 \int_{0}^{\infty} \frac{w\cos w - \sin w}{w^3} \cos wx\, dw$$

Thus, (ii) becomes,

$$2 \int_{0}^{\infty} \frac{w\cos w - \sin w}{w^3} \cos wx\, dw = -\frac{2\pi}{4} (1 - x^2) \qquad \text{for } |x| < 1.$$



Set $x = \frac{1}{2}$ (being $- < \frac{1}{2} < 1$) and w = x then,

$$\int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = -\frac{\pi}{4}\left(1 - \frac{1}{4}\right) = \frac{-3\pi}{16}.$$

**11. Find the Fourier transform of $f(x) = e^{-mx}$ m > 0 and then show that $\int_{0}^{\infty} \left(\frac{\cos kx}{1 + x^2}\right) dx = \frac{\pi}{2} e^{-k}$.**

**Solution:** Let, $f(x) = e^{-mx}$ for m > 0.

Then, the Fourier cosine transform of f(x) is,

$$F_C\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos wx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-mx} \cos wx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[ \frac{e^{-mx}}{m^2 + w^2} (-m \cos wx + w \sin wx) \right]_0^{\infty}$$

$$\left[ \because \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b\, \text{xinbx}) + c \right]$$

$$= \sqrt{\frac{2}{\pi}} \left( \frac{m}{m^2 + w^2} \right) \qquad \text{............ (i)}$$

This is the required Fourier cosine transform of f(x).

Next, we wish to show

$$\int_{0}^{\infty} \left( \frac{\cos kx}{1 + x^2} \right) dx = \frac{\pi e^{-k}}{2} \qquad \text{.............. (ii)}$$

Since, we have the inverse Fourier cosine transform of f(x) is,

$$F(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_C\{f(x)\} \cos wx\, dw$$

$$\Rightarrow e^{-mx} = \frac{2}{\pi} \int_{0}^{\infty} \left( \frac{m \cos wx}{m^2 + w^2} \right) dw \qquad [\because \text{using (i)}]$$

Set x = k, m = 1>0 and then, w = x. So that,

$$e^{-k} = \frac{2}{\pi} \int_{0}^{\infty} \left( \frac{\cos kx}{1 + x^2} \right) dx$$

$$\Rightarrow \int_{0}^{\infty} \left( \frac{\cos kx}{1 + x^2} \right) dx = \frac{\pi e^{-k}}{2}.$$

This is required form.

**12. Find the Fourier sine transform of $e^{-x}$ for $x > 0$ and then show that $\int_0^\infty \left(\frac{x \sin mx}{1+x^2}\right) = \frac{\pi}{2} e^{-m}$ for $m > 0$.**

**Solution:** Let, $f(x) = e^{-x}$ for $x > 0$.

Then, the Fourier sine transform of f(x) is

$$F_S\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin wx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+w^2}(-\sin wx - w \cos wx)\right]_0^\infty$$

$$\left[\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + c\right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{w}{1+w^2}\right) \quad \text{.................... (i)}$$

This is required Fourier sine transform of f(x)

Since, the inverse Fourier sine transform of f(x) is,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_S\{f(x)\} \sin wx \, dw$$

$$\Rightarrow e^{-x} = \frac{2}{\pi} \int_0^\infty \left(\frac{w \sin wx}{1+w^2}\right) dw \qquad [\because \text{using (i)}]$$

$$\Rightarrow \int_0^\infty \left(\frac{w \sin wx}{1+w^2}\right) dw = \frac{\pi e^{-x}}{2}$$

Set x = m and then w = x, so that,

$$\int_0^\infty \left(\frac{x \sin mx}{x^2+1}\right) dx = \frac{\pi e^{-x}}{2}.$$

This is required integral value.

**13. Find the Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$ for $x > 0$ and $a > 0$, then show that $\int_0^\infty \tan^{-1}\left(\frac{x}{a}\right) \sin x \, dx = \frac{\pi e^{-a}}{2}$.**

**Solution:** Let $f(x) = \frac{e^{-ax}}{x}$ for $x > 0$, $a > 0$.



Then, the Fourier sine transform of f(x) is,

$$F_S\{f(x)\} = \frac{2}{\pi} \int_0^\infty f(x) \sin wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin wx \, dx \quad \text{.............. (i)}$$

Let, $I = \int_0^\infty \frac{e^{-ax}}{x} \sin wx \, dx$

So, $\frac{dI}{dw} = \int_0^\infty \frac{e^{-ax}}{x} \frac{d}{dw}(\sin wx) \, dx$

$$= \int_0^\infty e^{-ax} \cos wx \, dx$$

$$= \left[\frac{e^{-ax}}{a^2+w^2}(-a \cos wx + w \sin wx)\right]_0^\infty \quad [\because \text{using integral formula}]$$

$$= \frac{a}{a^2+w^2}$$

Taking integration w. r. t. w then,

$$I = \int \frac{a}{a^2+w^2} dw = \tan^{-1}\left(\frac{w}{a}\right)$$

Then, (i) becomes,

$$F_S\{f(x)\} = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{w}{a}\right) \quad \text{................ (ii)}$$

This is required Fourier sine transform for f(x)

And, we have the inverse Fourier sine transform for f(x) is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_S\{f(x)\} \sin wx \, dw$$

$$\Rightarrow \frac{e^{-ax}}{x} = \frac{2}{\pi} \int_0^\infty \tan^{-1}\left(\frac{w}{a}\right) \sin wx \, dw$$

Setting x = 1 and then w = x we get,

$$\Rightarrow \frac{\pi e^{-a}}{2} = \int_0^\infty \tan^{-1}\left(\frac{x}{a}\right) \sin x \, dx$$

This is required integral form.

**14. Find the Fourier sine transform of $e^{-x}$ for x > 0 and then by using Parseval's identity, show that $\int_0^\infty \frac{x^2}{(1+x^2)^2} dx = \frac{\pi}{4}$.**

**Solution:** Let $f(x) = e^{-x}$ ................... (i)

Then, by Q. No. 12,

$$F_S\{f(x)\} = \sqrt{\frac{2}{\pi}}\left(\frac{w}{1+w^2}\right) \quad \text{............ (ii)}$$

By Parseval's identity for Fourier sine transform we have

$$\int_0^\infty |F_S\{f(x)\}|^2 dw = \int_0^\infty |f(x)|^2 dx$$

$$\Rightarrow \int_0^\infty \frac{2}{\pi}\left(\frac{w}{1+w^2}\right)^2 dw = \int_0^\infty |f(x)|^2 dx$$

For all $x \in R$, $e^{-x}$ has positive value. So, $|e^{-x}| = e^{-x}$

$$\Rightarrow \int_0^\infty \frac{w^2}{(1+w^2)^2} dw = \frac{\pi}{2}\left[\frac{e^{-2x}}{-2}\right]_0^\infty = \frac{\pi}{4}$$

Here the right part is free from x. So, set w = x then,

$$\int_0^\infty \frac{x^2}{(1+x^2)^2} dx = \frac{\pi}{4}$$

This is required integral value.

**15. Find the Fourier cosine transform of $e^{-x}$ for x > 0 and then show that $\int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$.**

**Solution:** Let, $f(x) = e^{-x}$ ............. (i)

Then the Fourier cosine transform of f(x) is,

$$F_C\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x) \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x} \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+w^2}(-\cos wx + w \sin wx)\right]_0^\infty$$



$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+w^2}\right) \quad \text{............ (ii)}$$

This is the Fourier cosine transform of f(x).

By Parseval's identity for Fourier cosine transform, we have,

$$\int_0^\infty |F_C\{f(x)\}|^2 dw = \int_0^\infty |f(x)|^2 dx$$

$$\Rightarrow \int_0^\infty \frac{2}{\pi}\left(\frac{1}{1+w^2}\right)^2 dw = \int_0^\infty e^{-2x} dx \quad [\because \text{using (i)}]$$

$$\Rightarrow \int_0^\infty \frac{dw}{(1+w^2)^2} = \frac{\pi}{2}\left[\frac{e^{-2x}}{-2}\right]_0^\infty = \frac{\pi}{4}$$

Here the right part is free from x. So, taking w = x we get,

$$\int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$$

This is the required integral form.

**16. Solve the integral equation $\int_0^\infty f(t) \sin wt \, dt = \begin{cases} 1 & \text{for } 0 \le t < 1 \\ 2 & \text{for } 1 \le t < 2 \\ 0 & \text{for } 2 \ge t \end{cases}$**

**Solution:** Let,

$$\int_0^\infty f(t) \sin wt \, dt = \begin{cases} 1 & \text{for } 0 \le t < 1 \\ 2 & \text{for } 1 \le t < 2 \\ 0 & \text{for } 2 \ge t \end{cases} \quad \text{................ (i)}$$

Since, we have the Fourier sine transform of f(t) is,

$$F_S\{f(t)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t) \sin wt \, dt = \sqrt{\frac{2}{\pi}}\begin{cases} 1 & \text{for } 0 \le t < 1 \\ 2 & \text{for } 1 \le t < 2 \\ 0 & \text{for } 2 \ge t \end{cases} \quad \text{................ (ii)}$$

Now, the inverse Fourier sine transform is,

$$f(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_S\{f(t)\} \sin wt \, dt$$

$$= \frac{2}{\pi}\left[\int_0^1 1 \sin wt \, dt + \int_1^2 2 \sin wt \, dt + \int_2^\infty 0.\sin wt \, dt\right] \quad [\because \text{using (ii)}]$$

$$= \frac{2}{\pi}\left\{\left[\frac{-\cos wt}{w}\right]_0^1 + 2\left[\frac{-\cos wt}{w}\right]_1^2 + 0\right\}$$

$$= \frac{2}{\pi}\left[\frac{1-\cos w}{w} + 2\left(\frac{\cos w - \cos 2w}{w}\right)\right]$$

$$= \frac{2}{\pi w}(1 + \cos w - 2\cos 2w)$$

Set w = x and t = x, which is possible. So,

$$f(x) = \frac{2}{\pi x}(1 + \cos x - 2\cos 2x).$$

This is required value of f(x).

## OTHER IMPORTANT QUESTION FROM FINAL EXAM FOR PRACTICE

**2002 Q. No. 6(b); 2011 Spring Q. No. 4(b)**

Find the Fourier transform of the function $f(x) = e^{-x^2}$

**2003 Fall Q. No. 6(a)**

Define Fourier sine and cosine transforms. If f(x) in continuous piecewise in each finite interval and absolutely integrable on x–axis, $f(x) \to 0$ as $x \to \infty$ then

$$F_c\{f'(x)\} = w F_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0) \qquad F_c\{f'(x)\} = w F_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0)$$

**2003 Fall Q. No. 6(b)**

Find the Fourier integral of f(x) defined by: $f(x) = \begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

**2004 (Spring) Q. No. 6(b)**

Define Fourier transform of the function f(x). Show that it is a linear operation.

Find the Fourier transform of the function $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

**Hint: For problem part: See exercise 6.3 Q. No. 3.**

**2004 Fall Q. No. 6(b)**

Define Fourier sine transform and Fourier transform of the function f(x). Find Fourier cosine transform and Fourier transform of the function:

$f(x) = \begin{cases} k & \text{if } 0 < x < b \\ 0 & \text{otherwise} \end{cases}$

**2005 Spring Q. No. 6(a)**

Show that $\int_0^\infty \frac{w^3 \sin wx}{w^4 + 4}\, dw = \frac{\pi}{2} e^{-x} \cos x$ . if $x > 0$



**2005 Spring Q. No. 6(b)**

Find Fourier transform of $f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

**2005 Fall Q. No.6(a)**

Derive Fourier integral of f(x) from Fourier series. Show that

$$\int_0^\infty \frac{\cos x\pi}{1+w^2}\, dw = \frac{\pi}{2} e^{-x} \text{ for } x > 0.$$

**Hint: Second Part: See Exercise 6.1 Q. No. 1(c).**

**2005 Fall Q. No. 6(b) OR**

Define Fourier transform and evaluate Fourier transform of $f(x) = e^{-x^2/2}$

**2006 Spring Q. No. 3(b)**

Find the Fourier cosine transform of $e^{-x^2}$.

**2006 Spring Q. No. 3(b) OR**

Find the Fourier transform of the function $f(x) = \begin{cases} 1 - x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$

Hint: See part of 6.3 Q. 10.

**2006 Spring Q. No. 6(a)**

Define Fourier sine and cosine integrals. Show that

$$\int_0^\infty \frac{\cos wx + w \sin wx}{1 + w^2}\, dx = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$$

Hint: For problem part, see exercise 6.1 Q. No. 1(a).

**2006 Fall Q. No. 6(a)**

Define Fourier transform of a function. Show that:

$$\int_0^\infty \left(\frac{\omega \sin \omega x}{a^2 + \omega^2}\right) d\omega = \frac{\pi}{2} e^{-ax} \text{ for } \begin{matrix} x > 0 \\ a > 0 \end{matrix}$$

**Hint: Problem Part: See Exercise 6.1 Q. No. 1(e).**

**2006 Fall Q. No. 6(b)**

Find the Fourier transform of $f(x) = xe^{-x^2}$

**2006 Spring Q. No. 3(a)**

Find the Fourier integral of the function $f(x) = \begin{cases} \pi/2 & \text{if } 0 \le x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$

**2007 Spring Q. No. 6(a)**

What is a Fourier Integral? Using Fourier integral, show that

$$\int_0^\infty \frac{\cos x\omega + \omega \sin x\omega}{1+\omega^2} d\omega = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$$

**Hint: Second Part: See Exercise 6.1 Q. No. 1(c).**

**2007 Spring Q. No. 6(b); 2008 Fall Q. No. 4(b); 2009 Fall Q. No. 4(b)**

Find the Fourier cosine transform of the function $f(x) = e^{-ax}$ $(a > 0)$.

**2007 Fall Q. No. 6(b); 2012 Fall Q. No. 2(b)**

Find the Fourier transform of $f(x) = \begin{cases} |x| & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

**2008 Spring Q. No. 6(a)**

Find the Fourier integral of $f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$

**2008 Spring Q. No. 6(b)**

Find the Fourier transform of $f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

**2008 Fall Q. No. 4(a); 2009 Spring Q. No. 6(b); 2009 Fall Q. No. 4(a)**

Show that: $\displaystyle\int_0^\infty \frac{\sin w \cos wx}{w} dw = \begin{cases} \pi/2 & \text{if } 0 \le x < 1 \\ \pi/4 & \text{at } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$

**2009 Spring Q. No. 6(a)**

Find Fourier sine and Cosine transform of $f(x) = \begin{cases} x^2 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

**Solution: See Exercise 6.2 Q. No. 1(d) & 2(b).**

**2011 Spring Q. No. 4(b) OR; 2011 Fall Q. No. 6(c)**

Find Fourier sine and cosine transform of $f(x) = e^{-ax}$ $(a > 0)$

**Hint: See 2007 Spring for Fourier Cosine transform. Process for F. S. T.**

**2011 Fall Q. No. 4(b)**

Define Fourier integral. Find the Fourier cosine and sine integrals of the function $f(x) = e^{-ax}$ for $x > 0$ and $a > 0$.



**2015 Fall Q. No. 4(b) OR**

Find Fourier transform of $f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

**2015 Fall Q. No. 6(b)**

Define Fourier integral. Choosing a suitable function, show that

$$\int_0^\infty \frac{\sin \pi\omega}{\omega} \sin wx \, dw = \begin{cases} \frac{\pi \sin \pi x}{2} & \text{if } 0 \le x \le \pi \\ 0 & \text{if } x > \pi \end{cases}$$

**2015 Fall Q. No. 6(b) OR**

Find the Fourier cosine transform of $e^{-x}$.

**2016 Fall Q. No. 5(a)**

Find the Fourier cosine and sine transform of $f(x) = e^{-ax}$, $a > 0$.

**2016 Fall Q. No. 5(a) OR**

Verify the convolution theorem for the functions $f(x) = e^{-x^2}$ and $g(x) = e^{-x^2}$.

**2016 Spring Q. No. 3(b)**

Find the Fourier transform of the function $f(x) = \begin{cases} 1 - x^2 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

**2017 Fall Q. No. 5(a)**

Starting from Fourier series, obtain the Fourier integral in complex form.

## SHORT QUESTIONS

**2002 Q. No. 7(c); 2004 Fall Q. No. 7(a)**

Show that Fourier sine transforms is a linear operation.

OR

**2008 Fall Q. No. 7(e); 2011 Spring Q. No. 7(b); 2012 Fall Q. No. 7(c)**

Show that $F_s\{af(x) + bg(x)\} = aF_s\{f(x)\} + bF_s\{g(x)\}$ where $F_s$ stands for the Fourier sine transform.

**2003 Fall Q. No. 7(c)**

If f(x) is even, find the Fourier integral.

**2004 Spring Q. No. 7(d)**

Find the Fourier cosine integral of the function $f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

**2007 Spring Q. No. 7(d)**

Find the Fourier sine transform of $e^{-5x}$.

❑❑❑

# Unit 8
# PARTIAL DIFFERENTIAL EQUATIONS

## Some Definitions

**Partial differential equation:**

An equation involving one or more partial derivatives of an (unknown) function of two or more independent variables is called a partial differential equation.

The order of highest derivative is called the order of the equation.

**Linear partial differential equation:**

A partial differential equation is said to be linear if it is of the first degree in the dependent variable and its partial derivatives.

**3. Homogeneous and non–homogeneous equation:**

All terms of an equation contains either the dependent variable or one of its derivatives, then this equation is said to be homogeneous otherwise it is said to be non–homogeneous.

## Important Linear Partial Differential Equation of the Second Order

1. One dimensional wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$
2. One dimensional heat equation: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$
3. Two–dimensional Laplace equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
4. Two dimensional Poisson equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$
5. Two dimensional heat equation: $\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$
6. Two dimensional wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$
7. Three dimensional Laplace equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

## Derivation of One Dimensional Wave Equation

Consider a tightly stretched elastic string of length $l$ and fix it at two ends O and B. Suppose that:

- The string is uniform and there is no resistance to bending;
- The tension caused by stretched the string before fixing, is so large that the action of gravitational force on the string can be neglected.
- The string performs the transverse motion in only vertical plane.

Under these assumptions, the wave produced by a vibrating string is as:

Let us suppose that the force acting on small portion of the string. Since the string does not offer resistance to bending, the tension is tangential to the curve of the string at each point.

Let $T_1$ and $T_2$ are tensions at the ends points P and Q of that portion. Since the points of the string move vertically, there is no motion in the horizontal direction. Therefore the horizontal components of the tension must be constant.

Therefore from figure, we get

$$T_1 \cos\alpha = T_2 \cos\beta = T \text{ (say)} = \text{constant} \quad \text{......(1)}$$

In the vertical direction, forces in vertical components are $-T_1 \sin\alpha$ and $T_2 \sin\beta$ of $T_1$ and $T_2$. Here negative sign appears because the component at P is directed downward.

Then, by Newton's second law, we get,

Resultant force is equals to mass of the portion times the acceleration.

That is,

$$T_2\sin\beta - T_1\sin\alpha = \rho\Delta x \frac{\partial^2 u}{\partial t^2} \quad \text{......(2)}$$

where ρ be the mass of undetected string per unit length, Δx is the length of the portion of the unperfected string.

Dividing equation (2) by equation (1) we get,

$$\frac{T_2\sin\beta}{T_2\cos\beta} - \frac{T_1\sin\alpha}{T_1\cos\alpha} = \frac{\rho\Delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \tan\beta - \tan\alpha = \frac{\rho\Delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2} \quad \text{.........(3)}$$

We know, $\tan\alpha$ and $\tan\beta$ are the slopes of the string at x and x + Δx respectively.

Where,

$$\tan\alpha = \left.\frac{\partial u}{\partial x}\right|_{\text{at } x} \quad \text{and,} \quad \tan\beta = \left.\frac{\partial u}{\partial x}\right|_{\text{at } x+\Delta x}$$

From equation (3), we get

$$\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right] = \frac{P\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{1}{\Delta x}\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right] = \frac{\rho}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

Taking limit Δx tends to zero on both sides, we get,

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$



$$\Rightarrow \frac{T}{\partial}\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

Thus, $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, where $c^2 = \frac{T}{\rho}$

This is the required one dimensional wave equation.

## Solution of Dimensional Wave equation under certain initial and Boundary conditions

We have one dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{........(1)}$$

where u(x, t) is the deflection of the string, with boundary condition

$$u(0, t) = 0 \text{ and } u(L, t) = 0 \text{ for all } t \quad \text{.......(2)}$$

And initial condition

$$u(x, 0) = f(x) \text{ (initial deflection)} \quad \text{.......(3)}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \text{ (initial velocity)} \quad \text{...... (4)}$$

Let, $u(x, t) = F(x)\, G(t)$ .........(5)

be the solution of (1). Then by differentiating we get

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = F''G$$

where dot denotes derivative with respect to t and prime denotes derivative with respect to x.

Putting these values in equation (1) we get

$$F\ddot{G} = c^2 F'' G$$

$$\Rightarrow \frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k \text{ (say)}$$

This gives

$$F'' - kF = 0 \quad \text{........(6)}$$

and

$$\ddot{G} - c^2 kG = 0 \quad \text{.......(7)}$$

which are ordinary differential equation.

We have to find F and G from equation (6) and (7) under the boundary condition (2).

$$0 = u(0, t) = F(0)\, G(t)$$

and

$$0 = u(L, t) = F(L)\, G(t) \quad \text{for all } t.$$

Solving equation (6),

If G = 0, then u = 0, which is of no interest. So G ≠ 0 and we get

$$F(0) = 0 = F(L) \quad \text{........(8)}$$

**Case – I**: Suppose k = 0 then the general solution of equation (6) is

$$F = ax + b$$

and from (8), we get a = 0 = b. Then F = 0, which is of no interest because u = 0.

**Case II:** Suppose that k > 0. Let $k = \lambda^2$. Then equation (6) can be written as

$$F'' - P^2F = 0$$

Its general solution is

$$F = Ae^{\lambda x} + Be^{-\lambda x}$$

and from equation (8), we get F = 0 which is of no interest. Thus we left with all possibilities except k is negative.

**Case III:** Suppose that k < 0. Let $k = -p^2$. Then equation (6) can be written as

$$F'' + P^2F = 0$$

Its general solution is

$$F(x) = A \cos px + B \sin px$$

From equation (8), we get

$$F(0) = A = 0$$

$$F(L) = B \sin pL = 0$$

Let B ≠ 0, since otherwise F = 0. So, we get sinp L = 0

$$\Rightarrow \quad PL = n\pi \quad \Rightarrow \quad P = \frac{n\pi}{L} \qquad \text{for n is an integer.}$$

We have B ≠ 0, setting B = 1, we get

$$F(x) = \sin \frac{n\pi}{L} x$$

We obtain infinitely many solutions

$F(x) = F_n(x)$, thus

$$F_n(x) = \sin \frac{n\pi}{L} x \text{ for } n = 1, 2, 3, \ldots \ldots \ldots (9)$$

Solving equation (7).

We have $k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$. Thus equation (7) is

$$\ddot{G} - c^2k\, G = 0$$

$$\Rightarrow \quad \ddot{G} + \lambda_n^2\, G = 0 \qquad \text{where } \lambda_n = \frac{cn\pi}{L}$$

Its solution is

$$G_n(t) = B_n \cos\lambda_n t + B^*n \sin \lambda_n t$$

Therefore required solution of equation (1) is

$$u_n(x, t) = (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin \frac{n\pi}{L} x$$



These functions are called the eigen functions or characteristics functions and the values $\lambda_n = \frac{cn\pi}{L}$ are called eigen values or characteristics values of the vibrating string. The set $\{\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\}$ is called the spectrum.

Again, a single solution $u_n(x, t)$ will not a general solution of given equation. Then by fundamental theorem the solution of given wave is the sum of finitely many $u_n$.

Thus,

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t)$$

$$\Rightarrow \quad u(x, t) = \sum_{n=1}^{\infty} (\cos\lambda_n t + B_n \sin \lambda_n t) \sin \frac{n\pi}{L} x \qquad \ldots\ldots(10)$$

From equation (3) and (4)

$$u(x, 0) = f(x) \qquad \text{and} \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x).$$

From equation (10) we get,

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x)$$

$$\Rightarrow \quad f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$$

which is Fourier sine series, the coefficient

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x\, dx$$

Also, we have $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$

$$\Rightarrow \quad \left[\sum_{n=1}^{\infty} (-B_n\lambda_n \sin\lambda_n t + B^*_n\lambda_n \cos\lambda_n t) \sin \frac{n\pi}{L} x\right]_{t=0} = g(x)$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} B^*_n\lambda_n \sin \frac{n\pi}{L} x = g(x)$$

Which is the Fourier sine series of g(x) with period 2L. Then the coefficient

$$B^*_n \lambda_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x\, dx$$

Therefore, we get required solution of one dimensional wave equation is

$$u = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B^*_n \sin \lambda_n t) \sin \frac{n\pi}{L} x$$

where $\lambda_n = \frac{cn\pi}{L}$.

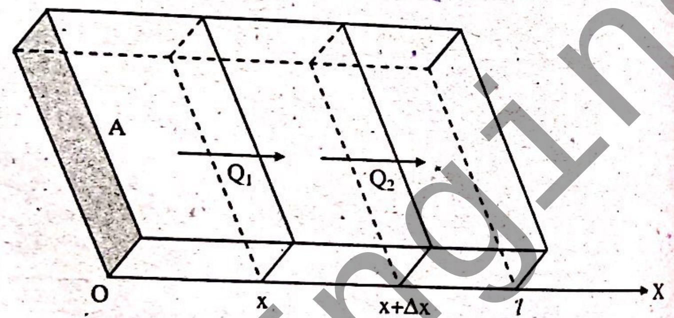
$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \quad \text{and} \quad B^*_n = \frac{2}{L\lambda_n} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

where f(x) and g(x) are initial deflection of string and initial velocity of the string respectively.

## Derivation of One Dimensional Heat Equation

Let us suppose that the flow of heat by conduction in a uniform bar. Suppose the sides of the bar are insulated and the loss of heat from the sides by conduction or radiation is negligible. Take one end of the bar as origin and the direction of flow as the positive x–axis.

The temperature u at any point of the bar depends on the distance x of the point from one end and the time t. Also, the temperature of all points of any cross-section is the same.




The amount of heat crossing any section of the bar per second depends on the area A of the cross section, the conductivity K of the material of the bar and the temperature gradient $\frac{\partial u}{\partial x}$.

Therefore $Q_1$, the quantity of heat flowing into the section at the distance x.

$$= -KA\left[\frac{\partial u}{\partial x}\right]_x \text{ per sec.}$$

(Here negative sign shows the sign on the right is attached because as x increase, u decreases)

Again $Q_2$, the quantity of heat flowing out of the section at the distance $x + \Delta x$



$$= -KA\left[\frac{\partial u}{\partial x}\right]_{x+\Delta x} \text{ per sec.}$$

Hence, the total amount of heat retained by the slab with thickness $\Delta x$ is

$$Q_1 - Q_2 = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \text{ per sec.} \quad \text{.......(1)}$$

But we have the rate of change of heat in the

$$\text{slab} = S\rho A \, \Delta x \frac{\partial u}{\partial t} \quad \text{............(2)}$$

where S is the specific heat and ρ be the density of material.

From equation (1) and (2) we get

$$S\rho A \, \Delta x \frac{\partial u}{\partial t} = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$\Rightarrow S\rho \frac{\partial u}{\partial t} = K\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right]$$

Taking limit as $\Delta x \to 0$ on both side, we get

$$S\rho \frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{\partial u}{\partial t} = \frac{K}{S\rho}\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $c^2 = \frac{K}{s\rho}$ is said to be diffusivity of the material.

Therefore the required one dimensional heat equation of the bar is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \text{ where } c^2 = \frac{K}{S\rho}.$$

**Solution of one dimensional heat equation under certain conditions**

Consider one dimensional heat equation,

$$\frac{\partial u}{dt} = c^2 \frac{\partial^2 u}{dx^2} \quad \text{............. (1)}$$

Under the boundary conditions $u(0, t) = 0 = u(l, t), \forall t$ ........... (2)

and initial conditions $u(x, 0) = f(x)$ ........... (3)

Suppose,

$$u(x, t) = X(x)\,T(t) \quad \text{........... (4)}$$

be the solution of (i) then by (i) we get

$$X\dot{T} = c^3 X'' T$$

where dot ( ˙ ) denotes the differentiation w. r. t. t and prime (') denotes the derivative w. r. t. x, then,

$$\frac{\dot{T}}{c^2 T} = \frac{X''}{X} = k \text{ (say)}$$

This gives us,

$$\dot{T} - kc^2T = 0 \quad \text{and} \quad X'' - kX = 0 \qquad \text{....... (5)}$$

Since k is a constant that may positive, negative or equal to zero.

**Case – I:** Suppose that k >0. So, let $k = p^2$. Then (5) gives,

$$\dot{T} - p^2c^2T = 0 \text{ and } X'' - p^2X = 0$$

Whose solution be,

$$T = c_1e^{pct} + c_2e^{-pct} \text{ and } X = c_3e^{px} + c_4 e^{-px}$$

Then, (4) becomes

$$u(x, t) = (c_1e^{pct} + c_2e^{-pct})(c_3e^{px} + c_4e^{-px}) \qquad \text{..........(6)}$$

By, (2), u(0, t) = 0 then (6) give,

$$0 = c_3 + c_4$$

as $c_1 e^{cpt} + c_2e^{-cpt} \neq 0$ ..................(*)

Also, by (2), u(*l*, t) = 0 then (6) gives

$$0 = c_3e^{lp} + c_4 e^{-pl}$$

as $c_1e^{cpt} + c_2e^{-cpt} \neq 0$ ................ (**)

Solving (*) and (**) we get, $c_3 = 0$, $c_4 = 0$.

With these values (6) gives us u(x, t) = 0, which is impossible.

Therefore, (6) is not a possible solution of (1).

**Case – II:** Suppose k = 0 then (5) becomes,

$$\dot{T} = 0 \text{ and } X'' = 0$$

Whose solution be,

$$T = c_5 \quad \text{and} \quad X = c_6x + c_7$$

Then (4) becomes

$$u(x, t) = c_5 (c_6 x + c_7) \qquad \text{.............. (7)}$$

By (2), u(0, t) = 0 then (7) gives;

$$0 = c_5 - c_7$$

$$\Rightarrow c_7 = 0 \text{ as } c_5 \neq 0, l \neq 0$$

Thus, $c_6 = 0$, $c_7 = 0$.

This forces to (7) as u(x, t) = 0, which is impossible

Thus, (7) is not a possible solution of (1).

**Case III** Suppose that k < 0. Let $k = -p^2$. So, that (5) becomes,

$$\dot{T} + p^2c^2T = 0 \text{ and } x'' + p^2 x = 0$$

Whose solution be,

$$T = c_8 e^{-p^2c^2t} \quad \text{and} \quad x = c_9 \cos px + c_{10} \sin px$$

Then, (4) becomes,

$$u(x, t) = c_8 e^{-p^2c^2t} (c_9 \cos px + c_{10} \sin px) \qquad \text{............ (8)}$$

By (2) u(0,t) = $c_9$ = 0 as $c_8 e^{-p^2c^2t} \neq 0$

Also, by (2), u(*l*, t) = 0 then, (8) gives us,

$$0 = c_{10} \sin pt \quad \text{as } c_8e^{-p^2c^2t} \neq 0$$



$$\Rightarrow \sin pl = 0 \quad \text{as } c_{10} \neq 0$$

$$= \sin n\pi$$

$$\Rightarrow p = \frac{n\pi}{l}$$

Since, the function holds on whole uniform rod. So,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) e^{-\lambda_n^2 t} \qquad \text{........... (9)}$$

For $\lambda_n = p^2c^2$ and $A_n = c_8c_{10}$

Also, by (3) u(x, 0) = f(x) then, (9) gives us,

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

this is half range Fourier sine series. So,

$$A_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \qquad \text{................ (10)}$$

Thus, (9) is the required solution of (1) with value of coefficient that is given in (10).

## Exercise 8.1

1. **Verify the given function to satisfy one dimension wave equation.**

(a) $u = \sin 9t \sin \frac{x}{4}$

**Solution:** Here, $u = \sin 9t \sin \frac{x}{4}$

We have, one dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 4}{\partial x^2} \qquad \text{... ... ... (i)}$$

Since, $u = \sin t \sin \frac{x}{4}$

So, $\frac{\partial u}{\partial t} = 9 \cos 9t \sin \frac{x}{4}$ $\qquad$ $\frac{\partial u}{\partial x} = \frac{1}{4} \sin 9t \cos \frac{x}{4}$

And, $\frac{\partial^2 u}{\partial t^2} = -81 \sin 9t \sin \frac{x}{4}$ $\qquad$ $\frac{\partial^2 u}{\partial x^2} = -\frac{1}{16} \sin 9t \sin \frac{x}{4}$

From (i)

$$-81 \sin 9t \sin \frac{x}{4} = c^2 \times -\frac{1}{16} \sin 9t \sin \frac{x}{4}$$

$$\Rightarrow c = \pm 36$$

Thus, u is satisfy (i) with c = ± 36.

(b) $u = \cos 4t \sin 2x$

**Solution:** Here, u = cos 4t sin 2x.

We have, one dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \ldots (i)$$

Since, $u = \cos 4t \sin 2x$

Then, $\frac{\partial u}{\partial t} = -4\sin 4t \sin 2x$ and $\frac{\partial^2 u}{\partial t^2} = -16\cos 4t \sin 2x$

Also, $\frac{\partial u}{\partial x} = 2\cos 4t \cos 2x$ and $\frac{\partial^2 u}{\partial t^2} = -4\cos 4t \sin 2x$

From equation (i),

$$-16 \cos 4t \text{ sain } 2x = c^2\{-4 \cos 4t \sin 2x\}$$

$$\Rightarrow 16 = 4c^2$$

$$\Rightarrow c = \pm 2$$

Thus, u is satisfy (i) with $c = \pm 2$.

**(c) $u = \sin ct \sin x$**

**Solution:** Here, $u = \sin ct \sin x$.

We have, one dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \ldots (i)$$

Here, $u = \sin ct \sin x$

Then, $\frac{\partial u}{\partial t} = c \cos ct \sin x$ and $\frac{\partial^2 u}{\partial t^2} = -c^2 \sin ct \sin x$

Also, $\frac{\partial u}{\partial x} = \sin ct \cos x$ and $\frac{\partial^2 u}{\partial x^2} = -\sin ct \sin x$

From equation (i),

$$-c^2 \sin ct \sin x = c^2\{-\sin ct \sin x\}$$

$\Rightarrow -c^2 = -c^2$ which is true.

Thus, u is satisfy (i) with for any c.

**2. Verify the given function to satisfy one dimensional heat equation.**

**Solution**

**(a) $u = e^{-t} \sin x$**

**Solution:** Here, $u = e^{-t} \sin x$.

We have, one dimensional wave equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \ldots \ldots (i)$$

Here, $u = e^{-t} \sin x$

Then, $\frac{\partial u}{\partial t} = -e^{-t} \sin x$ and $\frac{\partial u}{\partial x} = e^{-t} \cos x$

Also, $\frac{\partial^2 u}{\partial x^2} = -e^{-t} \sin x$



Now, from equation (i),

$$-e^{-t} \sin x = c^2\{-e^{-t} \sin x\}$$

$$\Rightarrow c = \pm 1$$

Thus, u is satisfy (i) with $c = \pm 1$.

**(b) $u = e^{-4t} \cos 3x$**

**Solution:** Here, $u = e^{-4t} \cos 3x$.

We have, one dimensional wave equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \ldots (i)$$

Here, $u = e^{-4t} \cos 3x$

Then, $\frac{\partial u}{\partial t} = -4e^{-4t} \cos 3x$

Also, $\frac{\partial u}{\partial x} = -3e^{-4t} \sin 3x$ and $\frac{\partial^2 u}{\partial x^2} = -9e^{-4t} \cos 3x$

From equation (i)

$$-3e^{-4t} \cos 3x = c^2(-9e^{-4t} \cos 3x)$$

$$\Rightarrow c = \pm \frac{1}{3}$$

Thus, u is satisfy (i) with $c = \pm \frac{1}{3}$.

**(c) $u = e^{-9t} \cos \omega x$**

**Solution:** Here, $u = e^{-9t} \cos \omega x$.

We have, one dimensional wave equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \ldots (i)$$

We have, $u = e^{-9t} \cos \omega x$

Then, $\frac{\partial u}{\partial t} = -9e^{-9t} \cos \omega x$

Also, $\frac{\partial u}{\partial x} = -\omega e^{-9t} \sin \omega t$ and $\frac{\partial^2 u}{\partial x^2} = -\omega^2 e^{-9t} \cos \omega x$

Now, from (i),

$$-e^{-it} \cos \omega x = c^2(-\omega^2 e^{-9t} \cos \omega x)$$

$$\Rightarrow c = \pm \frac{1}{\omega},$$

Thus, u is satisfy (i) with $c = \pm \frac{1}{\omega}$.

**3. Verify the given function to satisfy two dimensional Laplace equation.**

(a) $u = 2xy$ (b) $u = e^x \sin y$ (c) $u = \tan^{-1}\left(\frac{y}{x}\right)$

**Solution:**

(a) Here, $u = 2xy$.

We have, two dimensional Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots \text{(i)}$$

We have, $u = 2xy$

Then, $\frac{\partial u}{\partial x} = 2y$ and $\frac{\partial u}{\partial y} = 2x$.

Also, $\frac{\partial^2 u}{\partial x^2} = 0$ and $\frac{\partial^2 u}{\partial y^2} = 0$

Now, from (i),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Thus, u satisfies the Laplace equation.

(b) Here, $u = e^x \sin y$.

We have, two dimensional Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots \text{(i)}$$

We have, $u = e^x \sin y$

Then, $\frac{\partial u}{\partial x} = e^x \sin y$ and $\frac{\partial u}{\partial y} = e^x \cos y$.

Also, $\frac{\partial^2 u}{\partial x^2} = e^x \sin y$ and $\frac{\partial^2 u}{\partial y^2} = -e^x \sin y$

Now, from (i),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \sin y - e^x \sin y = 0.$$

Thus, u satisfies the Laplace equation.

(c) Here, $u = \tan^{-1}\left(\frac{y}{x}\right)$.

We have, one dimensional wave equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots \text{(i)}$$

We have, $u = \tan^{-1}\left(\frac{y}{x}\right)$

Then, $\frac{\partial u}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \left(\frac{-y}{x^2}\right) = \frac{-y}{x^2 + y^2}$ and $\frac{\partial^2 u}{\partial x^2} = \frac{+2xy}{(x^2 + y^2)^2}$

Also, $\frac{\partial u}{\partial y} = \frac{x}{x^2 + y^2}$, and $\frac{\partial^2 u}{\partial y^2} = -\frac{2xy}{(x^2 + y^2)^2}$

Now, from (i),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{+2xy}{(x^2 + y^2)^2} - \frac{2xy}{(x^2 + y^2)^2} = 0.$$

$\Rightarrow \quad 0 = 0$

Thus, u satisfies the Laplace equation.



4. Solve the following partial differential equations:

(a) $u_y = u$ (b) $u_{yy} = 0$ (c) $u_{xy} = u_x$ (d) $u_y = 2xy_u$ (e) $u_{xx} - u = 0$

**Solution:**

(a) Here, $u_y = u \Rightarrow \frac{\partial u}{\partial y} = u$

$\Rightarrow \frac{\partial u}{u} = \partial y$

Integrating we get, $\log(u) = y + \log(c)$, where c is function of x or constant.

$u = e^{y + \log(c)} = c(x)\, e^y$

(b) Here, $u_{yy} = 0 \Rightarrow \frac{\partial^2 y}{\partial y^2} = 0$

Integrating w. r. t. y, we get,

$\frac{\partial u}{\partial y} = C(x)$

Again, integrating w. r. t. y, we get,

$u = C(x)\, y + D(x)$.

(c) Here, $u_{xy} = u_x$

Put $u_x = v$ then, $v_y = v \Rightarrow \frac{\partial v}{\partial y} = v$

$\Rightarrow \frac{\partial v}{v} = \partial y$

Integrating we get, $\log(v) = y + \log(C)$, C is function of x on constant.

$\Rightarrow v = C(x)\, e^y$

$\Rightarrow u_x = C(x)\, e^y$

$\Rightarrow du = C(x)\, e^y\, dx$

Integrating, $u = C(x)\, e^y + D(y)$

(d) Here, $u_y = 2xy_u \Rightarrow \frac{\partial u}{\partial y} = 2xyu$

$\Rightarrow \frac{\partial u}{u} = 2xy\, dy$

Integrating,

$\log(u) = xy^2 + \log c(x)$

$\Rightarrow u = c(x) \exp(xy^2)$ where c(x) is function of x or constant.

(e) Here, $u_{xx} - u = 0$

Clearly the auxiliary equation is,

$m^2 - 1 = 0$

$\Rightarrow m = \pm$

… and B are function of y or constant.

## Exercise 8.2

The solution of one–dimensional wave equation is

$$u = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L}x$$

where,

$$\lambda_n = \frac{cn\pi}{L},\ B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

$$B_n^* = \frac{2}{L\lambda_n}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

**1. Find u(x, t) of the string of length $L = \pi$ when $c^2 = 1$, the initial velocity is zero and the initial deflection is**

**(a)** $k\left(\sin x - \frac{1}{2}\sin 2x\right)$

**Solution:** Here, $L = \pi$, $c^2 = 1$, initial velocity $g(x) = 0$

and, initial deflection $f(x) = k\left(\sin x - \frac{1}{2}\sin 2x\right)$.

Also, $\lambda_n = \frac{nc\pi}{L} = \pm n$.

Since we have the solution of the one dimensional wave equation is

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t)\sin\frac{n\pi}{L}x \quad \ldots\ldots\ldots (i)$$

where, $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$, and $B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}\right)xdx$

Since, $g(x) = 0$. So,

$$B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}\right)xdx = 0$$

Then (i) becomes,

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos nt \sin nx) \quad \ldots (ii)$$

So, $$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin nx$$

Since given that $u(x, 0) = f(x) = k\sin x - \frac{k}{2}\sin 2x$



Therefore,

$$k\sin x - \frac{k}{2}\sin 2x = B_1 \sin x + B_2 \sin 2x + B_3 \sin 3x + \ldots\ldots\ldots$$

$$\Rightarrow B_1 = k,\ B_2 = -\frac{k}{2},\ B_3 = 0 = B_4 = B_5 = \ldots\ldots\ldots$$

Therefore the required solution is given condition is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \cos nt \sin nx$$

$$= B_1 \cos t \sin x + B_2 \cos 2t \sin 2x$$

$$\Rightarrow u(x, t) = k\cos t \sin x - \frac{k}{2}\cos 2t \sin 2x$$

**(b)** $0.1\,x(\pi^2 - x^2)$

**Solution:** Here, $L = \pi$, $c^2 = 1$, initial velocity $g(x) = 0$

and, initial deflection $f(x) = 0.1\,x(\pi^2 - x^2)$.

Since we have the solution of the one dimensional wave equation is

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t)\sin\frac{n\pi}{L}x \quad \ldots\ldots\ldots (i)$$

where, $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$ and $B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}\right)xdx$

Since, $g(x) = 0$. So,

$$B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}\right)xdx = 0$$

And,

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \quad \left[\text{Since } L = \pi,\ c^2 = 1,\ \lambda_n = \frac{cn\pi}{L} = \pm n\right]$$

$$= \frac{2}{\pi}\int_0^{\pi} 0.1\,(\pi^2 x - x^3)\sin nx\,dx$$

$$= \frac{0.2}{\pi}\left[(\pi^2 x - x^3)\left(-\frac{\cos nx}{n}\right) - (\pi^2 - 3x^2)\left(\frac{\sin nx}{-n^2}\right) + (-6x)\left(\frac{\cos nx}{x^3}\right) - (-0.6)\frac{\sin nx}{n^4}\right]_0^{\pi}$$

$$= \frac{0.2}{\pi}\left[-6\pi \times \frac{(-1)^n}{n^3}\right]$$

$$= \frac{1.2}{n^3} \times (-1)^{n+1}$$

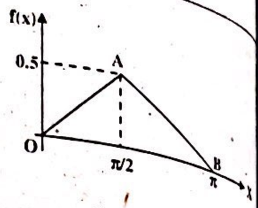
Therefore (i) becomes

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos nt \sin nx)$$

$$\Rightarrow u(x, t) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1.2}{n^3}\cos nt \sin nx.$$

(c)

**Solution:** Now, for initial deflection equation of




OA is $f(x) = \frac{x}{\pi}$ for $0 < x < \pi/2$

And the equation of AB is

$f(x) = 1 - \frac{x}{\pi}$ for $\pi/2 < x < \pi$.

Therefore,
$$f(x) = \begin{cases} \frac{x}{\pi} & \text{for } 0 < x < \frac{\pi}{2} \\ 1 - \frac{x}{\pi} & \text{for } \frac{\pi}{2} < x < \pi \end{cases} \quad \dots (1)$$

Given that, $L = \pi$, $c^2 = 1$, initial velocity $g(x) = 0$ and, initial deflection f(x) is given in (1).

Since, $\lambda_n = \frac{nc\pi}{L} = \frac{n(\pm 1)\pi}{\pi} = \pm n$

Since we have the solution of the one dimensional wave equation is

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin \frac{n\pi}{L} x \quad \dots\dots\dots (i)$$

where, $B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x\, dx$ and $B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}\right) x dx$

Since, $g(x) = 0$. So,

$$B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}\right) x dx = 0$$

And,

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$$
$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{x}{\pi} \sin nx\, dx + \int_{\pi/2}^{\pi} \left(1 - \frac{x}{\pi}\right) \sin nx\, dx\right]$$
$$= \frac{2}{\pi}\left[\left|\frac{x}{\pi}\left(\frac{\cos nx}{-n}\right) - \frac{1}{\pi}\left(\frac{\sin nx}{-n^2}\right)\right|_0^{\pi/2} + \left|\left(1 - \frac{x}{\pi}\right)\left(\frac{\cos nx}{-n}\right) + \frac{1}{\pi}\left(\frac{\sin nx}{-n^2}\right)\right|_{\pi/2}^{\pi}\right]$$
$$= \frac{2}{\pi}\left[-\frac{1}{2n}\left(\cos \frac{n\pi}{2}\right) + \frac{1}{\pi n^2}\left(\sin \frac{n\pi}{2}\right) + \frac{1}{2n}\left(\cos \frac{n\pi}{2}\right) + \frac{1}{\pi n^2}\left(\sin \frac{n\pi}{2}\right)\right]$$
$$= \frac{4}{(n\pi)^2}\left[\sin \frac{n\pi}{2}\right]$$

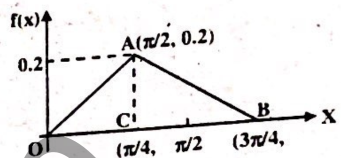
Thus, the equation (i) becomes,

$$u(x, t) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2} \sin \frac{n\pi}{2} \cos nt \sin nx.$$



(d)

**Solution:** From figure,




Since the equation of OA is

$$y - 0 = \frac{0.2 - 0}{\frac{\pi}{2} - \frac{\pi}{4}}\left(x - \frac{\pi}{4}\right)$$
$$\Rightarrow y = \frac{0.2}{\frac{\pi}{4}}\left(x - \frac{\pi}{4}\right)$$
$$\Rightarrow y = \frac{0.8}{\pi}\left(x - \frac{\pi}{4}\right) = \frac{0.8x}{\pi} - 0.2$$

So, $f(x) = \frac{0.8x}{\pi} - 0.2$ for $\frac{\pi}{4} < x < \frac{\pi}{2}$

And, the equation of AB is

$$y - 0.2 = \frac{0 - 0.2}{\frac{3\pi}{4} - \frac{\pi}{2}}\left(x - \frac{\pi}{2}\right)$$
$$y - 0.2 = \frac{-0.2}{\pi/4}\left(x - \frac{\pi}{2}\right) \Rightarrow y = -\frac{0.8}{\pi}x + 0.4 + 0.2$$
$$\Rightarrow y = -\frac{0.8}{\pi}x + 0.6 \quad \text{for } \frac{\pi}{2} < x < \frac{3\pi}{4}$$

Thus,

$$f(x) = \begin{cases} \frac{(0.8)x}{\pi} - (0.2) & \text{for } \frac{\pi}{4} < x < \frac{\pi}{2} \\ \frac{(-0.8)x}{\pi} + (0.6) & \text{for } \frac{\pi}{2} < x < \frac{3\pi}{4} \end{cases} \quad \dots (1)$$

Given that, $L = \pi$, $c^2 = 1$, initial velocity $g(x) = 0$ and the initial deflection f(x) is given in (1).

Since, $\lambda_n = \frac{nc\pi}{L} = \frac{n(\pm 1)\pi}{\pi} = \pm n$

Since we have the solution of the one dimensional wave equation is

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\left(\frac{n\pi}{L}\right) x \quad \dots\dots\dots (i)$$

where, $B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}\right) x\, dx$ and $B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}\right) x dx$

Since, $g(x) = 0$. So,

$$B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}\right) x dx = 0$$

And,

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$$

$$= \frac{2}{\pi}\left[\int_{\pi/4}^{\pi/2}\left(\frac{0.8x}{\pi} - 0.2\right)\sin nx\,dx + \int_{\pi/2}^{3\pi/4}\left(-\frac{0.8}{\pi} + 0.6\right)\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left|\left(\frac{0.8x}{\pi} - 0.2\right)\left(\frac{\cos nx}{-n}\right) - \frac{0.8}{\pi}\left(\frac{\sin nx}{-n^2}\right)\right|_{\pi/4}^{\pi/2} + \left|\left(-\frac{0.8x}{\pi} + 0.6\right)\left(\frac{\cos nx}{-n}\right) + \frac{0.8}{\pi}\left(\frac{\sin nx}{-n^2}\right)\right|_{\pi/2}^{3\pi/4}\right]$$

$$= \frac{2}{\pi}\left[\left|\frac{0.8\times\pi}{2\pi}\left(\frac{\cos\frac{n\pi}{2}}{-n}\right) + \frac{0.8}{\pi}\left(\frac{\sin\frac{n\pi}{2}}{n^2}\right) - \frac{0.8}{\pi n^2}\left(\sin\frac{n\pi}{4}\right)\right| + \left(\frac{-0.8}{\pi}\times\frac{\pi}{2} + 0.6\right)\left(\frac{\cos\left(\frac{n\pi}{2}\right)}{n}\right) - \frac{0.8}{\pi n^2}\left(\sin\left(\frac{3n\pi}{4}\right) - \left(\sin\left(\frac{n\pi}{2}\right)\right)\right)\right]$$

$$= \frac{1.6}{\pi^2 n^2}\sin\frac{n\pi}{2} - \frac{1.6}{\pi^2 n^2}\sin\frac{n\pi}{4} - \frac{1.6}{\pi^2 n^2}\sin\frac{3n\pi}{4}$$

$$\therefore\ B_1 = \frac{3.2}{\pi^2} - \frac{1.6}{\pi^2}\cdot\frac{1}{\sqrt{2}} - \frac{1.6}{\pi^2}\frac{1}{\sqrt{2}} = \frac{0.6}{\pi^2}(2 - \sqrt{2})$$

$$B_2 = 0 - \frac{1.6}{4\pi^2} + \frac{1.6}{4\pi^2} = 0$$

$$B_3 = -\frac{3.2}{9\pi^2} - \frac{1.6}{9\pi^2}\cdot\frac{1}{\sqrt{2}} - \frac{1.6}{9\pi^2}\frac{1}{\sqrt{2}} = -\frac{3.2}{9\pi^2} - \frac{3.2}{9\pi}\frac{1}{\sqrt{2}} = -\frac{1.6}{9\pi^2}(2 + \sqrt{2}).$$

Thus, the equation (i) becomes,

$$u(x, t) = \frac{1.6}{\pi^2}\left[2 - \sqrt{2}\cos t\sin x - \frac{1}{9}(2 + \sqrt{2})\cos 3t\sin 3x + \ldots\right]$$

**2. Find the deflection u(x, t) of a vibrating string of length π and $c^2 = 4$ for zero initial velocity and initial deflection sin5x.**

**Solution:** Given that, Initial deflection $f(x) = \sin 5x$

Initial velocity $g(x) = 0$ and Length $(L) = \pi$

Also, $c^2 = 4 \Rightarrow c = \pm 2$.

Then, $\lambda_n = \frac{cn\pi}{L} = \pm 2n$.

Since we have the solution of the one dimensional wave equation is

$$u(x, t) = \sum_{n=1}^{\infty}(B_n\cos\lambda_n t + B_n^*\sin\lambda_n t)\sin\frac{n\pi}{L}x \quad \ldots\ldots\ldots (i)$$

where, $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$ and $B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$

Since, $g(x) = 0$. So,

$$B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}\right)x\,dx = 0$$

Then (i) becomes,



$$u(x, t) = \sum_{n=1}^{\infty} B_n\cos 2nt\sin nx \quad \ldots\ldots\ldots (ii)$$

Since, we have,

$$f(x) = u(x, 0) = \sin 5x \quad \ldots\ldots\ldots (iii)$$

So (ii) gives,

$$u(x, 0) = \sum_{n=1}^{\infty} B_n\sin x$$

$$\Rightarrow \sin 5x = B_1\sin x + B_2\sin 2x + B_3\sin 3x + B_4\sin 4x + B_5\sin 5x + \ldots$$

Comparing coefficients, we get

$B_n = 0$ if $n \neq 5$ and $B_5 = 1$.

Therefore, (ii) becomes,

$$u(x, t) = B_5\cos 2(5)t\sin 5x$$

$$\Rightarrow u(x, t) = \cos 10t\sin 5x$$

**3. A tightly stretched string with fixed ends at x = 0 and x = L is initially at rest in its equilibrium position. Find u(x, t) if it is set vibrating by giving to each of its points a velocity $3(Lx - x^2)$.**

**Solution:** Given that, Length = L, initially defluxion i.e. $f(x) = u(x, 0) = 0$

and, initial velocity $g(x) = 3(Lx - x^2)$

Since we have the solution of the one dimensional wave equation is

$$u(x, t) = \sum_{n=1}^{\infty}(B_n\cos\lambda_n t + B_n^*\sin\lambda_n t)\sin\frac{n\pi}{L}x \quad \ldots\ldots\ldots (i)$$

where, $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$ and $B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$

Since, $f(x) = 0$. So,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx = 0.$$

And,

$$B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$$

$$= \frac{2}{\lambda_n L}\int_0^L 3(Lx - x^2)\sin\left(\frac{n\pi}{L}\right)x\,dx$$

$$= \frac{6}{\lambda_n L}\left[(Lx - x^2)\frac{\cos\left(\frac{n\pi}{L}\right)x}{-\left(\frac{n\pi}{L}\right)} - (L - 2x)\frac{\sin\left(\frac{n\pi}{L}\right)x}{-\left(\frac{n\pi}{L}\right)^2} + (-2)\frac{\cos\left(\frac{n\pi}{L}\right)}{\left(\frac{n\pi}{L}\right)^3}\right]_0^L$$

$$= \frac{6}{\lambda_n L}\left[-\frac{2\cdot L^3}{n^3\pi^3}(\cos n\pi - 1)\right]$$

$$= \frac{12}{\frac{cn\pi}{L}} \cdot \frac{L^2}{n^3\pi^3}[1-(-1)^n]$$

$$= \frac{12\,L^3}{c\pi^4}\left[\frac{1-(-1)^n}{n^4}\right]$$

$$= \frac{24\,L^3}{c\pi^4 n^4} \text{ if n is odd and has zero value if n is even.}$$

Therefore, (i) becomes,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{24\,L^3}{c\pi^4(2n-1)^4} \sin\left(\frac{(2n-1)\,c\pi}{2}\right) \sin\left(\frac{(2n-1)\,\pi x}{L}\right).$$

4. **If the string be fixed at both ends, find the solution with the following initial conditions. The initial displacement $u(x, 0) = u_0 \sin\frac{\pi x}{L}$ and initial velocity is zero.**

**Solution:** Given that,

initial displacement $f(x) = u(x, 0) = u_0 \sin\frac{\pi}{L}x$

And, initial velocity $g(x) = 0$

Here, $\lambda_n = \frac{nc\pi}{L}$

Since we have the solution of the one dimensional wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L}x \qquad \ldots\ldots\ldots \text{(i)}$$

where, $B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x\,dx$ and $B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}\right)xdx$

Since, $g(x) = 0$. So,

$$B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}\right)xdx = 0$$

Then (i) becomes,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t \sin\frac{n\pi}{L}x \qquad \ldots\ldots\ldots \text{(ii)}$$

Put $t = 0$ in (ii) then

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x$$

$$\Rightarrow u_0 \sin\frac{\pi x}{L} = B_1 \sin\frac{\pi}{L}x + B_2 \sin\frac{2\pi x}{L} + \ldots$$

Comparing coeff. of $B_n$'s then we get,

$$B_1 = u_0. \qquad B_2 = 0 = B_3 = \ldots\ldots$$



Therefore, (ii) becomes,

$$u(x,t) = B_1 \cos\lambda_1 t \sin\frac{\pi}{L}x$$

$$= u_0 \cos\frac{\pi ct}{L} \sin\frac{\pi}{L}x.$$

5. **Find the deflection $u(x, t)$ of the vibrating string of length $L = \pi$, $c^2 = 1$ and its initial deflection is zero and initial velocity is $\begin{cases} 0.01\,x & \text{if } 0 < x < \pi/2 \\ 0.01\,(\pi - x) & \text{if } \pi/2 < x < \pi \end{cases}$**

**Solution:** Given that, Length $L = \pi$, $c^2 = 1$.

Initial deflection $u(x, 0) = f(x) = 0$.

Initial velocity $g(x) = \begin{cases} 0.01\,x & \text{if } 0 < x < \pi/2 \\ 0.01\,(\pi - x) & \text{if } \pi/2 < x < \pi \end{cases}$

Also, $\lambda_n = \frac{cn\pi}{L} = \pm\frac{1n\pi}{\pi} = \pm n.$

Since we have the solution of the one dimensional wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L}x \qquad \ldots\ldots\ldots \text{(i)}$$

where, $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$ and $B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}\right)xdx$

Since, $f(x) = 0$. So,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx = 0.$$

Then (i) becomes,

$$u(x,t) = \sum_{n=1}^{\infty} B_n^* \sin\lambda_n t \sin nx \qquad \ldots\ldots\ldots \text{(ii)}$$

Here,

$$B_n^* = \frac{2}{\lambda_n\pi}\int_0^{\pi} g(x)\sin nx\,dx$$

$$= \frac{2}{\lambda_n\pi}\left[\int_0^{\pi/2} 0.01\,x\sin nx\,dx + \int_{\pi/2}^{\pi} 0.01\,(\pi - x)\sin nx\,dx\right]$$

$$= \frac{0.02}{\lambda_n\pi}\left[\left|\frac{x\cos nx}{-n} + \frac{\sin nx}{n^2}\right|_0^{\pi/2} + \left|(\pi - x)\frac{\cos nx}{-n} - \frac{\sin nx}{n^2}\right|_{\pi/2}^{\pi}\right]$$

$$= \frac{0.02}{\lambda_n\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2} + \frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{\pi n}{2}}{n^2}\right]$$

$$= \frac{0.02}{\lambda_n\pi}\frac{\sin\frac{n\pi}{2}}{n^2}$$

Since, $\lambda_n = \pm n$. So,

$$B_n^* = \pm \frac{0.04}{\pi} \frac{\sin\frac{n\pi}{2}}{n^3}$$

Therefore, (ii) becomes,

$$u(x, t) = \pm \frac{0.04}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi}{2}\right) \sin nt \sin nx$$

**6. A tightly stretched string of length L is drawn a side at its midpoint a distance $\frac{L}{2}$ perpendicular to the equilibrium position so that its initial position is given by** $f(x) = \begin{cases} x & \text{if } 0 < x < L/2 \\ (L - x) & \text{if } L/2 < x < L \end{cases}$

**It is initially at rest and suddenly released. Find u(x, t).**

**Solution:** Given that, Length = L

Initial deflection $f(x) = \begin{cases} x & \text{if } 0 < x < L/2 \\ (L - x) & \text{if } L/2 < x < L \end{cases}$

Initial velocity $g(x) = 0$

Since we have the solution of the one dimensional wave equation is

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin \frac{n\pi}{L} x \quad \dots\dots\dots (i)$$

where, $B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L} x\, dx$ and $B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}\right) x dx$

Since, $g(x) = 0$. So,

$$B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}\right) x dx = 0$$

Then (i) becomes,

$$u(x, t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi}{L} x \quad \dots\dots\dots (ii)$$

Here,

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L} x\, dx$$

$$= \frac{2}{L}\left[\int_0^{L/2} x \sin\left(\frac{n\pi}{L}\right) x\, dx + \int_{L/2}^{L} (L - x) \sin\left(\frac{n\pi}{L}\right) x\, dx\right]$$

$$= \frac{2}{L}\left[\left| x \frac{\cos\frac{n\pi}{L}x}{\left(-\frac{n\pi}{L}\right)} + \frac{\sin\frac{n\pi}{L}x}{\left(\frac{n\pi}{L}\right)^2}\right|_0^{L/2} + \left|(L - x)\frac{\cos\frac{n\pi}{L}x}{-\frac{n\pi}{L}} - \frac{\sin\frac{n\pi}{L}x}{\left(\frac{n\pi}{L}\right)^2}\right|_{L/2}^{L}\right]$$

$$= \frac{2}{L}\left[-\frac{L}{n\pi}\cdot\frac{L}{2}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{L}{2}\cdot\frac{L}{n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$



$$= \frac{4L}{\pi^2} \cdot \frac{\sin\frac{n\pi}{2}}{n^2}$$

Therefore, (ii) becomes,

$$u(x, t) = \sum_{n=1}^{\infty} \frac{4L}{\pi^2} \cdot \frac{1}{n^2} \sin\frac{n\pi}{2} \cos\frac{nc\pi t}{L} \sin\frac{2\pi}{L} x$$

**7. Find solution u(x, y) of the following equation by using separating of variables:** **[2005 Spring Q. No. 7(iii)]**

**(a) $u_x - u_y = 0$**

**Solution:** Given that,

$$u_x - u_y = 0 \quad \dots\dots\dots (i)$$

Let the solution of (i) be

$$u(x, y) = FG \quad \dots\dots\dots (ii)$$

where F is function of x and G is function of y only.

Now,

$$u_x = \mathring{F}G \quad \text{and} \quad u_y = FG'$$

where dot and prime represents partial derivative w. r. t x and y respectively.

Then from (i),

$$\mathring{F}G - FG' = 0$$

$$\Rightarrow \frac{\mathring{F}}{F} = \frac{G'}{G} = \lambda \text{ (say)}$$

This gives,

$$\frac{\mathring{F}}{F} = \lambda \quad \text{and} \quad \frac{G'}{G} = \lambda$$

Integrating we get,

$$\log F = \lambda x + \log C \quad \text{and} \quad \log G = \lambda y + D$$

$$\Rightarrow F = ce^{\lambda x} \qquad \Rightarrow G = De\lambda y$$

Then (ii) becomes,

$$u(x, y) = CD\, e^{\lambda(x + y)}$$

$$\Rightarrow u(x, y) = c\, e^{\lambda(x + y)} \quad \text{for } c = CD$$

**(b) $u_x + u_y = (x + y)u$**

**Solution:** Given that,

$$u_x + u_y = (x + y)\, u \quad \dots\dots\dots (i)$$

Let the solution of (i) be

$$u(x, y) = FG \quad \dots\dots\dots (ii)$$

where F is function of x and G is function of y only.

Now,

$$u_x = \mathring{F}G \quad \text{and} \quad u_y = FG'$$

where dot and prime represents partial derivative w. r. t x and y respectively.
Then from (i),

$$\overset{\circ}{F}G + FG' = (x+y)FG$$

$$\Rightarrow \frac{\overset{\circ}{F}}{F} + \frac{G'}{G} = (x+y)$$

$$\Rightarrow \frac{\overset{\circ}{F}}{F} - x = \frac{G'}{G} + y = k$$

This gives,

$$\frac{\overset{\circ}{F}}{F} = k + x \quad \text{and} \quad \frac{G'}{G} = -k + y$$

Integrating we get,

$$\log F = kx + \frac{x^2}{2} + \log C \quad \text{and} \quad \log G = -ky + \frac{y^2}{2} + \log D$$

$$\Rightarrow F = ce^{kx + x^2/2} \qquad \Rightarrow G = De^{-ky + y^2/2}$$

Then (ii) becomes,

$$u = FG = c\, e^{(kx + x^2/2 - ky + y^2/2)} \quad \text{for } c = CD.$$

**(d) $u_{xy} - u = 0$** **[2008 Fall Q. No. 2(b); 2009 Fall Q. No. 2(b); 2006 Spring Q. No. 7(b); 2006 Fall Q. No. 7(a)]**

**Solution:** Given that,

$$u_{xy} - u = 0 \qquad \ldots\ldots (i)$$

Let the solution of (i) be

$$u(x, y) = FG \qquad \ldots\ldots (ii)$$

where F is function of x and G is function of y only.
Now,

$$u_x = \overset{\circ}{F}G \quad \text{and} \quad u_y = FG'$$

where dot and prime represents partial derivative w. r. t x and y respectively.
Then from (i),

$$\overset{\circ}{F}G' - FG = 0$$

$$\Rightarrow \frac{\overset{\circ}{F}}{F} = \frac{G}{G'} = k \text{ (say)}$$

This gives

$$\frac{\overset{\circ}{F}}{F} = k \quad \text{and} \quad \frac{G'}{G} = \frac{1}{k}$$

Integrating we get,

$$\log F = kx + \log C \quad \text{and} \quad \log G = \frac{y}{k} + \log D$$

$$\Rightarrow F = c\, e^{kx} \qquad \Rightarrow G = De^{y/k}$$

Then (ii) becomes,

$$u(x, y) = c\, e^{(kx + y/k)} \quad \text{for } c = CD$$



**(d) $x\,u_{xy} + 2yu = 0$.** **[2012 Fall Q. No. 5(b)]**

**Solution:** Given that,

$$x\,u_{xy} + 2yu = 0 \qquad \ldots\ldots (i)$$

Let the solution of (i) be

$$u(x, y) = FG \qquad \ldots\ldots (ii)$$

where F is function of x and G is function of y only.
Now, $u_{xy} = \overset{\circ}{F}G'$.
where dot and prime represents partial derivative w. r. t x and y respectively.
Then from (i),

$$x\,\overset{\circ}{F}G' + 2y\,FG = 0$$

$$\Rightarrow \frac{x\overset{\circ}{F}}{F} = -2y\frac{G}{G'} = \lambda \text{ (say)}$$

This gives

$$\frac{\overset{\circ}{F}}{F} = \frac{\lambda}{x} \quad \text{and} \quad \frac{G'}{G} = -\frac{2y}{\lambda}$$

Integrating we get,

$$\log F = \lambda \log(x) + \log C \quad \text{and} \quad \log G = -\frac{y^2}{\lambda} + \log D$$

$$\Rightarrow F = C\,x^{\lambda} \qquad \Rightarrow G = e^{-y^2/\lambda} D$$

Then (ii) becomes,

$$u(x, y) = c\,x^{\lambda}\,e^{-y^2/\lambda} \quad \text{for } c = CD.$$

**(e) $u_{2x} = 9u = 0$**

**Solution:** Given that,

$$u_{2x} = 9u = 0$$

The auxiliary equation is

$$m^2 + 9 = 0$$

$$\Rightarrow m = \pm 3i$$

Therefore the general solution of given equation is,

$$u(x, y) = A\cos 3x + B\sin 3x$$

**(f) $u_x = y\,u_y$**

**Solution:** Given that,

$$u_x = y\,u_y \qquad \ldots\ldots (i)$$

Let the solution of (i) be

$$u(x, y) = FG \qquad \ldots\ldots (ii)$$

where F is function of x and G is function of y only.
Now,

$$u_x = \overset{\circ}{F}G \quad \text{and} \quad u_y = FG'$$

where dot and prime represents partial derivative w. r. t x and y respectively.

Then from (i),

$$\overset{\circ}{F}G = y\,FG'$$

$$\Rightarrow \frac{\overset{\circ}{F}}{F} = \frac{yG'}{G} = \lambda \quad \text{(say)}$$

This gives,

$$\frac{\overset{\circ}{F}}{F} = \lambda \quad \text{and} \quad \frac{G'}{G} = \frac{\lambda}{y}$$

Integrating we get,

$$\log F = \lambda x + \log C \qquad \text{and} \qquad \log G = \lambda \log y + \log D$$

$$\Rightarrow F = Ce^{\lambda x} \qquad\qquad \Rightarrow G = y^k D$$

Then (ii) becomes,

$$u(x, y) = c\,e^{\lambda x}\,y^k \qquad \text{for } c = CD.$$

**8. A tightly stretched string with fixed end points $x = 0$ and $x = L$ is initially in a position given by $u = u_0 \sin^3\left(\frac{\pi x}{L}\right)$. If it is released from the rest from this position. Find the displacement.**

**Solution:** Given that, Length = L

Initial deflection $f(x) = u(x, 0) = u_0 \sin^3\left(\frac{\pi x}{L}\right)$

And, the string is released from the rest. So, initial velocity $g(x) = 0$.

Also, $\lambda_n = \frac{nc\pi}{L}$

Since we have the solution of the one dimensional wave equation is

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L}x \qquad \ldots\ldots\ldots (i)$$

where, $B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x\,dx$ and $B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}\right) xdx$

Since, $g(x) = 0$. So,

$$B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}\right) xdx = 0$$

Then (i) becomes,

$$u(x, t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t \sin\frac{n\pi}{L}x \qquad \ldots\ldots\ldots (ii)$$

Put $t = 0$ we get,

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}\right)x \qquad \ldots\ldots\ldots (iii)$$



We know,

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\Rightarrow \sin^3 A = \frac{3\sin A - \sin 3A}{4}$$

So,

$$u(x, 0) = \frac{u_0}{4}\left(3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}\right)$$

Therefore, (iii) gives,

$$\frac{u_0}{4}\left(3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x$$

$$= B_1 \sin\frac{\pi x}{L} + B_2 \sin\frac{2\pi x}{L} + B_3 \sin\frac{3\pi x}{L} + B_4 \sin\frac{4\pi x}{L} + \ldots\ldots$$

Comparing the coefficient of $B_n$'s then we get,

$$B_1 = \frac{3u_0}{4}, \quad B_2 = 0, \quad B_3 = -\frac{u_0}{4}, \quad B_4 = 0 = B_5 = \ldots\ldots$$

Therefore, (ii) becomes,

$$u(x, t) = \frac{3u_0}{4}\cos\lambda_1 t \sin\frac{\pi x}{L} - \frac{u_0}{4}\cos\lambda_3 t \sin\frac{3\pi x}{L}$$

$$\Rightarrow u(x, t) = \frac{u_0}{4}\left(3\cos\frac{c\pi t}{L}\sin\frac{\pi x}{L} - \cos\frac{3c\pi t}{L}\sin\frac{3\pi x}{L}\right)$$

**9. A string is stretched and fastened to two points L apart. Motion is started by displacing the string into the form $u = k(Lx - x^2)$ from which it is released at time $t = 0$. Find the displacement of any point on the string at a distance of x from one end at time t.**

**Solution:** Given that, Length = L

Initial displacement $u(x, 0) = f(x) = k\,(Lx - x^2)$

And, the string is released from time $t = 0$. So, initial velocity $g(x) = 0$.

Also, $\lambda_n = \frac{nc\pi}{L}$

Since we have the solution of the one dimensional wave equation is

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L}x \qquad \ldots\ldots\ldots (i)$$

where, $B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x\,dx$ and $B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}\right) xdx$

Since, $g(x) = 0$. So,

$$B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}\right) xdx = 0$$

Then (i) becomes,

$$u(x, t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi}{L} x \qquad \ldots\ldots\ldots (ii)$$

Here,

$$B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x\, dx$$

$$= \frac{2}{L}\int_0^L k(Lx - x^2) \sin\left(\frac{n\pi}{L}\right) x\, dx$$

$$= \frac{2k}{L}\left[(Lx - x^2)\frac{\cos\left(\frac{n\pi}{L}\right)x}{-\left(\frac{n\pi}{L}\right)} - (k - 2x)\frac{\sin\left(\frac{n\pi}{L}\right)x}{-\left(\frac{n\pi}{L}\right)^2} + (-2)\frac{\cos\left(\frac{n\pi}{L}\right)x}{\left(\frac{n\pi}{L}\right)^3}\right]$$

$$= \frac{2k}{L}\left[-2\cos n\pi \cdot \frac{L^3}{n^3\pi^3} + 2\cdot\frac{L^3}{n^3\pi^3}\right]$$

$$= \frac{4kL^2}{n^3\pi^3}[(-1)^n + 1]$$

Therefore, (ii) becomes,

$$u(x, t) = \sum_{n=1}^{\infty} \frac{4L^2k}{n^3\pi^3}[-(-1)^n + 1] \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L}.$$

**10. The vibration of an elastic string is governed by the partial differential equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ with $c^2 = 1$. The length of string is $\pi$ and ends are fixed. The initial velocity is zero and initial deflection is $f(x) = 2(\sin x + \sin 3x)$. Find the deflection $u(x, t)$ of the vibrating string.**

**Solution:** Given that, Length $= L = \pi$, $c^2 = 1$.

Initial deflection $u(x, 0) = f(x) = 2(\sin x + \sin 3x)$

And, initial velocity $g(x) = 0$.

Also, $\lambda_n = \frac{nc\pi}{L} = n$.

Since we have the solution of the one dimensional wave equation is

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L} x \qquad \ldots\ldots\ldots (i)$$

where, $B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L} x\, dx$ and $B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}\right) x dx$

Since, $g(x) = 0$. So,

$$B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}\right) x dx = 0$$

Then (i) becomes,



$$u(x, t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t \sin\frac{n\pi}{L} x \qquad \ldots\ldots\ldots (ii)$$

Here,

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L} x\, dx$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$$

Now, put $B_n^* = 0$ and $L = \pi n = \pm n$ equation becomes

$$u(x, t) = \sum_{n=1}^{\infty} B_n \cos nt \sin nx$$

Put $t = 0$ then,

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin nx \qquad \ldots\ldots\ldots (ii)$$

$$\Rightarrow 2\sin x + 2\sin 3x = \sum_{n=1}^{\infty} B_n \sin nx = B_1 \sin x + B_2 \sin 2x + B_3 \sin 3x + B_4 \sin 4x$$

Comparing we get,

$B_1 = 2$, $B_3 = 2$, $B_2 = 0 = B_4 = B_5 = \ldots\ldots\ldots$

Then (ii) becomes,

$u(x, t) = 2(\cos t \sin x + \cos 3t \sin 3x)$.

**11. The vibration of an elastic string is governed by the partial differential equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ with length $\pi$ and ends are fixed. The initial deflection $u(x, 0) = 0$ and initial velocity $\left.\frac{d\theta}{dt}\right|_{t=0} = u_t(x, 0) = \begin{cases} kx & \text{for } 0 < x < \pi/2 \\ k(\pi - x) & \text{for } \pi/2 < x < \pi \end{cases}$**

**Solution:** Given that, Length L,

Initial deflection $u(x, 0) = f(x) = 0$,

Initial velocity $g(x) = \left.\frac{d\theta}{dt}\right|_{t=0} = u_t(x, 0) = \begin{cases} kx & \text{for } 0 < x < \pi/2 \\ k(\pi - x) & \text{for } \pi/2 < x < \pi \end{cases}$

Also, $\lambda_n = \frac{cn\pi}{L} = \pm\frac{1n\pi}{\pi} = \pm n$.

**Hint:** Same as Q. No. 5 with $k = 0.01$.

**12. Find the solution of one dimensional wave equation of the tight string of length $\pi$ under the [illegible] dition $u = (0, t) = 0$, $u(\pi, t) = 0$, $\left.\frac{du}{dt}\right|_{t=0} = 0$ and** [illegible]

**Solution:** Given that, Length = L = $\pi$, $u = (0, t) = 0$,

Initial deflection $u(x, 0) = f(x) = x$

And, initial velocity $g(x) = \left.\frac{du}{dt}\right|_{t=0} = 0.$

Also, $\lambda_n = \frac{nc\pi}{L} = cn.$

Since we have the solution of the one dimensional wave equation is

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L}x \quad \ldots\ldots\ldots (i)$$

where, $B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x\,dx$ and $B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}\right)xdx$

Since, g(x) = 0. So,

$$B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}\right)xdx = 0$$

Then (i) becomes,

$$u(x, t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t \sin\frac{n\pi}{L}x \quad \ldots\ldots\ldots (ii)$$

Here,

$$B_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx = \frac{2}{\pi}\left|x\frac{\cos nx}{-n} + \frac{\sin nx}{n^2}\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left\{-\pi\frac{\cos n\pi}{n}\right\} = \frac{2}{n}(-1)^{n+1}$$

Then (ii) becomes,

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \cos nct \sin nx$$

## Exercise 8.3

1. **Find the temperature u(x, t) in a bar of silver {length 10 cm, constant cross sectional of area 1 cm², density 10.6 gm/cm², thermal conductivity 1.04 cal/(gm sec °c), specific heat 0.056 cal/(gm °c)} that is perfectly insulated laterally, whose ends are kept at temperature 0°c and whose initial temperature (in °c) is f(x), where**

   **(a) f(x) = sin (0.1) πx (b) f(x) = x(10 – x)**

**Solution:** Given that, Length, L = 10 cm, Cross–section area, A = 1 cm²

Density $\rho = 10.6$ gm/cm², Specific hat, s = 0.056 cal/gm°c

Thermal conductivity, k = 1.04 cal/gm sec°c

Also, $u(0, t) = 0 = u(L, t)$



Now,

$$c^2 = \frac{k}{s\rho} = \frac{1.04}{0.056 \times 10.6} = 1.752 \text{ cm}^2/\text{sec}$$

$$\lambda_n = \frac{nc\pi}{L}\theta$$

We know solution of one dimensional heat equation under given initial and boundary condition is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x\, e^{-\lambda_n^2 t} \quad \ldots (i)$$

where $B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x\,dx$

Since, L = 10 cm. So, (i) gives,

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{10}\right) e^{-\lambda_n^2 t} \quad \ldots (ii)$$

Put t = 0 in equation (ii)

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin (0.1)\, n\pi x \quad \ldots (iii)$$

(a) Given that, $f(x) = u(x, 0) = \sin (0.1)\, \pi x$

Then (iii) gives,

$$\sin (0.1)x\pi = B_1 \sin 0.1x\pi + B_2 \sin 0.2\pi x + B_3 \sin 0.3\pi x + \ldots$$

Equating coefficient of like terms then we get,

$$B_1 = 1, \quad B_2 = 0 = B_3 = \ldots\ldots$$

Therefore (ii) becomes,

$$u(x, t) = \sin 0.1\pi x\, e^{-\left(\frac{\pi^2 c^2}{L^2}\right)t}$$

$$\Rightarrow u(x, t) = \sin 0.1\pi x\, e^{-1.752\pi^2 t/100}$$

(b) Given that, $f(x) = u(x, 0) = x(10 - x)$.

Here, with L = 10 cm,

$$B_n = 0.2\int_0^{10} x(10x - x^2) \sin 0.1n\pi x\,dx$$

$$= 0.2\left|(10x - x^2)\frac{\cos 0.1n\pi x}{-0.1n\pi} - (10 - 2x)\frac{\sin 0.1n\pi x}{-(0.1n\pi)^2} + (-2)\frac{\cos 0.1n\pi x}{(0.1n\pi)^3}\right|_0^{10}$$

$$= 0.2\left[-\frac{2\cos (n\pi)}{(0.1n\pi)^3} + \frac{2}{(0.1n\pi)^3}\right]$$

$$= \frac{0.4}{(0.1\pi)^3}[1 - (-1)^n]$$

Therefore (ii) becomes,

$$u(x, t) = \sum_{n=1}^{\infty} \frac{0.4}{(0.1\pi)^3}[1-(-1)^n] \sin(0.1)\, n\pi x e^{(-(1.752 n^2\pi^2/100)t)}$$

**2. A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is**

$$f(x) = \begin{cases} x & 0 \le x \le 50 \\ 100 - x & 50 \le x \le 100 \end{cases}$$

**Find the temperature u(x, t) at any time.**

**Solution:** Given that, Length = L = 100,

$$\text{Initial deflection } u(x, 0) = f(x) = \begin{cases} x & 0 \le x \le 50 \\ 100 - x & 50 \le x \le 100 \end{cases}$$

And, $u(0, t) = 0 = u(100, t)$.

Also, $\lambda_n = \frac{nc\pi}{L} = \frac{nc\pi}{100}$.

Since we have the solution of the one dimensional heat equation is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L} x\, e^{-\lambda_n^2 t} \quad \ldots\ldots\ldots (i)$$

where, $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$

Here with L = 100,

$$B_n = \frac{2}{100}\left[\int_0^{50} x\sin\frac{n\pi}{100}x\,dx + \int_{50}^{100}(100-x)\sin\frac{n\pi}{100}x\,dx\right]$$

$$= \frac{1}{50}\left[\left|\frac{x\cos\frac{n\pi}{100}x}{-\left(\frac{n\pi}{100}\right)} + \frac{\sin\left(\frac{n\pi}{100}\right)x}{\left(\frac{n\pi}{100}\right)^2}\right|_0^{50} + \left|(100-x)\frac{\cos\left(\frac{n\pi}{100}\right)x}{-\left(\frac{n\pi}{100}\right)} - \frac{\sin\left(\frac{n\pi}{100}\right)x}{\left(\frac{n\pi}{100}\right)^2}\right|_{50}^{0}\right]$$

$$= \frac{1}{50}\left[\frac{-50\,\lambda\,100}{n\pi}\cos\frac{n\pi}{2} + \frac{(100)^2}{(n\pi)^2}\sin\frac{n\pi}{2} + \frac{50\,\lambda\,100}{n\pi}\cos\frac{n\pi}{2} + \frac{(100)^2}{(n\pi)^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Therefore (i) becomes,

$$u(x, t) = \sum_{n=1}^{\infty} \frac{400}{\pi^2}\frac{\sin\frac{n\pi}{2}}{n^2}\sin\left(\frac{n\pi}{100}\right)x\, e^{-(nc\pi/100)^2 t}$$



**3. Find the temperature u(x, t) in a slab whose ends x = 0 and x = L are kept at zero temperature and whose initial temperature f(x) is given by**

$$f(x) = \begin{cases} k & \text{when } 0 < x < L/2 \\ 0 & \text{when } L/2 < x < L \end{cases}$$

**Solution:** Given that, Length = L.

$$\text{Initial deflection } u(x, 0) = f(x) = \begin{cases} k & \text{when } 0 < x < L/2 \\ 0 & \text{when } L/2 < x < L \end{cases}$$

And, $u(0, t) = 0 = u(L, t)$.

Also, $\lambda_n = \frac{nc\pi}{L}$.

Since we have the solution of the one dimensional heat equation is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L} x\, e^{-\lambda_n^2 t} \quad \ldots\ldots\ldots (i)$$

where, $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$

Here,

$$B_n = \frac{2}{L}\left[\int_0^{L} k\sin\left(\frac{n\pi}{L}\right)x\,dx\right]$$

$$= \frac{2k}{L}\left|-\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}\right)x\right|_0^{L/2} + 0.$$

$$= -\frac{2k}{n\pi}\left[\cos\frac{n\pi}{2} - 1\right]$$

$$= \frac{2k}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right) = \frac{2k}{n\pi}2\sin^2\frac{n\pi}{4} = \frac{4k}{n\pi}\sin^2\frac{n\pi}{4}$$

Therefore (i) becomes,

$$u(x, t) = \frac{4k}{\pi}\sum\frac{1}{n}\sin^2\frac{n\pi}{4}\sin\frac{n\pi x}{L}e^{-\left(\frac{nc\pi}{L}\right)^2 t}$$

**4. Find the temperature in a laterally insulated bar of length L whose ends are kept at temperature 0, assuming that the initial temperature is**

$$f(x) = \begin{cases} x & \text{if } 0 < x < L/2 \\ L - x & \text{if } L/2 < x < L \end{cases}$$

**[2004 Fall Q. No. 4(a); 2006 Fall Q. No. 3(a); 2006 Spring Q. No. 4(b); 2007 Fall Q. No. 4 (b); 2008 Fall Q. No. 3(b); 2011 Spring Q. No. 5(b); 2012 Fall Q. No. 4(b)]**

**Solution:** Given that, Length = L.

Initial deflection $u(x, 0) = f(x) = \begin{cases} x & \text{if } 0 < x < L/2 \\ L - x & \text{if } L/2 < x < L \end{cases}$

And, $u(0, t) = 0 = u(L, t)$.

Also, $\lambda_n = \frac{nc\pi}{L}$.

Since we have the solution of the one dimensional heat equation is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L} x\, e^{-\lambda_n^2 t} \quad \ldots\ldots\ldots (i)$$

where, $B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L} x\, dx$

Here,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

$$= \frac{2}{L}\left[\int_0^{L/2} x\sin\left(\frac{n\pi}{L}\right)x\,dx + \int_{L/2}^{L}(L-x)\sin\left(\frac{n\pi}{L}\right)x\,dx\right]$$

$$= \frac{2}{L}\left[\left| x\frac{\cos\frac{n\pi}{L}x}{\left(-\frac{n\pi}{L}\right)} + \frac{\sin\frac{n\pi}{L}x}{\left(\frac{n\pi}{L}\right)^2}\right|_0^{L/2} + \left|(L-x)\frac{\cos\frac{n\pi}{L}x}{-\frac{n\pi}{L}} - \frac{\sin\frac{n\pi}{L}x}{\left(\frac{n\pi}{L}\right)^2}\right|_{L/2}^{L}\right]$$

$$= \frac{2}{L}\left[-\frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{4L}{n^2\pi^2}\cdot\sin\left(\frac{n\pi}{2}\right)$$

Therefore (i) becomes,

$$u(x, t) = \sum_{n=1}^{\infty}\frac{4L}{\pi^2}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}\sin\left(\frac{n\pi}{L}\right)x\, e^{-(cn\pi/L)^2 t}$$

**5. Find the temperature distribution in a laterally insulated thin copper bar ($c^2$ = 1.158 $cm^2$/sec), 100 cm long and of constant cross section whose endpoints at x = 0 and x = 100 are kept at 0°c are kept at 0°c and whose initial temperature is, (i) f(x) = sin (0.01) πx (ii) f(x) = $\sin^3$ (0.01) πx.**

**Solution:** Given that,

$C^2 = 1.158\ cm^2/sec$, L = 100 cm, $u(0, t) = 0 = u(100, t)$

Then, $\lambda_n = \frac{cn\pi}{L} \Rightarrow \lambda_n^2 = \frac{1.158}{100}n^2\pi^2$



(i) Given that,

$$f(x) = \sin(0.01)\pi x = u(x, 0) \quad \ldots (i)$$

The required temperature is given by

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L} x\, e^{-\lambda_n^2 t} \quad \ldots (ii)$$

where, $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$

Put L = 10m and t = 0 in equation (i) then

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin(0.01)\,n\pi x \quad \ldots (iii)$$

From equation (i) and (iii)

$$\sin(0.01)\pi x = \sum_{n=1}^{\infty} B_n \sin(0.01)\,n\pi x$$

$$= B_1\sin(0.01)\pi x + B_2\sin(0.02)\pi x + B_3\sin(0.03)\pi x + \ldots$$

Comparing coefficients of like terms then we get,

$B_1 = 1, B_2 = 0 = B_3 = \ldots\ldots$

Therefore (ii) becomes,

$$u(x, t) = \sin(0.01)\pi x\, e^{-(1.158/100)\pi^2 t}$$

(b) Given that, **[2009 Spring Q. No.4 (b)]**

$$f(x) = u(x, 0) = \sin^3(0.01)\pi x$$

Now,

$$\sin^3(0.01x) = \frac{3\sin(0.01)\pi x - \sin(0.03)\pi x}{4} \quad [\because \sin 3A = 3\sin A - 4\sin^3 A]$$

Then, $u(x, 0) = \frac{3}{4}\sin(0.01)x\pi - \frac{1}{4}\sin(0.03)\pi x \quad \ldots (iv)$

Now, from equation (iii) and (iv)

$$\frac{3}{4}\sin(0.01)\pi x - \frac{1}{4}\sin(0.03)\pi x = \sum_{n=1}^{\infty} B_n \sin(0.01)n\pi x$$

$$= B_1\sin(0.01)\pi x + B_2\sin(0.02)\pi x + B_3\sin(0.03)\pi x + B_4\sin(0.04)\pi x + \ldots$$

Comparing coefficients of like terms then we get,

$B_1 = \frac{3}{4}$ $B_3 = -\frac{1}{4}$ $B_2 = 0 = B_4 + \ldots\ldots$

Therefore (ii) becomes,

$$u(x, t) = \frac{3}{4}\sin(0.01)\pi x\, e^{-(1.158/100)\pi^2 t} - \frac{1}{4}\sin(0.03)\pi x\, e^{-(1.158\times 9/100)t}$$

**6. Find the solution of one dimensional heat equation with boundary and initial condition u(0, t) = 0, u(L, t) = 0, u(x 0) = $\frac{100x}{L}$.**

**Solution:** Given that, Length = L.

Initial deflection $u(x, 0) = f(x) = \frac{100x}{L}$

And, $u(0, t) = 0 = u(L, t)$. Also, $\lambda_n = \frac{nc\pi}{L}$.

Since we have the solution of the one dimensional heat equation is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x\, e^{-\lambda^2 t} \quad \ldots\ldots\ldots (i)$$

where, $B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x\, dx$

Here,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

$$= \frac{2}{L}\times\frac{100}{L}\int_0^L x \sin\left(\frac{n\pi}{L}\right)x\,dx$$

$$= \frac{200}{L^2}\left[\frac{x\cos\left(\frac{n\pi}{L}\right)x}{-\left(\frac{n\pi}{L}\right)} + \frac{\sin\left(\frac{n\pi}{L}\right)x}{\left(\frac{n\pi}{L}\right)^2}\right]_0^L$$

$$= \frac{200}{L^2}\left[-\frac{L^2}{n\pi}\cos n\pi\right]$$

$$= \frac{200}{n\pi}(-1)^{n+1}$$

Therefore (i) becomes,

$$u(x, t) = \sum_{n=1}^{\infty}\frac{200}{\pi}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{L}\right)x\, e^{-(n^2c^2\pi^2/L^2)t}$$

**7. Solve the one dimensional heat equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ under the condition**

**(i) u is finite at t→∞ (ii) u(0, t) = 0 = u(π, t) (iii) u(x, 0) = πx − x²**

**Solution:** Given that, Length = L = π.

Initial deflection $u(x, 0) = f(x) = \pi x - x^2$

And, $u(0, t) = 0 = u(\pi, t)$. Also, $\lambda_n = nc$.



Since we have the solution of the one dimensional heat equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x\, e^{-\lambda^2 t}$$

$$= \sum_{n=1}^{\infty} B_n \sin nx\, e^{-n^2c^2t} \quad \ldots\ldots\ldots (i)$$

where, $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$

Here,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

$$= \frac{2}{L}\int_0^\pi (\pi x - x^2)\sin nx\,dx$$

$$= \frac{2}{\pi}\left|(\pi x - x^2)\frac{\cos nx}{-n} - (\pi - 2x)\frac{\sin nx}{-n^2} + (-2)\frac{\cos nx}{n^3}\right|_0^\pi$$

$$= \frac{2}{\pi}\left[-2\frac{\cos n\pi}{n^3} + \frac{2}{n^3}\right]$$

$$= \frac{4}{\pi n^3}[1 - (-1)^n]$$

$$= \frac{8}{\pi n^3} \text{ if n is odd and zero value if n is even.}$$

Therefore (i) becomes,

$$u(x, t) = \sum_{n=1}^{\infty}\frac{8}{\pi(2n-1)^3}\sin(2n-1)x\, e^{-c^2(2n-1)^2 t}$$

**8. A rod of length L has its ends A and B maintained at 0°c and 100°c respectively until steady state condition prevails. If the changes consists of raising the temperature of A to 20°c and reducing that of B to 80°c. Find the temperature distribution in the rod at time t.**

**Solution:** Given that, Length = L,

Initial deflection $u(x, 0) = f(x) = \pi x - x^2$

And, $u(0, t) = 0$ and $u(\pi, t) = 100$.

We have one dimensional heat equation is

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad \ldots (i)$$

For steady state condition flow is independent on time. So,

$$\frac{\partial u}{\partial t} = 0$$

Therefore, equation (i) becomes,

$$\frac{\partial^2 u}{\partial x^2} = 0$$

This imply $\quad u = Ax + B \quad \ldots$ (ii)

Since we have initial condition

$u(0, 0) = 0°$ then $B = 0$.

Also, $\quad u(L, 0) = 100°$ then $A = \frac{100}{L}$

Therefore,

$$u(x, 0) = f(x) = \frac{100}{L}x, \text{ initial temperature.}$$

Again we have boundary condition (after some time)

$u(0, t) = 20° \quad$ and $\quad u(L, t) = 80°$

From Equation (ii),

$$20 = B \quad \text{and} \quad 80 = AL + 20 \Rightarrow A = \frac{60}{L}$$

Thus, $\quad u = \frac{60}{L}x + 20.$

Splitting temperature function into two parts

$$u(x, t) = u_1(x, t) + u_2(x) \quad \ldots \text{(iii)}$$

Here

$u_2(x) \Rightarrow$ steady state component

$u_1(x, t) \Rightarrow$ transient component & tends to 0.

$$\therefore \quad u_2(x) = \frac{60}{L}x + 20$$

To find transient temperature we have to find initial and boundary condition

From eq$^n$. (iii)

$$u(0, t) = u_1(0, t) + u_2(0)$$
$$\Rightarrow \quad 20 = u_1(0, t) = 20$$
$$\Rightarrow \quad u_1(0, t) = 0$$

Again,

$$u(L, t) = u_1(L, t) + u_2(L)$$
$$\Rightarrow \quad 80 = u_1(L, t) + 80$$
$$\Rightarrow \quad u_1(L, t) = 0$$

Here, boundary conditions are $u_1(L, t) = 0 = u_1(0, t)$

For initial condition,

Put t = 0 in (iii)

$$u(x, 0) = u_1(x_1, 0) + u_2(x)$$
$$\Rightarrow \quad \frac{100x}{L} = u_1(x, 0) + \frac{60}{L}x + 20$$



$$\Rightarrow \quad u_1(x, 0) = \frac{40x}{L} - 20 = f_1(x)$$

Hence, the required transient solution is

$$u_1(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}\right) x\, e^{-\lambda^2 t}$$

where,

$$B_n = \frac{2}{L}\int_0^L \left(\frac{40x}{L} - 20\right) \sin\left(\frac{n\pi}{L}\right) x\, dx$$

$$= \frac{2}{L}\left[\left(\frac{40x}{L} - 20\right)\frac{\cos\left(\frac{n\pi}{L}\right)x}{-\left(\frac{n\pi}{L}\right)} + \frac{40}{L}\frac{\sin\left(\frac{n\pi}{L}\right)x}{\left(\frac{n\pi}{L}\right)^2}\right]_0^L$$

$$= \frac{2}{L}\left[-\frac{20L}{n\pi}\cos n\pi - \frac{L}{n\pi}\right]$$

$$= \frac{-40}{n\pi}[1 + (-1)^n]$$

Then $\quad u_1(x, t) = \sum_{n=1}^{\infty} \frac{-40}{n\pi}[1 + (-1)^n] \sin\left(\frac{n\pi}{L}\right) x\, e^{-\lambda n^2 t}$

Again,

$$u(x, t) = u_1(x, t) + u_2(x)$$

$$\Rightarrow \quad u(x, t) = \frac{60x}{L} + 20 - \frac{40}{\pi}\sum_{n=1}^{\infty} \frac{[(-1)^n + 1]}{n} \sin\left(\frac{n\pi}{L}\right) x\, e^{-\lambda n^2 t}$$

**9. The ends A and B of a rod 20 cm long having temperature at 30°c and 80°c until steady state prevails. If the change consists of raising the temperature of A to 40°c and reducing that of B to 60°c. Find the temperature distribution in the bar at time t.**

**Solution: Similar as Q. No. 8.**

**10. Find the solution of one dimensional heat equation such that u is finite at $t \to \infty$, $\frac{\partial u}{\partial x} = 0$ at $x = 0$ and $x = L$, and $u = Lx - x^2$ for $t = 0$ between $x = 0$ to $x = L$.**

**Solution:** Given that, $\quad$ Length = L,

Initial deflection $u(x, 0) = f(x) = Lx - x^2$

And, $u_x(0, t) = 0 = u_x(L, t)$.

Since we have the solution of the one dimensional heat equation is

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi}{L} x\, e^{-\lambda n^2 t}$$

where, $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$ and $A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}\right) x\,dx$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L (Lx - x^2)\,dx = \frac{1}{L}\left[\frac{Lx^2}{2} - \frac{x^3}{3}\right]_0^L = \frac{L^2}{6}$$

And,

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}\right) x\,dx$$

$$= \frac{2}{L}\int_0^L (Lx - x^2)\cos\frac{n\pi}{L} x\,dx$$

$$= \frac{2}{L}\left|(lx - x^2)\frac{\sin\left(\frac{n\pi}{L}\right)x}{\frac{n\pi}{L}} - (L - 2x)\frac{\cos\left(\frac{n\pi}{L}\right)x}{-\left(\frac{n\pi}{L}\right)^2} + (+2)\frac{\sin\frac{n\pi}{L}x}{\left(\frac{n\pi}{L}\right)^3}\right|_0^L$$

$$= \frac{2}{L}\left(-\frac{L^3}{n^2\pi^2}\cos n\pi - \frac{L^3}{n^2\pi^2}\right)$$

$$= \frac{-4L^2}{n^2\pi^2}[1 + (-1)^n]$$

Thus the required solution under this boundary condition is

$$u(x, t) = \frac{L^2}{6} - \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n + 1]\cos\frac{n\pi}{L} x\, e^{-\lambda^2 t}$$

**11. Solve $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$, with initial and boundary condition $\frac{\partial u}{\partial x} = 0$ at $x = 0$ and $x = \pi$ and $u(x, 0) = x^2$ for $0 < x < \pi$.**

**Solution:** Given that, Length = L = π,

Initial deflection $u(x, 0) = f(x) = x^2$ for $0 < x < \pi$.

And, $u_x(0, t) = 0 = u_x(\pi, t)$.

Since we have the solution of the one dimensional heat equation is

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi}{L} x\, e^{-\lambda n^2 t} \quad \ldots\ldots (i)$$

where, $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$ and $A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}\right) x\,dx$



Here,

$$A_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{3}$$

And,

$$A_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx = \frac{2}{\pi}\left|x^2\frac{\sin nx}{n} - 2x\frac{\cos nx}{-n^2} + 2\frac{\sin nx}{-n^3}\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left|+\frac{2\pi}{n^2}\cos n\pi\right|$$

$$= \frac{4}{\pi^2}(-1)^n$$

Thus (i) becomes,

$$u(x, t) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx\, e^{-\lambda n^2 t}$$

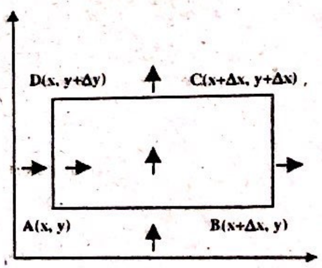
## Theoretical Part

**Derivation of two dimensional heat equations**

Consider the heat flow in a metal plate ABCD.

Suppose, that the plate has sides $\Delta x$ and $\Delta y$ with thickness $\alpha$.

Then, the heat enters through the die of the plate be,




$$-k\alpha\left(\frac{\partial u}{\partial y}\right)_y \Delta x$$

and $$-\alpha k \Delta y\left(\frac{\partial u}{\partial y}\right)_x$$

The negative sign is used for heat is decreasing form.

Also, the heat flow out through another sides of the plate be,

$$-k\alpha\Delta x\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} \quad \text{and} \quad -k\alpha\Delta y\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

Thus, total heat retained by the plate be,

= heat enters through its sides – heat flow out through its remaining sides

$$= k\alpha\Delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial y}\right)_x\right] + k\alpha\Delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)_y\right] \quad \ldots\ldots (i)$$

Since, we have the heat gain by the plate of length Δx, Δy and thickness α is

$$s\rho\Delta x\Delta y\frac{\partial u}{\partial t}$$

where, s = specific heat, ρ = density of metal plate

Then, (i) should equal to $s\rho\Delta x\Delta y \frac{\partial u}{\partial t}$

Therefore,

$$s\rho\frac{\partial u}{\partial t} = k\alpha\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] + \frac{k\alpha}{\Delta y}\left[\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)_y\right]$$

as, $\Delta x \to 0, \Delta y \to 0$ then,

$$s\rho\frac{\partial u}{\partial t} = k\alpha\frac{\partial^2 u}{\partial x^2} + k\alpha\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{k\alpha}{s\rho}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

Set $c^2 = \frac{k\alpha}{s\rho}$ then, $\frac{\partial u}{\partial t} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$

This is required equation.

## Derivation of two dimensional wave equation

Consider a stretched elastic membrane in two dimensional of vibrating string has,

(a) The mass of membrane, per unit area is constant. And, no resistance forces are allow

(b) The tension per unit length T caused by the membrane same at all points and in all directions and does not change during the motion.

(c) The deflection in the small portion is compared to the size of membrane and all angles of inclination are small.

Consider ΔxΔy be small element of membrane. Let T be the tension on per unit length. Then, the forces acting on the sides are TΔx and TΔy.

Since, the all angles of inclination are small. So, suppose the angle is nearly to zero and the horizontal components are nearly to 1. Thus, each particle moves vertically.

Suppose, that the angle of inclination varies along the sides be α and β. Then, the vertical components are –TΔy sinα and TΔy sinβ when T acts along at Δy.

Thus, the resultant vertical component be,

$$T\Delta y\sin\beta - T\Delta y\sin\alpha$$

$$= T\Delta y\tan\beta - T\Delta y\tan\alpha \quad [\because \text{Horizontal components} \to 1]$$

$$= T\Delta y(\tan\beta - \tan\alpha)$$

$$= T\,\Delta x\,\Delta y\left[\frac{1}{\Delta x}\left(\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}\Big|_{x}\right)\right] \quad \text{.............. (1)}$$

Being tangent is slope at the point.

Similarly, the vertical component act along Δx be,

$$= T\,\Delta x\,\Delta y\left[\frac{1}{\Delta y}\left(\frac{\partial u}{\partial y}\Big|_{y+\Delta y} - \frac{\partial u}{\partial y}\Big|_{x}\right)\right] \quad \text{.......... (2)}$$



By Newton's second law, the force is equal to the product of mass and its acceleration.

Since, the mass = $\rho\Delta x\Delta y\frac{\partial^2 u}{\partial t^2}$

Since, F is equal to force act along Δx and Δy. So, F is sum of (1) and (2).

Therefore,

$$T\,\Delta x\,\Delta y\left[\frac{1}{\Delta x}\left(\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}\Big|_{x}\right)\right] + \left[\frac{1}{\Delta y}\left(\frac{\partial u}{\partial y}\Big|_{y+\Delta y} - \frac{\partial u}{\partial y}\Big|_{x}\right)\right] = \rho\Delta x\,\Delta y\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{1}{\Delta x}\left(\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}\Big|_{x}\right) + \frac{1}{\Delta y}\left(\frac{\partial u}{\partial y}\Big|_{y+\Delta y} - \frac{\partial u}{\partial y}\Big|_{x}\right) = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

Taking $\Delta x \to 0, \Delta y \to 0$ then,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

Set, $\frac{\rho}{T} = \frac{1}{c^2}$ Then,

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

This is required two dimensional wave equation.

## Solution of two dimensional wave equation under certain condition

Consider two dimensional wave equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{.............. (1)}$$

With u = 0 for all t ≥ 0 ........ (2)

$$u(x, y, 0) = f(x, y); \quad u_t(x, y, 0) = g(x, y) \quad \text{.............. (2)}$$

assume that $u(x, y, t) = F(x, y)\,G(t)$ .............. (3)

be solution of (1). Then, (1) gives,

$$F\ddot{G} = c^2 G(F_{xx} + F_{yy})$$

where, $\dot{G} = \frac{\partial G}{\partial t}$, $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$

$$\Rightarrow \frac{\ddot{G}}{c^2 G} = \frac{1}{F}(F_{xx} + F_{yy}) = -\alpha^2 \text{ (say)}$$

This gives,

$$\ddot{G} + \lambda^2 G = 0 \quad \text{.............. (5)}$$

And $F_{xx} + F_{yy} + \alpha^2 F = 0$ .............. (6)

where, $\lambda = \alpha$

The equation (6) is called Helmholtz equation.

Let, $F(x, y) = X(x)\,Y(y)$ .............. (7)

be solution of (6). Then, (6) becomes,

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + \alpha^2 XY = 0$$

$$\Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} - \alpha^2 = -k^2 \text{ (say)}$$

For otherwise the solution is not possible.

That gives,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k^2 \quad \text{and} \quad -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} - \alpha^2 = -k^2$$

$$\Rightarrow \frac{\partial^2 X}{\partial x^2} + k^2 X = 0 \quad \text{and} \quad \frac{\partial^2 Y}{\partial y^2} + p^2 Y = 0$$

with $p^2 = \alpha^2 - k^2$

whose solution be

$$X(x) = A\cos kx + B\sin kx$$

And $\quad Y(y) = C\cos py + D\sin py$

where, A, B, C, D are constants.

Then, (7) becomes,

$$F(x, y) = (A\cos kx + B\sin kx)(C\cos py + D\sin py) \quad \text{............ (8)}$$

Since, u = 0, for all t. So, F = 0 at x = 0, a; y = 0, b

As x = 0 then A = 0. Also, as y = 0 then c = 0.

So that, at x = a,

$$B\sin ka = 0$$

$\Rightarrow \sin ka = 0$ as $B \neq 0$ otherwise it leads impossible solution.

$$= \sin n\pi$$

$$\Rightarrow k = \frac{n\pi}{a}$$

Similarly, at y = b, $p = \frac{m\pi}{b}$ as $\sin m\pi = 0$

Thus, (8) becomes,

$$F_{mn}(x, y) = A_{mn}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \quad \text{where, } A_{mn} = B.D$$

be required solution of (6)

And, we have $p^2 = \alpha^2 - k^2$ and $\lambda = c\alpha$

So, $\quad \lambda = c\alpha = c\sqrt{p^2 + k^2}$

$$\Rightarrow \lambda mn = \lambda = c\pi = \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \quad [\because \text{ using value of p and k}]$$

Therefore, the solution of (5) be,

$$G_{mn}(t) = B_{mn}\cos\lambda_{mn}t + B_{mn}^*\sin\lambda_{mn}t$$

Thus, (4) becomes,

$$u(x, y, t) = \Sigma u_{mn}(x, y, t)$$
$$= \Sigma F_{mn}(x, y)\, G_{mn}(t)$$
$$= \Sigma(B_{mn}\cos\lambda_{mn}(t) + B_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi}{b}\sin\frac{n\pi}{a}$$



As the constants includes $A_{mn}$n i.e. u(x, y, t)

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(B_{mn}\cos\lambda_{mn}t + B_{mn}^*\sin\lambda_{mn}t)\sin\frac{n\pi x}{b}\sin\frac{n\pi y}{a} \quad \text{... (10)}$$

This is required solution of (1)

We have, u(x, y, 0) = f(x, y). Then, (10) gives,

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

which is half range multiple Fourier sine series. So,

$$B_{mn} = \frac{2}{a}\frac{2}{b}\int_0^a\int_0^b f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dy\,dx$$

$$\Rightarrow B_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dy\,dx \quad \text{....... (11)}$$

Also, we have, $u_t(x, y, 0) = g(x, y)$ then, (10) gives,

$$g(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}^*\lambda_{mn}\sin\sin\left(\frac{n\pi x}{a}\right)$$

which is half range multiple Fourier sine series. So,

$$B_{mn}^* = \frac{4}{ab\lambda_{mn}}\int_0^a\int_0^b g(x, y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dy\,dx \quad \text{........ (12)}$$

Hence, (10) is required solution of (1) with the coefficients value given in (11) and (12).

**Laplacian in polar form**

We have, the Laplacian in Cartesian coordinates be,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{.............. (1)}$$

Set $x = r\cos\theta$, $y = r\sin\theta$

Then,

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$
$$= u_r r_x + u_\theta \theta_x$$

And,

$$\frac{\partial^2 u}{\partial x^2} = u_{xx} = u_{rr}r_x^2 + u_r r_{xx} + u_{\theta\theta}\theta_x^2 + u_\theta\theta_{xx}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = u_{yy} = u_{rr}r_y^2 + u_r r_{yy} + u_{\theta\theta}\theta_y^2 + u_\theta\theta_{yy}$$

Since, we have

$$x = r\cos\theta, \qquad y = r\sin\theta$$

So, $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Therefore,

$$r_x = \frac{x}{r}, \quad r_{xx} = \left(\frac{x}{r}\right)_x = \frac{r - x\,r_x}{r^2} = \frac{r - x.x/r}{r^2} = \frac{r^2 - x^2}{r^3}$$

i.e. $r_x = \frac{x}{r}$, $r_{xx} = \frac{y^2}{r^3}$

Also,

$$r_y = \frac{y}{r} \quad \text{and} \quad r_{yy} = \frac{x^2}{r^3}$$

And,

$$\theta_x = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2 + y^2} = -\frac{y}{r^2}$$

So, $\theta_{xx} = \left(-\frac{y}{r^2}\right)_x = -y\left(\frac{-2}{r^3}.r_x\right) = \frac{2y}{r^3}.\frac{x}{r} = \frac{2xy}{r^4}$

Similarly,

$$\theta_y = +\frac{x}{r^2} \quad \text{and} \quad \theta_{yy} = \frac{-2xy}{r^4}$$

Now,

$$\frac{\partial^2 u}{\partial x^2} = u_{rr}\left(\frac{x}{r}\right)^2 + u_r\left(\frac{y^2}{r^3}\right) + u_{\theta\theta}\left(\frac{-y}{r^2}\right)^2 + u_\theta\left(\frac{-2xy}{r^4}\right)$$

And,

$$\frac{\partial^2 u}{\partial y^2} = u_{rr}\left(\frac{y}{r}\right)^2 + u_r\left(\frac{x^2}{r^3}\right) + u_{\theta\theta}\left(\frac{x}{r^2}\right)^2 + u_\theta\left(\frac{-2xy}{r^4}\right)$$

Thus,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{rr}\left(\frac{x^2 + y^2}{r^2}\right) + u_r\left(\frac{x^2 + y^2}{r^3}\right) + u_{\theta\theta}\left(\frac{x^2 + y^2}{r^4}\right) + u_\theta . 0$$

$$= u_{rr}\left(\frac{r^2}{r^2}\right) + u_r\left(\frac{r^2}{r^3}\right) + u_{\theta\theta}\left(\frac{r^2}{r^4}\right)$$

$$= u_{rr} + \frac{1}{r}u_r + u_{\theta\theta}.\frac{1}{r^2}$$

This is required form.

***Note:*** The Laplace equation is polar form be,

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

being, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

***Note:*** The Laplacian in polar form in three dimensional geometry be,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$$

Being $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$

Such form is also known as in cylindrical coordinate form.



## Solution of two dimensional wave equation in circular membrane under certain conditions

Consider two dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{.............. (1)}$$

In polar form,

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) \quad \text{.............. (2)}$$

Suppose that the circular membrane has radius B and is radically symmetrical function u(r, t). That means u does not depends up on θ. So that

$$\frac{\partial^2 u}{\partial \theta^2} = 0$$

Then (2) becomes,

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial y}\right) \quad \text{.............. (3)}$$

Consider (3) satisfies the conditions

$u(R, t) = 0$, for all $t \geq 0$. .............. (4)

$u(r, 0) = f(r)$, $u\,t(r,0) = g(r)$ .............. (5)

Let,

$u(r, t) = F(r)\,G(t)$ .............. (6)

be the solution of (3) then (6) becomes,

$$F\ddot{G} = c^2\left(GF'' + \frac{1}{r}GF'\right)$$

This gives,

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} + \frac{F'}{rF} = -k^2 \text{ (say)}$$

That gives,

$\ddot{G} + \lambda^2 G = 0$ .............. (7)

And $F'' + \frac{F'}{r} + k^2 F = 0$ .............. (8)

where, $\lambda = ck$

Since (7) is second order differential equation whose solution be,

$G_m(t) = A_m\cos\lambda_m t + B_m\sin\lambda_m t$ .............. (9)

And, for (8), set $\frac{1}{r} = \frac{k}{s}$. So that,

$$F' = \frac{\partial F}{\partial r} = \frac{\partial F}{\partial s}\frac{\partial s}{\partial r} = k\frac{\partial F}{\partial s}$$

And

$$F'' = \frac{\partial^2 F}{\partial r} = \frac{\partial}{\partial s}\left(\frac{\partial F}{\partial r}\right)\frac{\partial s}{\partial r} = k^2\frac{\partial^2 F}{\partial s^2}$$

Then (8) becomes,

$$k^2\frac{\partial^2 F}{\partial s^2}+\frac{1}{s}\frac{\partial F}{\partial s}+F=0 \qquad \text{.................. (10)}$$

This is Bessel's equation with n = 0.

Clearly, (10) has solution $J_0$ and $Y_0$ where $J_0$ and $Y_0$ are Bessel's functions of first and second kind, respectively.

Since, at n = 0, $Y_0$ has no finite value. That means $Y_0$ has no meaningful solution as f(r) is finite in (5).

Therefore,

$$F(r)=J_0(s)=J_0(kr)$$

Be the solution of (10).

At boundary, F(R) = 0, being $G \neq 0$ and by (4)

So, s has infinite roots. Say,

$$s=\alpha_m \qquad \text{for } m=1, 2, 3, \text{.........}$$

$$\Rightarrow \quad k_m=\frac{\alpha m}{R} \qquad \text{for } m=1, 2, 3, \text{......... being } s=kR$$

Thus, $F_m(r)=J_0\left(\frac{r\,\alpha\, m}{R}\right)$ for m = 1, 2, ..............

Hence, (6) becomes,

$$u(r, t)=\sum_{m=1}^{\infty}(A_m\cos\lambda_m t+B_m\sin\lambda_m t)+J_0\left(\frac{r\,\alpha\, m}{R}\right) \qquad \text{.............. (11)}$$

By (5), u = (r, 0) = F(r) then, (11) gives,

$$f(r)=\sum_{m=1}^{\infty}A_m J_0\left(\frac{r\,\alpha\, m}{R}\right)$$

Which is Fourier Bessel series. So,

$$A_m=\frac{2}{R^2 J_1^2(\alpha m)}\int_0^R r\, g(r)\, J_0\left(\frac{r\,\alpha\, m}{R}\right)dr \qquad \text{.............. (13)}$$

for m = 1, 2,......

Thus, (11) is required solution with the value of $A_m$ and $B_m$ given in (12) and (13).

## *Potential function and its solution by spherical membrane*

**Potential function:**

If the Laplacian form is equal to zero then it called potential equation. That is the equation $\nabla^2 u = 0$ is called potential equation. And the solution of the equation is known as potential function.

**Solution:** Let $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$

Then,

$$\nabla^2 u=\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)+\frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}\right]=0 \qquad \text{.............. (1)}$$

Consider the boundary condition, $u(R, \theta, \phi) = f(\phi)$



Then, $u_{\theta\theta} = 0$ as temperature is fixed.

So that (1) becomes,

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)\right]=0 \qquad \text{.............. (2)}$$

At infinity, the potential will be zero, that is

$$\lim_{\Delta z\to\infty} u(r, \phi)=0 \qquad \text{.............. (3)}$$

Suppose that $u(r, \phi) = A(r)\,B(\phi)$ .............. (4)

Be solution of (2). Then we get,

$$B\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right)+\frac{A}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial B}{\partial\phi}\right)=0$$

$$\Rightarrow \quad \frac{1}{A}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right)=-\frac{1}{B\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial B}{\partial\phi}\right)=k \text{ (say)}$$

This gives,

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right)=kA \qquad \text{.............. (5)}$$

And,

$$\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial B}{\partial\phi}\right)+kB=0 \qquad \text{.............. (6)}$$

Set $\cos\phi = 2$ the, $1 - w^2 = \sin^2\phi$.

Also,

$$\frac{d}{dQ}=\frac{d}{dw}\left(\frac{dw}{d\phi}\right)=-\sin\phi\frac{d}{dw}$$

So that (6) becomes,

$$\frac{d}{dw}\left[(1-w^2)\frac{dB}{dw}\right]+n(n+1)B=0. \qquad \text{for } k=n(n+1)$$

$$\Rightarrow \quad (1-w^2)\frac{d^2B}{dw^2}-2w\frac{dB}{dw}+n(n+1)B=0$$

Which is Lagendre's equation that has solution,

$$B=P_n(w)=P_n(\cos\phi) \qquad \text{for } n\geq 0$$

Also, (5) becomes

$$r^2A''+2rA'-n(n+1)A=0$$

which is called Euler–cauchy equation.

set $A = r^\alpha$ then,

$$\alpha(\alpha-1)r^\alpha+2\alpha r^\alpha-n(n+1)^\alpha=0$$

$$\Rightarrow \quad \alpha(\alpha-1)+2\alpha-n(n+1)=0$$

$$\Rightarrow \quad \alpha(\alpha+1)-n(n+1)=0$$

This gives $\alpha = n$ and $\alpha = -(n+1)$

Thus, $A_n = r^n$ and $A_n^* = r^{-(n+1)}$

Hence, (4) becomes,

$$u(r, \phi)=AB$$

$$\Rightarrow \quad u_n(r,\phi) = c_n r^n P_n(\cos\phi) \quad \text{and} \quad u_n^*(r,\phi) = D_n r^{-(n+1)} P_n(\cos\phi)$$

So that,

$$u(r,\phi) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\phi) \quad \text{and} \quad u(r,\phi) = \sum_{n=0}^{\infty} D_n r^{-(n+1)} P_n(\cos\phi)$$

This is required solution.

## Exercise 8.4

**1. Find the deflection u(x, y, t) of the square membrane with a = b = 1 and c = 1. If the initial velocity is zero and the initial deflection is**

**(a) $0.1 \sin 3\pi x \sin 4\pi y$ (b) $kxy(1-x)(1-y)$**

**Solution:** Given that, a = b = c = 1.

(a) $g(x, y) = 0$, $f(x, y) = 0.1 \sin 3\pi x \sin 4\pi y$ **[2004 Spring Q. No. 4(b) OR]**

We have the solution of two dimensional wave equation is

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (B_{mn}\cos\lambda_{mn}t + B_{mn}^*\sin\lambda_{mn}t)\sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \quad \ldots (i)$$

where $B_{mn}^* = 0$, being $g(x, y) = 0$

Now,

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi}{a}x.\sin\frac{n\pi}{b}y\,dx\,dy$$

$$= 4\int_0^1\int_0^1 0.1\sin 3\pi x\sin 4\pi y\sin m\pi x\sin n\pi y\,dx\,dy$$

$$= 0.1\left[\int_0^1 2\sin 3\pi x\sin m\pi x\,dx\right]\left[\int_0^1 2\sin 4\pi y\sin n\pi y\,dy\right]$$

$$= 0.1\int_0^1 [\cos(3-m)\pi x - \cos(3+m)\pi x]\,dx$$

$$\int_0^1 [\cos(4-n)\pi y - \cos(4+n)\pi y]\,dy$$

$$= 0.1\left[\frac{\sin(3-m)\pi x}{(3-m)\pi} - \frac{\sin(3+m)\pi x}{(3+m)\pi}\right]_0^1\left[\frac{\sin(4-n)\pi y}{(4-n)\pi} - \frac{\sin(4+n)\pi y}{(4+n)\pi}\right]_0^1$$

$$= 0.1\left[\left\{\frac{\sin(3-m)\pi}{(3-m)\pi} - \frac{\sin(3+m)\pi}{(3+m)\pi}\right\}\right]\left[\left\{\frac{\sin(4-n)\pi}{(4-n)\pi} - \frac{\sin(4+n)\pi}{(4+n)\pi}\right\}\right]$$

$$= 0.1\left(\frac{\sin(3-m)\pi}{(3-m)\pi}\right)\left(\frac{\sin(4-n)\pi}{(4-n)\pi}\right)$$

Therefore,



$$B_{mn} = \frac{0.1}{\pi^2}\left(\frac{\sin(3-m)\pi}{(3-m)}.\frac{\sin(4-n)\pi}{(4-n)}\right) \text{ for } m, n = 1, 2, 3, 4, \ldots\ldots$$

Now, for $B_{34}$ the term has 0/0 form. So, using L'hopital rule then

$B_{34} = 0.1$ and others zero.

Now,

$$\lambda_{34} = c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} = c\pi\sqrt{\frac{3^2}{1^2}+\frac{4^2}{1^2}} = 5c\pi = 5\pi.$$

Thus, the required solution from Equation (i) is

$$u(x,y,t) = B_{34}\cos\lambda_{34}t\sin 3\pi x.\sin 4\pi y$$
$$= 0.1\cos 5\pi t\sin 3\pi x\sin 4\pi y$$

(b) $kxy(1-x)(1-y)$

**Solution:** Given,

$$f(x,y) = kxy(1-x)(1-y)$$
$$= k(x-x^2)(y-y^2)$$

And, $g(x, y) = 0$

The solution of 2 – D wave eq$^n$. is

$$u(x,y,t) = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin m\pi x.\sin n\pi y\,dx\,dy$$

$$= 4\left[\int_0^1\int_0^1 f(x,y)\sin m\pi x\sin n\pi y\,dx\,dy\right]$$

$$= 4k\left[\int_0^1 (x-x^2)\sin m\pi x\,dx\right]\left[\int_0^1 (y-y^2)\sin n\pi y\,dy\right]$$

$$= 4k\left[(x-x^2).\frac{-\cos m\pi x}{m\pi} - (1-2x).\frac{-\sin m\pi x}{m^2\pi^2} + (-2).\frac{\cos m\pi x}{m^3\pi^3}\right]_0^1$$

$$\left[(y-y^2).\frac{-\cos n\pi y}{n\pi} - (1-2y).\frac{-\sin n\pi y}{n^2\pi^2} + (-2).\frac{\cos n\pi y}{n^3\pi^3}\right]_0^1$$

$$= 4k\left(\frac{-\sin m\pi}{m^2\pi^2} - \frac{2\cos m\pi}{m^3\pi^3} + \frac{2}{m^3\pi^3}\right)\left(-\frac{\sin n\pi}{n^2\pi^2} - \frac{2\cos n\pi}{n^3\pi^3} + \frac{2}{n^3\pi^3}\right)_0^1$$

For any value of m, $\sin m\pi = 0$ $[\because m = 1, 2, 3, \ldots\ldots]$

Then,

$$B_{mn} = 4k\left(-\frac{2\cos m\pi}{m^3\pi^3} + \frac{2}{m^3\pi^3}\right)\left(-\frac{2\cos n\pi}{n^3\pi^3} + \frac{2}{n^3\pi^3}\right)$$

$$= \frac{16k}{m^3n^3\pi^6}(1-\cos m\pi)(1-\cos n\pi)$$

Again,

$1 - \cos m\pi = 0$, when m is even

$1 - \cos n\pi = 0$, when n is even

Now, when m and n are odd

$1 - \cos m\pi = 1 - (-1) = 2$

$1 - \cos n\pi = 1 - (-1) = 2$

Hence, $B_{mn} = \frac{16k}{m^3 n^3 \pi^6} \cdot 2 \cdot 2 \cdot = \frac{64k}{m^3 n^3 \pi^6}$

and, $\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} = 1\,\pi\sqrt{m^2 + n^2} = \pi\sqrt{m^2 + n^2}$

The required solution is

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{64k}{m^3 n^3 \pi^6} \cos(\sqrt{m^2+n^2}\,.\,t) \sin m\pi x \sin n\pi y,$$

for m and n are odd.

Thus,

$$u(x,y,t) = \frac{64k}{\pi^6} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^3 n^3} \cos(\pi\sqrt{m^2+n^2}\,.\,t) \sin m\pi x \sin n\pi y$$

for odd value of m and n.

**2. Find u(x, y, t) for the rectangular membrane with sides a and b with c = 1, if the initial velocity is zero and initial deflection is $\sin\frac{2\pi x}{a} \sin\frac{3\pi y}{b}$.**

**Solution:** Given initial deflection $f(x, y) = \sin\frac{2\pi x}{a} \sin\frac{3\pi y}{b}$ initial velocity, $g(x, y) = 0$.

The required solution is

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \cos \lambda_{mn} t \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

Since $B_{mn}^* = 0$ $[\because g(x, y) = 0]$

Here,

$$B_{mn} = \frac{4}{ab}\int_0^b \int_0^a f(x, y) \sin\frac{m\pi x}{a} \,.\, \sin\frac{n\pi y}{b}\, dx\, dy$$

$$= \frac{4}{ab}\int_0^b \int_0^a \sin\frac{2\pi x}{a} \,.\, \sin\frac{3\pi y}{b} \,.\, \sin\frac{m\pi x}{a} \,.\, \sin\frac{n\pi y}{b}\, dx\, dy$$

$$= \frac{1}{ab}\left[\int_0^a 2\sin\frac{2\pi x}{a} \,.\, \sin\frac{m\pi x}{a}\, dx\right]\left[\int_0^b 2\sin\frac{3\pi y}{b} \,.\, \sin\frac{n\pi y}{b}\, dy\right]$$



$$= \frac{1}{ab}\int_0^a \left[\cos\frac{(2-m)}{a}\pi x - \cos\frac{(2+m)}{a}\pi x\right] dx$$

$$\int_0^a \left[\cos\frac{(3-n)}{b}\pi y - \cos\frac{(3+n)}{b}\pi y\right] dy$$

$$= \frac{1}{ab}\left\{\frac{\sin\left(\frac{2-m}{a}\right)\pi x}{\left(\frac{2-m}{a}\right)\pi} - \frac{\sin\left(\frac{2+m}{a}\right)\pi n}{\left(\frac{2+m}{a}\right)\pi}\right\}_0^a$$

$$\left\{\frac{\sin\left(\frac{3-n}{b}\right)\pi y}{\left(\frac{3-n}{b}\right)\pi} - \frac{\sin\left(\frac{3+n}{b}\right)\pi y}{\left(\frac{3+n}{b}\right)\pi}\right\}_0^b$$

On further calculation, we get

$B_{23} = 1$ and

$\lambda_{23} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} = \pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} = \pi\sqrt{\frac{4}{n^2} + \frac{a}{b^2}}$

Thus the required solution is

$$u(x, y, t) = \cos\left[\pi t\sqrt{\frac{4}{a^2} + \frac{9}{b^2}}\right] \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{3\pi y}{b}\right)$$

**3. Find the vibration of a rectangular membrane of sides a = 4ft and b = 2ft, if the tension is 12.5 lb/ft, the density is 2.5 slugs/ft², the initial velocity is 0 and the initial displacement is $f(x, y) = 0.1\,(4x - x^2)\,(2y - y^2)$ ft**

**[2007 Spring Q. No. 5(a) OR]**

**Solution:** Given that,

$f(x, y) = 0.1(4x - x^2)(2y - y^2)$ ft and $g(x, y) = 0$.

Also, $c^2 = \frac{T}{\rho} = \frac{12.5}{2.5} = 5$

Then, $B_{mn}^* = 0$ since $g(x, y) = 0$

Here,

$$B_{mn} = \frac{4}{4.2}\int_0^2 \int_0^4 0.1\,(4x - x^2)\,(2y - y^2) \sin\frac{m\pi x}{4} \,.\, \sin\frac{n\pi y}{2}\, dx\, dy$$

$$= \frac{1}{20}\int_0^4 (4x - x^2) \sin\frac{m\pi x}{4}\, dx \int_0^2 (2y - y^2) \sin\frac{n\pi y}{2}\, dy$$

Integrating first integral we get,

$$= \frac{128}{m^3\pi^3}[1-(-1)^m]$$

$$= \frac{256}{m^3\pi^3} \quad (\text{for m odd})$$

and for the second integral

$$= \frac{16}{n^3\pi^3}(1-(-1)^n]$$

$$= \frac{32}{\pi^3 n^3} \quad (\text{for n odd})$$

for even m or n we get zero value of the ingral.

Now,

$$B_{mn} = \frac{256 \times 32}{20\, m^3 n^3 \pi^6} \approx \frac{0.426050}{m^3 n^3} \quad (\text{for m \& n both odd})$$

Thus, the solution of the problem is

$$u(x, y, t) = 0.426050 \sum_{m, n\ odd} \sum \frac{1}{m^3 n^3} \cos\left(\frac{5\pi}{4}\sqrt{m^2+4n^2}\right)t \sin\frac{m\pi x}{4} \sin\frac{n\pi y}{2}$$

**OTHER IMPORTANT QUESTION FROM FINAL EXAM FOR PRACTICE**

### Problems related to one dimensional wave equation

## Theoretical Problems:

**2004 Spring Q. No. 4(b); 2011 Spring Q. No. 5(a); 2011 Fall Q. No. 3(a)**

Solve one dimensional, wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ under the conditions

(i) $u(0, t) = 0$ (ii) $u(l, t) = 0$ (iii) $u(x, 0) = f(x)$ and (iv) $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$

**OR 2008 Fall Q. No. 3(a)**

Show that the solutions of the one dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2}$ satisfying the boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$ is $u_n(x, t) = (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\left(\frac{n\pi}{L}\right)x \quad (n = 1, 2, 3, ...)$

**OR 2009 Spring Q. No. 4(a)**

Find u(x, t) from one dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$.



**2006 Spring Q. No. 4(a); 2007 Spring Q. No. 4(b); 2007 Fall Q. No. 4(a); 2009 Fall Q. No. 3(a); 2012 Fall Q. No. 4(a)**

Derive one dimensional wave equation of an elastic string which is tightly stretched to the length L and two ends are fixed.

**2006 Fall Q. No. 2(b)**

Derive one–dimensional wave equation. Solve $u_x + u_y = 0$ by using separation of variables.

**2007 Spring Q. No. 2(a) OR**

Solve one–dimensional wave equation completely.

## Problems

**2002 Q. No. 4(b); 2008 Spring Q. No. 4(b)**

Find u(x, t) of the string of length $l = \pi$ when $c^2 = 1$, the initial velocity is zero and the initial deflection is: $f(x) = \begin{cases} kx/a & \text{for } 0 < x < a \\ k(\pi - x)/\pi - a & \text{for } a < x < \pi \end{cases}$

**2003 Fall Q. No. 2(b)**

Find u(x, t) of vibrating string of length $L = \pi$, initial velocity $g(x) = 0$, $c^2 = 1$ and initial deflection is $0.1x(\pi - x)$.

**2004 Fall Q. No. 4(b) OR; 2005 Spring Q. No. 4(b); 2009 Fall Q. No. 3(b)**

Solve one dimensional wave equation with initial deflection $\frac{1}{100}\sin 3x$ and initial velocity if zero with length $L = \pi$ and $c^2 = 1$.

OR

Find the deflection u(x, t) of the string of length $L = \pi$ when $c^2 = 1$, the initial velocity is zero and the initial deflection is $0.01 \sin 3x$.

**2005 Fall Q. No. 4(a)**

Define partial differential equation. Write down one dimensional wave equation and heat equation and similarly two dimensional heat equation and wave equations. Solve one dimensional wave equation with initial deflection 0.01 sin3x and initial velocity zero and $L = \pi$, $c^2 = 1$.

**2007 Spring Q. No. 4(b) OR**

Find the solution of one dimensional wave equation with initial deflection $\frac{1}{2}\sin 3x + \sin x$ and initial velocity is zero.

**2011 Fall Q. No. 2(b)**

Find the solution of one dimensional wave equation corresponding to the triangular initial deflection: $f(x) = \frac{2K}{L}x$, if $0 < x < L/2$, $= \frac{2K}{L}(L - x)$, if $L/2 < x < L$; with its initial velocity zero and $c = 1$.

**2015 Fall Q. No. 6(a)**

Express the Laplacian $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ in polar co-ordinates. Solve one dimerisional wave equation wave equation completely.

**2016 Fall Q. No. 6(b)**

Find u(x, t) of the string of length $l = \pi$ when $c^2 = 1$, the initial velocity is zero and the initial deflection is $0.1 (\pi - x)$.

**2016 Spring Q. No. 4(a)**

Find the solution of one dimensional wave equation by using D'Alembert's method.

**2016 Spring Q. No. 5(a)**

A string of length 20 cm is fastened at both ends is displaced from its position of equilibrium by imparting to its points and initial velocity,

$$g(x) = \begin{cases} x & \text{if } 0 \le x \le 10 \\ 20 - x & \text{if } 10 \le x \le 20 \end{cases}$$

Find the deflection U(x, t)

**2017 Fall Q. No. 4(a)**

Derive one dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ with necessary assumptions.

## Problems related to one dimensional heat equation

## Theoretical Problems:

**2003 Fall Q. No. 3(a)**

Writing one–dimensional heat equation with corresponding initial and boundary conditions solve it by using separation of variables.

**OR 2011 Spring Q. No. 6(a)**

Write one–dimensional heat equation and solve it completely.

**2005 Spring Q. No. 4(a)**

Find the solution of one dimensional heat equation of a rod of length L. With boundary condition $u(0, t) = 0 = u(L, t)$ for all t and initial condition $u(x, 0) = f(x)$.

**OR 2007 Spring Q. No. 4(a)**

Find the solution of one dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, having zero temperature in end points and initial temperature f(x).



**2008 Spring Q. No. 4(a); 2011 Fall Q. No. 3(b)**

Derive one dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ with required assumption.

## PROBLEMS

**2002 Q. No. 4(a)**

A rod of length l with insulated sides is initially at a uniform temperature $u_0$. Its ends are suddenly cooled to 0°C and are kept at that temperature. Find the temperature function u(x, t).

**2004 Spring Q. No. 4(a)**

Find the temperature u(x, t) in a laterally insulated copper bar 80 cm long if the initial temperature is $100 \sin\left(\frac{\pi x}{80}\right)^\circ$C and the ends are kept at 0°C. How long will it take for the maximum temperature in the bar to drop to 50°C?

Physical data for Copper: Density: 8.92 gm/cm$^3$

Specific heat: 0.092 cal/gm°C Thermal conductivity: 0.95 cal/cm sec°C

**2016 Fall Q. No. 4(b)**

A homogeneous rod of conducting material of length 100 cm has its ends, kept at zero temperature and the initial temperature be defined by $f(x) = \begin{cases} x & \text{for } 0 \le x \le 50 \\ 100 - x & \text{for } 50 < x \le 100 \end{cases}$. Find the temperature u(x, y) at any time t.

**2016 Spring Q. No. 4(b)**

Find the temperature distribution in laterally insulated thin copper bar ($c^2 = 1.158$ cm$^2$/sec), 100 cm long and of constant thickness whose end points at x = 0 and x = 100 are kept at 0°C and initial temperature is $f(x) = \sin^3(0.01)\pi x$.

**2017 Fall Q. No. 6(b); 2016 Fall Q. No. 6(b)**

Obtain the solution of one dimensional heat equation completely.

## Problems related to two dimensional equation

## Theoretical Problems:

**2003 Fall Q. No. 3(b)**

Write the Laplacian in spherical coordination. Also write the potential function with boundary and condition at infinity and obtain Legendre equation from it.

**2004 Fall Q. No. 4(b)**

Solve the equation: $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial t^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$ where u(r, t) represents the vibration of the circular membrane in time t, under the conditions

(i) $u(R, t) = 0$ for all $t \ge \theta$ (ii) $u(r, 0) = f(r)$ (iii) $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r)$

**2005 Spring Q. No. 5(a) OR**

Define potential function and then find the solution of potential function by spherical membrane.

**2005 Fall Q. No. 4(b)**

Derive two dimensional heat equation, and solve $u_x - u_y = 0$ by using separation of variables.

**2005 Fall Q. No. 5(a); 2007 Spring Q. No. 5(a)**

What is Helmholtz's equation on F(x, y) and solve it subject to $F(0, y) = 0 = F(a, y) = F(x, 0) = F(x, b)$.

**OR**

Derive Helmholtz equation and then solve it using initial and boundary equation.

**2005 Fall Q. No. 5(a) OR**

Write down Laplacian in two dimensions. Express Laplacian polar coordinate. Write down Laplacian in cylindrical co–ordinate.

**OR 2005 Spring Q. No. 5(a)**

Write down Laplacian in polar co–ordinate and cylindrical ordinates.

**2006 Fall Q. No. 3(b); 2011 Spring 6(a) OR**

Define potential function and find its solution by spherical membrane.

**2008 Spring Q. No.5(a)**

Derive two dimensional heat equation and wave equation with required assumptions.

**2008 Spring Q. No. 5(a) OR**

Solve two dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$, in circular membrane.

**2008 Fall Q. No. 3(a) OR**

Show that the Laplacian in cylindrical polar coordinate is $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$

**2009 Fall Q. No. 3(b) OR**

Show that the Laplacian in uin polar coordinate is $\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$

**2009 Spring Q. No. 5(a)**

Find the solution of two dimensional wave equation in circular membrane with necessary initial and boundary condition.

**2012 Fall Q. No. 1(b)**

Write the wave equation for vibrating circular membrane together with its initial and boundary conditions and solve it.



**2016 Fall Q. No. 6(a)**

What is Helmholtz's equation of F(x, y) and solve it subject to $F(0, y) = 0 = F(a, y) = F(x, 0) = F(x, b)$.

**2016 Fall Q. No. 6(a) OR**

Find the deflection u(x, y, t) of the square membrane with a = b = 1 and c = 1, if the initial velocity is zero and the initial deflection is $(0.1)\sin 3\pi x \sin 4\pi y$.

**2016 Spring Q. No. 5(b)**

Derive two dimensional heat equation and solve completely.

## Separation

**2015 Fall Q. No. 5(a)**

Define partial differential equation with suitable example. By separating the variables solve $u_{xx} + u_{yy} = 0$.

**2015 Fall Q. No. 5(b)**

(b) Define partial differential equation with suitable example. By separating the variables solve $u_{xx} + u_{yy} = 0$.

**2017 Fall Q. No. 6(a)**

Solve $U_{xx} + U_{yy} = 0$

## SHORT QUESTIONS

**2004 Fall Q. No. 7(c)**

Write down the one–dimensional wave equation stating boundary and initial conditions. What is the Laplacian of u?

**2004 Spring 7(e)**

What is Laplacian of u? Write down the Laplacian of u in polar and cylindrical coordinates.

**2008 Fall Q. No. 7(d); 2009 Fall Q. No. 7(d)**

State the two dimensional wave equations in polar form.

**2007 Spring Q. No. 7(a)**

Solve $U_{xy} + U_x = 0$.

**2007 Fall Q. No. 7(c)**

Solve $u_x + u_y = 0$, by using separation of variables.

**2008 Spring Q. No. 7(e); 2011 Spring Q. No.7(c)**

Solve the partial differential equation $u_{xx} - u_{yy} = 0$, by using separation of variables methods.

**2015 Fall Q. No. 7(a)**

Solve the partial differential equation $u_x = 2xy$.

**2015 Fall Q. No. 7(c)**

Verify that $u = x^2 + t^2$ is the solution of one dimensional wave equation.

**2016 Fall Q. No. 7(a)**

Solve by using separation of variables $u_x - u_y = 0$.

□□□

# Unit 9 Z TRANSFORM AND ITS APPLICATION

**Definition (Z–Transform)**

Let f(t) be the function of time t defined in the discrete time invariant system 0, T, 2T, ..., then the Z–Transform of f(t) is defined by

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT)z^{-n}.$$

where z be the complex number in the region where $\left|\frac{1}{z}\right| < R$ and $t = nT$.

**Maclaurin's series and Binomial expansion gives**

1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
2. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$
3. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$
4. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
5. $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$
6. $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$
7. $\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ for $|x| < 1$
8. $\log(1-x) = -\sum_{n=0}^{\infty} \frac{x^n}{n}$
9. $\frac{a}{1-r} = a + ar + ar^2 + \ldots$ for $|r| < 1$ and $a \neq 0$.
10. $(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots$ for $|x| < 1$
11. $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots$
12. $\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - \ldots$

**Some functions:**

1. ***Unit step function: (Unit Heaviside Function)***

A function f(x) is called unit step function if it satisfies,

$$f(x) = \begin{cases} 1 & \text{for } x = 0, 1, 2, 3, \ldots \\ 0 & \text{for } x < 0 \end{cases}$$

OR $$f_a(x) = f(x-a) = \begin{cases} 1 & \text{for } x < a \\ 0 & \text{for } x > a \end{cases} \quad \text{with } x \geq 0.$$

2. ***Unit Ramp function:***

A function f(x) is called unit Ramp function if it satisfies the condition,

$$f(x) = \begin{cases} x & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

3. ***Polynomial function:***
A function f(x) is called polynomial function if it defined as

$$f(x) = \begin{cases} a^x & \text{for} \quad x = 0, 1, 2, \ldots\ldots \\ 0 & \text{for} \quad x < 0 \end{cases}$$

where, a is a constant.

4. ***Exponential function:***
An exponential function f(x) is defined as,

$$f(x) = \begin{cases} e^{-ax} & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 \end{cases}$$

where, a be a constant.

5. ***Sinusoidal function:***
A sinusoidal function f(x) is defined as,

$$f(x) = \begin{cases} \sin wx & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 \end{cases}$$

6. ***Unit impulse or Direc–delta function:***
A function f(x) is called unit impulse function, is defined as,

$$f(x) = \begin{cases} 1 & \text{for} \quad x = 0 \\ 0 & \text{for} \quad x \neq 0 \end{cases}$$

## Linearity property of Z–Transform

**Statement:** ***If f(t) and g(t) are any two function of t in the discrete time period 0, T, 2T, ... then,*** **Z [af(t) + bg(t) = a Z[f(t)] + b Z[g(t)],** ***where a and b are two constants.***

**Proof:** Let f(t) and g(t) are any two function of t in the discrete time period 0, T, 2T,……then,

$$Z\,[af(t) + bg(t) = az\,[f(t)] + bg(t)]\,z^{-n}$$

$$= a \sum_{n=0}^{\infty} f(t)\,z^{-n} + b \sum_{n=0}^{\infty} g\,(t)\,z^{-n}$$

$$= a\,z[f(t)] + b\,z[g(t)]$$

This proves that Z–Transform is linear.

## First Shifting Theorem

**Statement: If** $Z[f(t)] = F(z)$ **then** $Z[e^{-at}\,f(t)] = F(e^{aT}z) = [f(z)]_{z \to z e^{aT}}$.

**Proof:** Let,

$$Z\,[f(t)] = F(z) = \sum_{n=0}^{\infty} f(nT)\,z^{-n}$$

Now,

$$Z\,[e^{-at}\,f(t)] = \sum_{n=0}^{\infty} e^{-at}\,f(t)\,z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-anT}\,f(nT)\,z^{-n} = \sum_{n=0}^{\infty} f(nT)\,(e^{aT}z)^{-n} = F[ze^{aT}] = [f(z)]_{z \to ze^{aT}}$$

Thus, $Z\,[e^{-at}\,f(t)] = [f(z)]_{z \to ze^{aT}}$.



## Second Shifting Theorem:

**Statement: If** $Z[f(t)] = F(z)$ **then** $Z[f(t + T)] = z\,[F(z) - f(0)]$

**Proof:** Let

$$Z[f(t)] = F(z) = \sum_{n=0}^{\infty} f(t)\,z^{-n} = \sum_{n=0}^{\infty} f(nT)z^{-n}$$

Now,

$$Z\,[f(t + T)] = \sum_{n=0}^{\infty} f(t + T)\,z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT + T)\,z^{-n}$$

$$= \sum_{n=0}^{\infty} f[(n + 1)T]\,z^{-n}$$

Put, n + 1 = k then,

$$= \sum_{k=1}^{\infty} f(kT)\,z^{-(k-1)}$$

$$= z \sum_{k=1}^{\infty} f(kT)\,z^{-k}$$

$$= z \left[ \sum_{n=0}^{\infty} f(kT)z^{-k} - f(0) \right] = z\,[F(z) - f(0)].$$

Thus, $Z\,[f(t + T)] = z\,[F(z) - f(0)]$.

**Theorem 1: Show that** $Z\{t^k\} = -zT \dfrac{d}{dz}[Z\{t^{k-1}\}]$

**Proof:** We have,

$$Z\{t^k\} = \sum_{n=0}^{\infty} t^k\,z^{-n} = \sum_{n=0}^{\infty} (nT)^k\,z^{-n} \quad \text{and} \quad Z\{t^{k-1}\} = \sum_{n=0}^{\infty} (nT)^{k-1}\,z^{-n}$$

Here,

$$\frac{d}{dz}[Z\{t^{k-1}\}] = \sum_{n=0}^{\infty} (nT)^{k-1} \frac{d}{dz}(z^{-n})$$

$$= \sum_{n=0}^{\infty} (-n)\,(nT)^{k-1}\,z^{-(n+1)}$$

$$= \left(\frac{-1}{zT}\right) \sum_{n=0}^{\infty} (nT)^k\,z^{-n} = \left(\frac{-1}{zT}\right) Z\{t^k\}$$

Thus, $Z\{t^k\} = -zT \dfrac{d}{dz}[Z\{t^{k-1}\}]$.

**Theorem 2: Show that** $Z\{a^n f(t)\} = F\left(\frac{z}{a}\right) = (F(z))_{z\to z/a}$ **where** $Z\{f(t)\} = F(z)$.

**Proof:** We have.

$$Z\{f(t)\} = F(z) = \sum_{n=0}^{\infty} f(t)\, z^{-n} = \sum_{n=0}^{\infty} f(nT) z^{-n}.$$

Now,

$$Z\{a^n f(t)\} = \sum_{n=0}^{\infty} a^n f(t) z^{-n} = \sum_{n=0}^{\infty} f(t)\left(\frac{z}{a}\right)^{-n} = F\left(\frac{z}{a}\right) = (F(z))_{z\to z/a}.$$

**Theorem 3: If f(n) = 0 for n< 0 and if Z{f(t)} = F(z) then** $Z\{f(t-kT)\} = z^{-k} F(z)$ **for n > 0, k < 0.**

**Proof:** Let, Z{f(t)} = F(z).

Now,

$$Z\{f(t-kT)\} = \sum_{n=0}^{\infty} f(t-kT)\, z^{-n} = \sum_{n=0}^{\infty} f[(n-k)T]\, z^{-n}$$

Put, n–k = m then,

$$Z\{f(t-kT)\} = \sum_{m=-k}^{\infty} f(mT)\, z^{-(m+k)}$$

$$= z^{-k} \sum_{m=0}^{\infty} f(mT)\, z^{-m} \qquad [\because f(n) = 0, \text{ for all } n < 0]$$

$$= z^{-k} Z\{f(t)\}.$$

Thus, $Z\{f(t-kT)\} = z^{-k} Z\{f(t)\} = z^{-k} F(z)$.

**Theorem: If** $Z\{y_n\} = \bar{y}$ **then** $Z\{y_{n+k}\} = z^k\left[\bar{y} - \sum_{m=0}^{k-1}\left(\frac{y_m}{z^m}\right)\right]$.

**Proof:** Let, $Z\{y_n\} = \bar{y}$.

Now,

$$Z\{y_{n+k}\} = \sum_{n=0}^{\infty} y_{n+k}\, z^{-n} = z^k \sum_{n=0}^{\infty} y_{n+k}\, z^{-(n+k)}$$

Put, n + k = m then,

$$= z^k \sum_{m=k}^{\infty} y_m z^{-m}$$

$$= z^k\left(\sum_{m=0}^{\infty} y_m z^{-m} - \sum_{m=0}^{k-1} y_m z^{-m}\right)$$

$$= z^k\left(\bar{y} - \sum_{m=0}^{k-1}\left(\frac{y_m}{z^m}\right)\right).$$



**Initial value theorem:**

**If Z{f(t)} = F(z) then,** $f(0) = \lim_{z\to\infty} F(z)$ **and** $\lim_{z\to\infty} z F(z) = f(1)$ **as f(0) = 0.**

**Proof:** Let, $Z\{f(t)\} = F(z) = \sum_{n=0}^{\infty} f(t)\, z^{-n}$

$$= f(0) + f(1) z^{-1} + f(2) z^{-2} + \ldots\ldots$$

So,

$$\lim_{z\to\infty} F(z) = f(0) \qquad [\because 1/z \to 0 \text{ as } z \to \infty]$$

Also, set f(0) = 0 then,

$$F(z) = f(1) z^{-1} + f(2) z^{-2} + \ldots\ldots$$

So, $z F(z) = f(1) + f(2) z^{-1} + \ldots\ldots$

Then, $\lim_{z\to\infty} z F(z) = f(1)$.

**Final value theorem:**

**If Z{f(t)} = F(z) then,** $\lim_{t\to\infty} f(t) = \lim_{z\to 1} \{(z-1) F(z)\}$

**Proof:** Let, Z{f(t)} = F(z) and we have $Z\{f(t+T)\} = z(F(z) - f(0))$.

Now,

$$\lim_{z\to 1} [z F(z) - z f(0) - F(z)]$$

$$= \lim_{z\to 1} [Z\{f(t+T)\} - F(z)]$$

$$= \lim_{z\to 1}\left(\sum_{n=0}^{\infty} f(t+T)\, z^{-n} - F(z)\right)$$

$$= \lim_{z\to 1} \sum_{n=0}^{\infty} [f(nT+T) - f(nT)]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] \qquad [\because z^{-n} \to 1 \text{ as } z \to 1]$$

$$= \lim_{n\to\infty} [f(T) + f(2T) + f(3T) + \ldots - f(0) - f(T) - f(2T) - \ldots]$$

$$= \lim_{n\to\infty} f[(n+1)T] - f(0)$$

$$= f(\infty) - f(0)$$

$$\Rightarrow \lim_{z\to 1} (z-1) F(z) - f(0) = \lim_{t\to\infty} f(t) - f(0)$$

$$\Rightarrow \lim_{z\to 1} (z-1) F(z) = \lim_{t\to\infty} f(t).$$

**Convolution of functions:**

Let f(t) and g(t) are any two functions. Then, the convolution of the functions is denoted by f∗g and defined as

$$(f*g)(n) = \sum_{k=0}^{\infty} f(kT)\, g[(n-k)\,T]$$

**Convolution Theorem**

**If F(z) and G(z) are Z–Transform of f(t) and g(t) respectively then**
$$Z[f(t)*g(t)] = F(z)\,G(z).$$

**Proof:** Let, F(z) and G(z) are Z–Transform of f(t) and g(t) respectively. Then,

$$F(z)\,G(z) = \sum_{n=0}^{\infty} f(kt)\, z^{-n} \sum_{n=0}^{\infty} g\,(mT)\, z^{-n}$$
$$= [f(0) + f(T)z^{-1} + f(2T)z^{-2} + \ldots]\;[g(0) + g(T)z^{-1} + g(2T)z^{-2} + \ldots]$$
$$= [f(0)\,g(0)] + [f(0)\,g(T) + f(T)\,g(0)]z^{-1} + [f(0)g(2T) + f(T)g(T) + f(2T)\,g(0)]z^{-2} + \ldots$$
$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty} f(kT)g[(n-k)T]\right] z^{-n}$$
$$= \sum_{n=0}^{\infty} (f*g)(t)z^{-n}$$
$$= Z\{f*g\}$$

Thus, $Z[f(t)*g(t)] = F(z)\,G(z)$.

## Exercise 9.1

**1. Obtain the Z–Transform of**

**(i) $Z\{e^n\}$**

**Solution:** Since we have,

$$Z\{a^n f(t)\} = (Z\{F(z)\})_{z\to z/a} \quad \text{for } F(z) = Z\{f(t)\}.$$

Now,

$$Z\{e^n\} = (Z\{1\})_{z\to z/e} = \left(\frac{z}{z-1}\right)_{z\to z/e} = \frac{z/e}{(z/e)-1} = \frac{z}{z-e}$$

**(ii) $n\,3^n$**

**Solution:** Since we have,

$$Z\{a^n\} = \frac{z}{z-a} \quad \text{and} \quad Z\{n\,f(t)\} = -z\frac{d}{dz}Z\{f(t)\}.$$

Now,

$$Z\{n\,3^n\} = -z\frac{d}{dz}Z\{3^n\} = -z\frac{d}{dz}\left(\frac{z}{z-3}\right) = -z\left(\frac{(z-3)-z}{(z-3)^2}\right) = \frac{3z}{(z-3)^2}.$$

**(iii) $r^n \cos n\theta$ (iv) $r^n \sin n\theta$**

**Solution:** Since we have,

$$Z\{a^n\} = \frac{z}{z-a}$$



Now,

$$Z\{r^n e^{in\theta}\} = Z\{(re^{i\theta})^n\} = \frac{z}{z-re^{i\theta}}$$
$$= \frac{z}{z - r\cos\theta - i\,r\sin\theta} \times \frac{z - r\cos\theta + ir\sin\theta}{z - r\cos\theta + ir\sin\theta}$$
$$= \frac{z(z - r\cos\theta) + izr\sin\theta}{(z - r\cos\theta)^2 + r^2\sin^2\theta}$$
$$= \frac{z(z - r\cos\theta) + izr\sin\theta}{z^2 - 2rz\cos\theta + r^2}$$
$$\Rightarrow z\{r^n(\cos n\theta + i\sin n\theta)\} = \frac{z(z - r\cos\theta)}{z^2 - 2rz\cos\theta + r^2} + i\frac{zr\sin\theta}{z^2 - 2rz\cos\theta + r^2}.$$

Equating real and imaginary part then,

$$Z\{r^n\cos n\theta\} = \frac{z(z - r\cos\theta)}{z^2 - 2rz\cos\theta + r^2} \quad \text{and} \quad Z\{r^n\sin n\theta\} = \frac{zr\sin\theta}{z^2 - 2rz\cos\theta + r^2}.$$

**(v) n**

**Solution:** Since we have,

$$Z\{n\,f(t)\} = -z\frac{d}{dz}(Z\{f(t)\}) \quad \text{and} \quad Z\{1\} = \frac{z}{z-1}$$

Now,

$$Z\{n\} = -z\frac{d}{dz}(z\{1\})$$
$$= -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = -z\left[\frac{z-1-z}{(z-1)^2}\right] = \frac{z}{(z-1)^2}$$

**(vi) $\frac{1}{n+2}$.**

**Solution:** Let $f(t) = \frac{1}{n+2}$

Now,

$$Z\{f(t)\} = \sum_{n=0}^{\infty}\left(\frac{1}{n+2}\right)z^{-n}$$
$$= \frac{1}{2} + \frac{1}{3z} + \frac{1}{4z^2} + \frac{1}{5z^3} + \ldots\ldots$$
$$= z^2\left[\frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \frac{1}{5s^5} + \ldots\ldots\right] + z - z$$
$$= z^2\left[\frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \frac{1}{5s^5} + \ldots\ldots\right] - z$$
$$= -z^2\log(1 - 1/z) - z$$
$$= -z - z^2.\log(z - 1/z)$$

**(vii)** $\left\{\frac{1-e^{-bt}}{b}\right\}$

**Solution:** Since we have,

$$Z\{1\} = \frac{z}{z-1} \quad \text{and} \quad Z\{a^n\} = \frac{z}{z-a}$$

Now,

$$Z\left\{\frac{1-e^{-bt}}{b}\right\} = -\frac{1}{b}\,[Z\{1\} - Z\{e^{-bt}\}]$$

$$= \frac{1}{b}\left[\frac{z}{z-1} - Z\{(e^{-bT})^n\}\right] \quad \text{for } t = nT.$$

$$= \frac{1}{b}\left[\frac{z}{z-1} - \frac{z}{z-e^{-bT}}\right].$$

**(viii)** $t^2 e^{-bt}$

**Solution:** Since we have,

$$Z\{t^k\} = -zT\frac{d}{dz}(Z\{t^{k-1}\}), \qquad Z\{1\} = \frac{z}{z-1}$$

and $Z\{e^{at} f(t)\} = (Z\{f(t)\})_{z\to ze^{-aT}}$

Now,

$$Z\{t^2 e^{-bt}\} = (Z\{t^2\})_{z\to ze^{bT}}$$

$$= \left(-zT\frac{d}{dz}Z\{t\}\right)_{z\to ze^{bT}}$$

$$= \left[-zT\frac{d}{dz}\left\{-zT\frac{d}{dz}(Z\{1\})\right\}\right]_{z\to ze^{bT}}$$

$$= \left[-zT\frac{d}{dz}\left\{-zT\frac{d}{dz}\left(\frac{z}{z-1}\right)\right\}\right]_{z\to ze^{bT}}$$

$$= \left[zT^2\frac{d}{dz}\left\{z\frac{-1}{(z-1)^2}\right\}\right]_{z\to ze^{bT}}$$

$$= \left[zT^2\left(\frac{(z-1)^2(-1)-(-2z)(z-1)}{(z-1)^4}\right)\right]_{z\to ze^{bT}}$$

$$= T^2\left[z\left(\frac{-z+1+2z}{(z-1)^3}\right)\right]_{z\to ze^{bT}}$$

$$= T^2\left[\frac{z^2+z}{(z-1)^3}\right]_{z\to ze^{bT}}$$

$$= T^2\left[\frac{z^2e^{2bT}+ze^{bT}}{(ze^{bT}-1)^3}\right]$$

$$= T^2e^{-bT}\left[\frac{z(z+e^{-bT})}{(z-e^{-bT})^3}\right]$$



**(ix)** $\cos 4\pi n$

**Solution:** Since we have,

$$Z\{a^n\} = \frac{z}{z-a} \text{ and } e^{4i\pi n} = \cos 4\pi n + i\sin 4\pi n = 1 \text{ for n is integer.}$$

[∵ $\cos 4\pi = 1$ and $\sin 4\pi = 0$]

Here,

$$Z\{e^{4i\pi n}\} = \frac{z}{z-e^{4i\pi}} = \frac{z}{z-1}$$

$$\Rightarrow Z\{\cos 4\pi n + i\sin 4\pi n\} = \frac{z}{z-1}$$

Comparing the real part,

$$Z\{\cos 4\pi n\} = \frac{z}{z-1}$$

**(x)** $\cos\left(\frac{n\pi}{3}\right)$

**Solution:** Since we have,

$$Z\{a^n\} = \frac{z}{z-a}$$

Now,

$$Z\{e^{in\pi/3}\} = \frac{z}{z-e^{i\pi/3}} = \frac{z}{z-\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}}$$

$$= \frac{z}{z-\frac{1}{2}-\frac{i\sqrt{3}}{2}}$$

$$= \frac{2z}{2z-1-i\sqrt{3}}\times\frac{2z-1+i\sqrt{3}}{2z-1+i\sqrt{3}}$$

$$= \frac{2z(2z-1)+i2\sqrt{3}z}{(2z-1)^2+3}$$

$$= \frac{2z(2z-1)+i.2\sqrt{3}z}{4z^2-4z+4}$$

$$= \frac{z(2z-1)+iz\sqrt{3}}{2z^2-2z+2}$$

$$\Rightarrow Z\left\{\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{2}\right\} = \frac{z(2z-1)+iz\sqrt{3}}{2z^2-2z+2}$$

Comparing the real part then,

$$Z\left\{\cos\frac{n\pi}{3}\right\} = \frac{z(2z-1)}{2z^2-2z+2}$$

**(xi)** $\sin 4n$

**Solution:** Since, $\sin 4n = \text{Im}(e^{i4n})$

We have, $Z\{a^n\} = \frac{z}{z-a}$

Now.

$$Z\{e^{i4n}\} = z\{(e^{4i})^n\} = \frac{z}{z - e^{i4}} = \frac{z}{z - \cos4 - i\sin4} \times \frac{z - \cos4 + i\sin4}{z - \cos4 + i\sin4}$$

$$= \frac{z(z - \cos4 + i\sin4)}{(z - \cos4)^2 + \sin^2 4}$$

$$= \frac{z(z - \cos4) + iz\sin4}{z^2 - 2z\cos4 + 1}$$

$$\Rightarrow \quad Z\{\cos4n + i\sin4n\} = \frac{z(z - \cos4) + iz\sin4}{z^2 - 2z\cos4 + 1}$$

Comparing the imaginary part from both sides then,

$$Z\{\sin 4n\} = \frac{z\sin4}{z^2 - 2z\cos4 + 1}$$

**(xii) cosh 7n**

**Solution:** We know that,

$$\cosh 7n = \frac{e^{7n} + e^{-7n}}{2} = \frac{1}{2}[(e^7)^n + (e^{-7})^n]$$

Since we have, $Z\{a^n\} = \dfrac{z}{z-a}$

Now,

$$Z\{\cosh 7n\} = \frac{1}{2}[Z\{(e^7)^n\} + Z\{(e^{-7})^n\}]$$

$$= \frac{1}{2}\left[\frac{z}{z - e^7} + \frac{z}{z - e^{-7}}\right]$$

$$= \frac{z}{2}\left[\frac{2z - e^7 - e^{-7}}{(z - e^7)(z - e^{-7})}\right]$$

$$= \frac{z}{2}\left[\frac{2z - 2\cosh7}{z^2 - z(e^7 + e^{-7}) + e^7 e^{-7}}\right]$$

$$= \frac{z(z - \cosh7)}{z^2 - 2z\cosh7 + 1} \qquad \left[\because \cosh 7 = \frac{e^{7n} + e^{-7n}}{2}\right]$$

Thus,

$$Z\{\cosh 7n\} = \frac{z(z - \cosh7)}{z^2 - 2z\cosh7 + 1}$$

**(xiii) sinh 8n**

**Solution:** Since, $\sinh 8n = \dfrac{e^{8n} - e^{-8n}}{2}$ and process as in (xii).

**(xiv) $e^{bn}$**

**Solution:** Since we have, $Z\{a^n\} = \dfrac{z}{z-a}$

Now,

$$Z\{e^{bn}\} = Z\{(e^b)^n\} = \frac{z}{z - e^b}$$

**(xv) $t^3 e^{-bt}$**

**Solution:** Since we have,



$$Z\{t^k f(t)\} = (-1)^k \left(z\frac{d}{dz}\right)^k [z\{f(t)\}] \text{ and } \qquad Z\{a^n\} = \frac{z}{z-a}$$

Now.

$$Z\{t^3 e^{-bt}\} = Z\{t^3 e^{-bnT}\}$$

$$= (-1)^3 T\left(z\frac{d}{dz}\right)^3 Z\{(e^{-bT})^n\}$$

$$= -T^2\left(z\frac{d}{dz}\right)^3 \left[z\frac{d}{dz}\left(\frac{z}{z - e^{-bT}}\right)\right]$$

$$= -T^2\left(z\frac{d}{dz}\right)^2 \left[z\left(\frac{-e^{-bT}}{(z - e^{-bT})^2}\right)\right]$$

$$= -T^3(-e^{-bT})\left(z\frac{d}{dz}\right)\left[z\frac{d}{dz}\left(\frac{z}{(z - e^{-bT})^2}\right)\right]$$

$$= T^3 e^{-bT}\left(z\frac{d}{dz}\right)\left[z\left(\frac{(z - e^{-bT})^2 - z\,2(z - e^{-bT})}{(z - e^{-bT})^4}\right)\right]$$

$$= T^3 e^{-bT} z\frac{d}{dz}\left[\frac{z^2 - ze^{-bT} - 2z^2}{(z - e^{-bT})^3}\right]$$

$$= T^3 \frac{ze^{-bT}[(z - e^{-bT})^3(2z - e^{-bT} - 4z) - (z^2 - e^{-bT}.z - 2z^2).3(z - e^{-bT})^2]}{(z - e^{-bT})^6}$$

$$= T^3 \frac{ze^{-bT}[(z - e^{-bT})(2z - e^{-bT} - 4z) - 3z^2 + 3e^{-bT}z + 6z^2]}{(z - e^{-bT})^4}$$

$$= T^3 \frac{ze^{-bT}[2z^2 - ze^{-bT} - 4z^2 - 2ze^{-bT} + e^{-2bT} + 4ze^{-bT} - 3z^2 + 3e^{-bT}z + 6z^2]}{(z - e^{-bT})^4}$$

$$= T^3 \frac{ze^{-bT}[z^2 + e^{-2bT} + 4ze^{-bT}]}{(z - e^{-bT})^4}$$

$$= T^3 \frac{ze^{bT}[z^2 e^{2bT} + 4ze^{bT} + 1]}{(ze^{bT} - 1)^4}$$

**(xvi) $e^{-at} - e^{-bt}$**

**Solution:** Since we have,

$$Z\{a^n\} = \frac{z}{z-a}$$

Now,

$$z\{e^{-at} - e^{-bt}\} = Z\{e^{-at}\} - Z\{e^{-bt}\}$$

$$= \frac{z}{z - e^{-aT}} - \frac{z}{z - e^{-bT}} \qquad \text{Being } t = nT$$

$$= \frac{z(z - e^{-bT}) - z(z - e^{-aT})}{(z - e^{-aT})(z - e^{-bT})}$$

$$= \frac{z(e^{-aT} - e^{-bT})}{(z - e^{-aT})(z - e^{-bT})}$$

$$= \frac{z(e^{bT} - e^{aT})}{(ze^{aT} - 1)(ze^{bT} - 1)}$$

**(xvii) $(1 - bt)e^{-bt}$**

**Solution:** Since we have,

$$Z\{e^{-aT} f(t)\} = (Z\{f(t)\})_{z\to ze^{aT}} \quad \text{and} \quad Z\{t\,f(t)\} = -Tz\frac{d}{dz}(Z\{f(t)\}),\ Z\{1\} = \frac{z}{z-1}$$

Now,

$$Z\{(1-bt)\,e^{-bt}\} = (Z\{1-bt\})_{z\to ze^{aT}}$$
$$= (Z\{1\} - bZ\{t\})_{z\to ze^{aT}}$$
$$= \left[\frac{z}{z-1} + bTz\frac{d}{dz}(Z\{1\})\right]_{z\to ze^{aT}}$$
$$= \left[\frac{z}{z-1} + bTz\frac{d}{dz}\left(\frac{z}{z-1}\right)\right]_{z\to ze^{aT}}$$
$$= \left[\frac{z}{z-1} + bTz\left(\frac{-1}{(z-1)^2}\right)\right]_{z\to ze^{aT}}$$
$$= \frac{ze^{bT}}{ze^{bT}-1} - \frac{bTze^{bT}}{(ze^{bT}-1)^2}$$
$$= \frac{ze^{bT}(ze^{bT}-1) - bTze^{bT}}{(ze^{bT}-1)^2}$$
$$= \frac{e^{-2bT}ze^{bT}(ze^{bT}-1-Tb)}{(z-e^{-bT})^2}$$
$$= \frac{z(z-(1+Tb)\,e^{-bT})}{(z-e^{-bT})^2}$$

**(xviii) $n\,a^{n-1}$**

**Solution:** Since we have,

$$Z\{n\,f(t)\} = -z\frac{d}{dz}(Z\{f(t)\}),\ Z\{a^n\} = \frac{z}{z-a}$$

Now,

$$Z\{n\,a^{n-1}\} = a^{-1}Z\{n\,a^n\}$$
$$= -a^{-1}z\frac{d}{dz}(Z\{a^n\})$$
$$= -a^{-1}z\frac{d}{dz}\left(\frac{z}{z-a}\right) = -a^{-1}z\left(\frac{-a}{(z-a)^2}\right) = \frac{z}{(z-a)^2}$$

**(xix) $n^2a^{n-1}$ for $n \geq 1$**

**Solution:** Since we have,

$$Z\{n\,f(t)\} = -z\frac{d}{dz}(Z\{f(t)\}) \quad \text{and} \quad Z\{a^n f(t)\} = (Z\{f(t)\})_{z\to z/a},\ Z\{1\} = \frac{z}{z-1}$$

Now, for $n \geq 1$,

$$Z\{n^2a^{n-1}\} = -z\frac{d}{dz}(Z\{n\,a^{n-1}\})$$
$$= -z\frac{d}{dz}\left(\frac{z}{(z-a)^2}\right)$$
$$= -z\left(\frac{(z-a)^2 - z\,2(z-a)}{(z-a)^4}\right)$$



$$= -z\left(\frac{-a-z}{(z-a)^3}\right)$$
$$= \frac{z(z+a)}{(z-a)^3}$$

**(xx) $[3(2)^n - 4(3)^n]\,u(n)$**

**Solution:** Since we have,

$$Z\{a^n f(t)\} = (Z\{f(t)\})_{z\to z/a} \quad \text{and} \quad Z\{u(n)\} = \frac{z}{z-1}$$

Now,

$$Z\{[3(2)^n - 4(3)^n]\,u(n)\} = 3[Z\{u(n)\}]_{z\to z/2} - 4[Z\{u(n)\}]_{z\to z/3}$$
$$= 3\left(\frac{z}{z-1}\right)_{z\to z/2} - 4\left(\frac{z}{z-1}\right)_{z\to z/3}$$
$$= 3\left(\frac{z}{z-2}\right) - 4\left(\frac{z}{z-3}\right)$$
$$= \frac{3z(z-3) - 4z(z-2)}{(z-2)(z-3)}$$
$$= \frac{z(3z-9-4z+8)}{(z-2)(z-3)}$$
$$= \frac{-z(z+1)}{(z-2)(z-3)}$$

**(xxi) $\sin w_0 n.\ u(n)$**

**Solution:** Since we have,

$$Z\{e^{-at} f(t)\} = (Z\{f(t)\})_{z\to ze^{aT}} \quad \text{and} \quad Z\{u(n)\} = \frac{z}{z-1}$$

Now,

$$Z\{\sin w_0 n.\ u(n)\} = Z\left\{\frac{e^{iw_0 n} - e^{-iw_0 n}}{2i}\,u(n)\right\}$$
$$= \frac{1}{2i}\left[(z\{u(n)\})_{z\to ze^{-iw_0}} - (z\{u(n)\})_{z\to ze^{iw_0}}\right]$$
$$= \frac{1}{2i}\left[\left(\frac{z}{z-1}\right)_{z\to ze^{-iw0}} - \left(\frac{z}{z-1}\right)_{z\to ze^{iw0}}\right]$$
$$= \frac{1}{2i}\left[\frac{z}{z-e^{iw0}} - \frac{z}{z-e^{-iw0}}\right]$$
$$= \frac{1}{2i}\left[\frac{z(e^{iw0} - e^{-iw0})}{z^2 - z(e^{iw0} + e^{-iw0}) + 1}\right]$$
$$= \frac{z\sin w_0}{z^2 - 2z\cos w_0 + 1}$$

**(xxii) $\delta(n-k)$ for $k > 0$.**

**Solution:** Since we have, $\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$

Now, for $k > 0$,

$$Z\{\delta(n-k)\} = \sum_{n=0}^{\infty} \delta(n-k)\, z^{-n} = z^{-k} \quad \text{being } \delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$$

**(xxiii) (1 + n) u(n)**

**Solution:** Since we have,

$$Z\{n\, f(t)\} = -z \frac{d}{dz}(Z\{f(t)\}) \quad \text{and} \quad Z\{u(n)\} = \frac{z}{z-1}$$

Now,

$$Z\{(1+n)\, u(n)\} = Z\{u(n)\} + Z\{n.u(n)\}$$

$$= \frac{z}{z-1} - z\frac{d}{dz}(Z\{u(n)\})$$

$$= \frac{z}{z-1} - z\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= \frac{z}{z-1} - z\left(\frac{-1}{(z-1)^2}\right)$$

$$= \frac{z(z-1)+z}{(z-1)^2}$$

$$= \frac{z^2}{(z-1)^2}$$

**(xxiv) $[4(3)^n - (2)^n]\, u(n)$**

**Solution:** Similar to (xx).

**2. Find the inverse Z–Transform of**

**(i) $\frac{z}{(z-1)(z^2+1)}$**

**Solution:** Let,

$$F(z) = \frac{z}{(z-1)(z^2+1)} = \frac{z}{(z-1)(z-i)(z+i)}$$

Then,

$$z^{n-1} F(z) = \frac{z^n}{(z-1)(z-i)(z+i)} \qquad \text{..........(i)}$$

Clearly the poles of $z^{n-1}F(z)$ are $z = 1$, $z = i$ and $z = -i$.

Since we know that,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1} F(z) \qquad \text{..........(ii)}$$

Here, residue of $z^{n-1}F(z)$ at $z = 1$ is

$$\operatorname*{Res}_{z=1} z^{n-1} F(z) = \lim_{z\to 1} (z-1)\, z^{n-1} F(z)$$

$$= \lim_{z\to 1} \frac{z^n}{(z-i)(z+i)} = \lim_{z\to 1} \frac{z^n}{z^2+1} = \frac{1^n}{1+1} = \frac{1}{2}$$

And,

$$\operatorname*{Res}_{z=i} z^{n-1} F(z) = \lim_{z\to i} (z-i)\, z^{n-1} F(z)$$



$$= \lim_{z\to i} \frac{z^n}{(z-1)(z+i)} = \frac{i^n}{(i-1)(i+i)} = \frac{i^n}{(i-1)(2i)}$$

Also,

$$\operatorname*{Res}_{z=-i} z^{n-i} F(z) = \lim_{z\to -i} (z+i)\, z^{n-1} F(z)$$

$$= \lim_{z\to -i} \frac{z^n}{(z-1)(z-i)} = \frac{(-i)^n}{(-i-1)(-2i)}$$

Now (ii) becomes,

$$Z^{-1}\{F(z)\} = \frac{1}{2} + \frac{i^n}{(i-1)\,2i} + \frac{(-1)^n i^n}{2i\,(i+1)}$$

$$= \frac{1}{2} + \frac{i^n(i+1) + (-i)^n(i-1)}{2i\,(i^2-1)}$$

$$= \frac{1}{2} + \frac{i^{n+1} + i^n + (-1)^n i^{n+1} - i^n}{-4i}$$

$$= \frac{1}{2} + \frac{i^{n+1}(1+(-1)^n)}{-4i}$$

$$= \frac{1}{2} + \frac{i^n(1+(-1)^n)}{-4}$$

$$= \frac{1}{2} + \frac{(1+(-1)^n)\left[\cos\left(\frac{n\pi}{2}\right) + i\sin\left(\frac{n\pi}{2}\right)\right]}{-4}$$

$$= \frac{1}{2}\left[1 - \left(\cos\left(\frac{n\pi}{2}\right) + i\sin\left(\frac{n\pi}{2}\right)\right)\right]$$

**(ii) $\frac{z^2-3z}{(z-5)(z+2)}$** **[2003 Fall Q. No. 4(b)]**

**Solution:** Let

$$F(z) = \frac{z^2-3z}{(z-5)(z+2)}$$

Then,

$$z^{n-1} F(z) = \frac{z^{n+1} - 3z^n}{(z-5)(z+2)} \qquad \text{..........(1)}$$

Clearly $z^{n-1}F(z)$ has poles at $z = 5$ and $z = -2$

Since we have,

$$z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1} F(z) \qquad \text{.......(2)}$$

Here, residue of $z^{n-1}F(z)$ at $z = 5$ is

$$\operatorname*{Res}_{z=5} z^{n-1} F(z) = \lim_{z\to 5} (z-5)\, z^{n-1} F(z)$$

$$= \lim_{z\to 5} \frac{z^n(z-3)}{z+2} = \frac{5^n(5-3)}{5+2} = \frac{2}{7}(5^n)$$

And residue of $z^{n-1}F(z)$ at $z = -2$ is,

$$\operatorname*{Res}_{z=-2} z^{n-1} F(z) = \lim_{z\to(-2)} (z+2)\, z^{n-1} F(z)$$

$$= \lim_{z\to(-2)} \frac{z^n(z-3)}{z-5} = \frac{(-2)^n(-2-3)}{-2-5} = -\frac{5(-2)^n}{7}$$

Now, (1) becomes,

$$Z^{-1}\{F(z)\} = \frac{2}{7}(5^n) - \left(\frac{5}{7}\right)(-2)^n = \frac{1}{7}[2(5)^n - 5(-2)^n].$$

(iii) $\frac{z}{(z-1)(z-2)}$

**Solution:** Let,

$$F(z) = \frac{z}{(z-1)(z-2)} \quad ........(1)$$

Then,

$$z^{n-1}F(z) = \frac{z^n}{(z-1)(z-2)} \quad .........(2)$$

Since we have,

$$z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z) \quad .........(3)$$

Clearly, (2) has poles at z = 1 and z = 2.

Here, residue of (2) at z = 1 is,

$$\operatorname*{Res}_{z=1} z^{n-1}F(z) = \lim_{z\to1}(z-1)z^{n-1}F(z)$$

$$= \lim_{z\to1}\left(\frac{z^n}{z-2}\right) = \frac{1^n}{1-2} = -1^n$$

And residue of (2) at z = 2 is,

$$\operatorname*{Res}_{z=2} z^{n-1}F(z) = \lim_{z\to2}(z-2)z^{n-1}F(z)$$

$$= \lim_{z\to2}\left(\frac{z^n}{z-1}\right) = \frac{2^n}{2-1} = 2^n$$

Now (3) becomes,

$$Z^{-1}\{F(z)\} = -1^n + 2^n$$

$$\Rightarrow Z^{-1}\{F(z)\} = 2^n - 1^n = 2^n - 1.$$

(iv) $\frac{z^2-4z}{(z-2)^2}$

**Solution:** Let,

$$F(z) = \frac{z^2-4z}{(z-2)^2}$$

Then,

$$z^{n-1}F(z) = \frac{z^{n+1}-4z^n}{(z-2)^2}$$

Clearly, $z^{n-1}F(z)$ pole at z = 2 of order 2.

So, residue of $z^{n-1}F(z)$ at z = 2 is,

$$\operatorname*{Res}_{z=2} z^{n-1}F(z) = \lim_{z\to2}\frac{d}{dz}[(z-2)^2 z^{n-1}F(z)]$$



$$= \lim_{z\to2}\frac{d}{dz}(z^{n+1}-4z^n)$$

$$= \lim_{z\to2}[(n+1)z^n - 4nz^{n-1}]$$

$$= (n+1)2^n - 4n\,2^{n-1}$$

$$= (n+1)2^n - 2n.2^n$$

$$= 2^n(n+1-2n)$$

$$= (1-n)2^n$$

Since we know that,

$Z^{-1}\{F(z)\}$ = sum of residues of $z^{n-1}F(z)$ at z = 2 of all orders.

So, $Z^{-1}\{F(z)\} = (1-n)2^n$

(v) $\frac{z^2-z}{(z-2)(z-3)}$

**Solution:** Let,

$$F(z) = \frac{z^2-z}{(z-2)(z-3)}$$

Then,

$$z^{n-1}F(z) = \frac{z^{n+1}-z^n}{(z-2)(z-3)}$$

Clearly the form $z^{n-1}F(z)$ has poles at z = 2 and z = 3 of simple order.

Here, residue of $z^{n-1}F(z)$ at z = 2 is,

$$\operatorname*{Res}_{z=2} z^{n-1}F(z) = \lim_{z\to2}(z-2)z^{n-1}F(z)$$

$$= \lim_{z\to2}\frac{z^{n+1}-z^n}{z-3} = \frac{2^{n+1}-2^n}{2-3} = \frac{2^n(2-1)}{-1} = -2^n$$

And the residue of $z^{n-1}F(z)$ at z = 3 is,

$$\operatorname*{Res}_{z=3} z^{n-1}F(z) = \lim_{z\to3}(z-3)z^{n-1}F(z)$$

$$= \lim_{z\to3}\frac{z^{n+1}-z^n}{z-2} = \frac{3^{n+1}-3^n}{3-2} = \frac{3^n(3-1)}{1} = 2(3^n)$$

Since we have,

$Z^{-1}\{F(z)\}$ = sum of residues of $z^{n-1}F(z)$

Therefore, $Z^{-1}\{F(z)\} = 2(3^n) - 2^n$

(vi) $\frac{z}{z^2-3z+2}$

**Solution:** Let, $F(z) = \frac{z}{z^2-3z+2} = \frac{z}{(z-2)(z-1)}$

Repeated question to Q.(iii)

(vii) $\frac{10z+5}{(z-1)(z-1/5)}$

**Solution:** Let,

$$F(z) = \frac{10z + 5}{(z-1)(z-1/5)}$$

Then,

$$z^{n-1} F(z) = \frac{10z^n + 5z^{n-1}}{(z-1)(z-1/5)}$$

Clearly the function $z^{n-1} F(z)$ has poles at $z = 1$ and at $z = 1/5$ of single order.

Here, the residue of $z^{n-1} F(z)$ at $z = 1$ is,

$$\underset{z=1}{\text{Res}}\ z^{n-1} F(z) = \lim_{z\to 1} (z-1) z^{n-1} F(z)$$

$$= \lim_{z\to 1} \frac{10z^n + 5z^{n-1}}{(z-1/5)} = \frac{10(1)^n + 5(1)^{n-1}}{1 - 1/5} = \frac{10+5}{4/5} = \frac{75}{4}$$

And the residue of $z^{n-1} F(z)$ at $z = \frac{1}{5}$ is,

$$\underset{z=1/5}{\text{Res}}\ z^{n-1} F(z) = \lim_{z\to 1/5} (z-1/5) z^{n-1} F(z)$$

$$= \lim_{z\to 1/5} \left(\frac{10z^n + 5z^{n-1}}{z-1}\right)$$

$$= \frac{10(1/5)^n + 5(1/5)^{n-1}}{1/5 - 1}$$

$$= \frac{1}{5^{n-1}}\left[\frac{10(1/5) + 5}{-4/5}\right] = \frac{1}{5^{n-2}}\left(\frac{7}{-4}\right) = \frac{-7}{4(5^{n-2})}$$

Since we have,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1} F(z).$$

Therefore, $Z^{-1}\{F(z)\} = \frac{1}{4}\left[75 - \frac{7}{5^{n-2}}\right]$

**(viii)** $\frac{z+1}{z^2 - 2z + 1}$

**Solution:** Here,

$$F(z) = \frac{z+1}{z^2 - 2z + 1} = \frac{z+1}{(z-1)^2}$$

Then, $z^{n-1} F(z) = \frac{z^n + z^{n-1}}{(z-1)^2}$

Clearly $z^{n-1} F(z)$ has poles at $z = 1$ of order 2.

Here, residue of $z^{n-1} F(z)$ at $z = 1$ is,

$$\lim_{z\to 1} z^{n-1} F(z) = \lim_{z\to 1} \frac{d}{dz}\left((z-1)^2 \frac{z^n + z^{n-1}}{(z-1)^2}\right)$$

$$= \lim_{z\to 1} [nz^{n-1} + (n-1) z^{n-2}] = n + n - 1 = 2n - 1$$

Since we know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1} F(z).$$

Therefore, $Z^{-1}\{F(z)\} = 2n - 1$



**(ix)** $\frac{z(z+2)}{(z-1)^2}$

**Solution:** Let, $F(z) = \frac{z(z+2)}{(z-1)^2}$

Then $z^{n-1} F(z) = \frac{z^{n+1} + 2z^n}{(z-1)^2}$

Clearly $z^{n-1} F(z)$ has at $z = 1$ of order of 2.

Here, the residue of $z^{n-1} F(z)$ at $z = 1$ is,

$$\lim_{z\to 1} z^{n-1} F(z) = \lim_{z\to 1} \frac{d}{dz}\left((z-1)^2 \frac{z^{n+1} + 2z^n}{(z-1)^2}\right)$$

$$= \lim_{z\to 1} [(n+1)z^n + 2n z^{n-1}] = 3n + 1$$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1} F(z)$$

Therefore, $Z^{-1}\{F(z)\} = 3n+1$

**(x)** $\frac{z+2}{(z-2) z^2}$

**Solution:** Let, $F(z) = \frac{z+2}{(z-2) z^2}$

Then $z^{n-1} F(z) = \frac{z^n + 2n^{n-1}}{(z-2) z^2}$

Clearly $z^{n-1} F(z)$ has at $z = 2$ of simple pole and at $z = 0$ of order of 2.

Here, the residue of $z^{n-1} F(z)$ at $z = 2$ is

$$\underset{z=2}{\text{Res}}\ z^{n-1} F(z) = \lim_{z\to 2} \frac{z^n + 2z^{n-1}}{z^2}$$

$$= \frac{2^n + 2^{n-1} \cdot 2}{2^2} = \frac{2 \cdot 2^n}{2^2} = 2^{n-1}$$

And, the residue of $z^{n-1} F(z)$ at $z = 0$ is

$$\underset{z=0}{\text{Res}}\ z^{n-1} F(z) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{z^n + 2z^{n-1}}{z-2}\right)$$

$$= \lim_{z\to 0}\left(\frac{(nz^{n-1} + 2(n-1) z^{n-2})(z-2) - z^n - 2z^{n-1}}{(z-2)^2}\right)$$

$$= 0$$

Since we know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1} F(z)$$

Therefore, $Z^{-1}\{F(z)\} = 2^{n-1} + 0 = 2^{n-1}$

**(xi)** $\frac{z}{(z+1)^2}$

**Solution:** Let $F(z) = \frac{z}{(z+1)^2}$

Then, $z^{n-1}F(z) = \frac{z^n}{(z+1)^2}$

Clearly $z^{n-1}F(z)$ has pole at $z = -1$ of order 2.

Then, the residue of $z^{n-1}F(z)$ at $z = -1$ be,

$$\underset{z=-1}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to -1} \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\{z^{n-1}F(z)(z+1)^2\}$$

$$= \lim_{z\to -1} \frac{d}{dz}\{z^n\}$$

$$= \lim_{z\to -1} nz^{n-1} = n(-1)^{n-1}$$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z)$$

Therefore, $Z^{-1}\{F(z)\} = n(-1)^{n-1}$.

**(xii)** $\frac{1}{(z-2)(z-3)}$

**Solution:** Let, $F(z) = \frac{1}{(z-2)(z-3)}$

Then, $z^{n-1}F(z) = \frac{z^{n-1}}{(z-2)(z-3)}$

Clearly $z^{n-1}F(z)$ has simple poles at $z = 2$ and $z = 3$.

Here the residue of $z^{n-1}F(z)$ at $z = 2$ is

$$\underset{z=2}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to 2}(z-2)z^{n-1}F(z) = \lim_{z\to 2}\frac{z^{n-1}}{z-3} = -(2)^{n-1}$$

And, the residue of $z^{n-1}F(z)$ at $z = 3$ is

$$\underset{z=3}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to 3}(z-3)z^{n-1}F(z) = \lim_{z\to 3}\frac{z^{n-1}}{z-2} = (3)^{n-1}$$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z)$$

Therefore, $Z^{-1}\{F(z)\} = (3)^{n-1} - (2)^{n-1}$

**(xiii)** $\frac{2z^2+3z}{(z+2)(z-4)}$

**Solution:** Let,

$$F(z) = \frac{2z^2+3z}{(z+2)(z-4)}$$

Then, $z^{n-1}F(z) = \frac{2z^{n+1}+3z^n}{(z+2)(z-4)}$

Clearly $z^{n-1}F(z)$ has simple poles at $z = -2, 4$.

Here, residue of $z^{n-1}F(z)$ at $z = -2$ is,

$$\underset{z=-2}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to -2}(z+2)z^{n-1}F(z)$$



$$= \lim_{z\to -2}\frac{2z^{n+1}+3z^n}{z-4}$$

$$= \frac{2(-2)^{n+1}+3(-2)^n}{-6} = \frac{(-2)^n(-4+3)}{-6} = \frac{1}{6}(-2)^n$$

And, residue of $z^{n-1}F(z)$ at $z = 4$ is,

$$\underset{z=4}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to 4}(z-4)z^{n-1}F(z)$$

$$= \lim_{z\to 4}\frac{2z^{n+1}+3z^n}{z+2} = \frac{2(4)^{n+1}+3\cdot 4^n}{6} = \frac{11}{6}(4)^n$$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z)$$

Therefore, $Z^{-1}\{F(z)\} = \frac{11}{6}(4)^n + \frac{1}{6}(-2)^n$

**(xiv)** $\frac{2z^2-5z}{(z-2)(z-3)}$

**Solution:** Let,

$$F(z) = \frac{2z^2-5z}{(z-2)(z-3)}$$

Then, $z^{n-1}F(z) = \frac{2z^{n+1}-5z^n}{(z-2)(z-3)}$

Clearly $z^{n-1}F(z)$ has simple poles at $z = 2$ and 3.

Here, residue of $z^{n-1}F(z)$ at $z = 2$ is,

$$\underset{z=2}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to 2}(z-2)z^{n-1}F(z)$$

$$= \lim_{z\to 2}\frac{2z^{n+1}-5z^n}{(z-3)} = \frac{2(2^{n+1})-5(2^n)}{-1} = \frac{2^n(4-5)}{-1} = 2^n$$

And, residue of $z^{n-1}F(z)$ at $z = 3$ is,

$$\underset{z=3}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to 3}(z-3)z^{n-1}F(z)$$

$$= \lim_{z\to 3}\frac{2z^{n+1}-5z^n}{(z-2)} = \frac{2(3)^{n+1}-5(3^n)}{1} = \frac{3^n(6-5)}{1} = 3^n$$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z)$$

Therefore, $Z^{-1}\{F(z)\} = 2^n + 3^n$.

**(xv)** $\frac{2z^3+z}{(z-2)^2(z-1)}$

**Solution:** Here,

$$F(z) = \frac{2z^3+z}{(z-2)^2(z-1)}$$

Then, $z^{n-1}F(z) = \frac{2z^{n+2}+z^n}{(z-2)^2(z-1)}$

Clearly, $z^{n-1}F(z)$ has simple poles at z = 1 and at z = 2 of degree 2.

Here, residue of $z^{n-1}F(z)$ at z = 1 is.

$$\underset{z=1}{\text{Res}}\ z^{n-1}F(z) = \lim_{z\to 1}(z-1)z^{n-1}F(z)$$

$$= \lim_{z\to 1}(z-1)\left(\frac{2z^{n+2}+z^n}{(z-1)(z-2)^2}\right) = \frac{2+1}{(1-2)^2} = 3.$$

And, residue of $z^{n-1}F(z)$ at z = 2 is,

$$\underset{z=2}{\text{Res}}\ z^{n-1}F(z) = \lim_{z\to 2}(z-2)z^{n-1}F(z)$$

$$= \lim_{z\to 2}\frac{d}{dz}[(z-2)^2 z^{n-1}F(z)]$$

$$= \lim_{z\to 2}\frac{d}{dz}\left(\frac{2z^{n+2}+z^n}{z-1}\right)$$

$$= \lim_{z\to 2}\left[\frac{\{2(n+2)z^{n+1}+nz^{n-1}\}(z-1)-\{2z^{n+2}+z^n\}}{(z-1)^2}\right]$$

$$= 2(n+2)2^{n+1}+n(2)^{n-1}-2(2)^{n+2}-2^n$$

$$= 2^{n-1}(8n+16+n-16-2)$$

$$= 2^{n-1}(9n-2)$$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z)$$

Therefore, $Z^{-1}\{F(z)\} = 9n(2)^{n-1} - 2^n + 3$

**(xvi)** $\dfrac{z}{(z-1)^2(z-2)}$

**Solution:** Let,

$$F(z) = \frac{z}{(z-1)^2(z-2)}$$

Then,

$$z^{n-1}F(z) = \frac{z^n}{(z-1)^2(z-2)}$$

Clearly $z^{n-1}F(z)$ has simple pole at z = 2 and z = 1 of degree 2.

Here, residue of $z^{n-1}F(z)$ at z = 2 is,

$$\underset{z=2}{\text{Res}}\ z^{n-1}F(z) = \lim_{z\to 2}((z-2)z^{n-1}F(z)) = \lim_{z\to 2}\frac{z^n}{(z-1)^2} = 2^n$$

And, residue of $z^{n-1}F(z)$ at z = 1 is,

$$\underset{z=1}{\text{Res}}\ z^{n-1}F(z) = \lim_{z\to 1}\frac{d}{dz}((z-1)^2 z^{n-1}F(z))$$

$$= \lim_{z\to 1}\frac{d}{dz}\left(\frac{z^n}{z-2}\right) = \lim_{z\to 1}\frac{nz^{n-1}(z-2)-z^n}{(z-2)^2} = -n-1$$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z)$$

Therefore, $Z^{-1}\{F(z)\} = 2^n - n - 1$



**(xvii)** $\dfrac{3z^2+2z+1}{z^2+3z+2}$

**Solution:** Let,

$$F(z) = \frac{3z^2+2z+1}{z^2+3z+2}$$

Then, $$z^{n-1}F(z) = \frac{3z^{n+1}+2z^n+z^{n-1}}{(z+1)(z+2)}$$

Clearly $z^{n-1}F(z)$ has simple poles at z = –1, –2.

Here, residue of $z^{n-1}F(z)$ at z = –1 is,

$$\underset{z=-1}{\text{Res}}\ z^{n-1}F(z) = \lim_{z\to -1}(z+1)z^{n-1}F(z)$$

$$= \lim_{z\to -1}\left(\frac{3z^{n+1}+2z^n+z^{n-1}}{z+2}\right)$$

$$= 3(-1)^{n+1}+2(-1)^n+(-1)^{n-1}$$

$$= (-1)^{n-1}\{3-2+1\}$$

$$= 2(-1)^{n-1}$$

And, residue of $z^{n-1}F(z)$ at z = –2 is,

$$\underset{z=-2}{\text{Res}}\ z^{n-1}F(z) = \lim_{z\to -2}\left(\frac{3z^{n+1}+2z^n+z^{n-1}}{z+1}\right)$$

$$= \frac{3(-2)^{n+1}+2(-2)^n+(-2)^{n-1}}{-1}$$

$$= -(-2)^{n-1}\{12-4+1\}$$

$$= -9(-2)^{n-1}$$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z)$$

Therefore, $Z^{-1}\{F(z)\} = -9(-2)^{n-1} + 2(-1)^{n-1}$

**(xviii)** $\dfrac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$

**Solution:** Let,

$$F(z) = \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$$

Then, $$z^{n-1}F(z) = \frac{z^n(1-e^{-aT})}{(z-1)(z-e^{-aT})}$$

Clearly, $z^{n-1}F(z)$ has simple poles at z = 1 and $e^{-aT}$.

Here, residue of $z^{n-1}F(z)$ at z = 1 is,

Now,

$$\underset{z=1}{\text{Res}}\ z^{n-1}F(z) = \lim_{z\to 1}(z-1)z^{n-1}F(z) = \lim_{z\to 1}\frac{z^n(1-e^{-aT})}{z-e^{-aT}} = 1$$

And, residue of $z^{n-1}F(z)$ at $z = e^{-aT}$ is,

$$\underset{z=e^{-aT}}{\text{Res}}\ z^{n-1}F(z) = \lim_{z\to e^{-aT}}\frac{z^n(1-e^{-aT})}{z-1} = -e^{-aTn}$$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z)$$

Therefore, $Z^{-1}\{F(z)\} = 1 - e^{-anT} = 1 - e^{-an}$.

(xix) $\dfrac{z^{-2}}{(1-z^{-1})^3}$

**Solution:** Let,

$$F(z) = \frac{z^{-2}}{(1-z^{-1})^3} = \frac{z}{(z-1)^3}$$

Then,

$$z^{n-1}F(z) = \frac{z^n}{(z-1)^3}$$

Clearly, $z^{n-1}F(z)$ has pole at $z = 1$ of degree 3.

Here, residue of $z^{n-1}F(z)$ at $z = 1$ is,

> residue of $z^{n-1}F(z)$ at $z = p$ of order r is,
>
> $$\underset{z=p}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to p}\frac{1}{(r-1)!}\frac{d^{r-1}}{dz^{r-1}}[(z-p)^r F(z)]$$

$$\underset{z=1}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to 1}\frac{1}{(3-1)!}\frac{d^2}{dz^2}[z^n]$$

$$= \frac{1}{2}n(n-1)\lim_{z\to 1}(z^{n-2}) = \frac{1}{2}(n^2-n)$$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z)$$

Therefore, $Z^{-1}\{F(z)\} = \frac{1}{2}(n^2 - n)$

(xx) $\dfrac{z^{-1}(1-z^{-2})}{(1+z^{-2})^2}$

**Solution:** Let,

$$F(z) = \frac{z^{-1}(1-z^{-2})}{(1+z^{-2})^2} = \frac{z^4(z^2-1)}{z^3(z^2+1)^2} = \frac{z(z^2-1)}{(z^2+1)^2} = \frac{z^3-z}{(z-i)^2(z+i)^2}$$

Clearly, F(z) has poles at $z = i$ of order 2 and at $z = -i$ of order 2.

Here, residue of $z^{n-1}F(z)$ at $z = i$ of order 2 is,

$$\underset{z=i}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to i}\frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}(z^{n-1}F(z)(z-i)^2)$$

$$= \lim_{z\to i}\frac{d}{dz}\left(\frac{z^{n+2}-z^n}{(z+i)^2}\right)$$

$$= \lim_{z\to i}\frac{(z+i)^2[(n+2)z^{n+1} - nz^{n-1}] - (z^{n+2}-z^n)\,2(z+i)}{(z+i)^4}$$

$$= \frac{(2i)^2[(n+2)\,i^{n+1} - ni^{n-1}] - (i^{n+2}-i^n)\,2.\,2i}{(2i)^4}$$

$$= \frac{-2n\,i^{n+1} - 4\,i^{n+1} - 2n\,i^{n+1} + 4\,i^{n+1} + 4\,i^{n+1}}{16}$$



$$= \frac{i^{n+1}(4-4n)}{16} = \frac{i^{n+1}(1-n)}{4}$$

And residue of $z^{n-1}F(z)$ at $z = -i$ of order 2 be,

$$\underset{z=-i}{\text{Res}}\, z^{n-1}F(z) = \frac{(-i)^{n+1}(1-n)}{4} \quad \text{[as above]}$$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z)$$

Therefore, $Z^{-1}\{F(z)\} = \dfrac{i^{n+1}(1-n)}{4} + \dfrac{(-i)^{n+1}(1-n)}{4}$

$$= \frac{i^{n+1}}{4}(1-n)[1+(-1)^{n+1}]$$

$$= \frac{i^{2n+1}}{2}(1-2n)$$

$$= \frac{(-1)^n i}{2}(1-2n).$$

(xxi) $\dfrac{z^2+z+2}{(z-1)(z^2-z+1)}$

**Solution:** Let,

$$F(z) = \frac{z^2+z+2}{(z-1)(z^2-z+1)}$$

$$= \frac{z^2+z+2}{(z-1)\left(z^2-2.\frac{1}{2}.z+\frac{1}{4}-\frac{1}{4}+1\right)}$$

$$= \frac{z^2+z+2}{(z-1)\left(\left(z-\frac{1}{2}\right)^2-\frac{3}{4}i^2\right)}$$

$$= \frac{z^2+z+2}{(z-1)\left(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}$$

Then, $z^{n-1}F(z) = \dfrac{z^{n+1}+z^n+2z^{n-1}}{(z-1)\left(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}$

Clearly $z^{n-1}F(z)$ has poles at $z = 1, \frac{1}{2}+\frac{\sqrt{3}}{2}i$ and $\frac{1}{2}-\frac{\sqrt{3}}{2}i$ of simple order.

Here, residue of $z^{n-1}F(z)$ at $z = 1$ of order 1 is,

$$\underset{z=1}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to 1}\frac{z^{n-1}(z^2+z+2)}{z^2-z+1} = 1^{n-1}\frac{(1^2+1+2)}{1^2-1+1} = 4.$$

Also, residue of $z^{n-1}F(z)$ at $z = \frac{1}{2}+\frac{\sqrt{3}}{2}i$ of order 1 be,

$$\underset{z=(1/2)+i(\sqrt{3}/2)}{\text{Res}}\, z^{n-1}F(z)$$

$$= \lim_{z\to\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)} \frac{z^{n-1}(z^2+z+2)}{(z-1)\left(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}$$

$$= \frac{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{n-1}\left[\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2+\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)+2\right]}{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i-1\right)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}$$

Also, residue of $F(z)\, z^{n-1}$ at $z = \frac{1}{2} - \frac{\sqrt{3}}{4}i$ of simple order be,

$$\underset{z=(1/2)-i(\sqrt{3}/2)}{\text{Res}} F(z)\, z^{n-1}$$

$$= \lim_{z\to\frac{1}{2}-\frac{\sqrt{3}}{2}i} \frac{z^{n-1}(z^2+z+2)}{(z-1)\left(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)}$$

$$= \frac{\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{n-1}\left[\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^2+\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)+2\right]}{\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i-1\right)\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)}$$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z)$$

Therefore, $$Z^{-1}\{F(z)\} = 4 + \frac{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{n-1}\left[\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2+\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)+2\right]}{\frac{\sqrt{3}}{2}i-\left(\frac{\sqrt{3}}{2}i-\frac{1}{2}\right)} + \frac{\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{n-1}\left[\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^2+\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)+2\right]}{-\frac{\sqrt{3}}{2}i-\left(\frac{\sqrt{3}}{2}i-\frac{1}{2}\right)}$$

Since, $\frac{1}{2}+\frac{\sqrt{3}}{2}i = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$,

and $\frac{1}{2}-\frac{\sqrt{3}}{2}i = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$

*and process for book answer.*

**(xxii)** $\dfrac{z^{-3}}{(1-z^{-1})\left(1-\frac{1}{5}z^{-1}\right)}$

**Solution:** Let,

$$F(z) = \frac{z^{-3}}{(1-z^{-1})\left(1-\frac{1}{5}z^{-1}\right)} = \frac{1}{z(z-1)(z-1/5)}$$



Then,

$$z^{n-1}F(z) = \frac{z^{n-2}}{(z-1)(z-1/5)}$$

Clearly $z^{n-1}F(z)$ has poles are $z = 1, \frac{1}{5}$

Here, residue of $z^{n-1}F(z)$ at $z = 1$ is,

$$\underset{z=1}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to 1}\frac{z^{n-2}}{(z-1/5)} = \frac{1}{4/5} = \frac{5}{4}$$

And, residue of $z^{n-1}F(z)$ at $z = 1/5$ is,

$$\underset{z=1/5}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to 1/5}\left(\frac{z^{n-2}}{z-1}\right)$$

$$= \frac{(1/5)^{n-2}}{-4/5} = \left(\frac{1}{5}\right)^n \times -\frac{5}{4}\times 25 = -\frac{125}{4}\left(\frac{1}{5}\right)^4$$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z)$$

Therefore, $$Z^{-1}\{F(z)\} = \frac{5}{4} - \frac{125}{4}\left(\frac{1}{5}\right)^n$$

**(xxiii)** $\dfrac{z(3z^2-6z+4)}{(z-1)^2(z-2)}$

**Solution:** Let,

$$F(z) = \frac{z(3z^2-6z+4)}{(z-1)^2(z-2)}$$

Then,

$$z^{n-1}F(z) = \frac{3z^{n+2}-6z^{n+1}+4z^n}{(z-1)^2(z-2)}$$

Clearly $z^{n-1}F(z)$ has poles at $z = 1$ of order 2 and $z = 2$ of order 1.

Here, residue of $z^{n-1}F(z)$ at $z = 2$ is,

$$\underset{z=2}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to 2}\frac{3z^{n+2}-6z^{n+1}+4z^n}{(z-1)^2}$$

$$= 3(2)^{n+2} - 6(2)^{n+1} + 4(2)^n$$

$$= 2^n(12-12+4)$$

$$= 4\,.\,2^n$$

$$= 2^{n+2}$$

And,

$$\underset{z=1}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to 1}\frac{d}{dz}\left((z-1)^2\frac{3z^{n+2}-6z^{n+1}+4z^n}{(z-1)^2(z-2)}\right)$$

$$= \lim_{z\to 1}\left(\frac{\{3(n+2)z^{n+1}-6(n+1)z+n4z^{n-1}\}(z-2)-(3z^{n+2}-6z^{n+1}+4z^n)}{(z-2)^2}\right)$$

$$= \{3(n+2)-6(n+1)+4n\}\times(-1)-3+6-4$$

$$= -3n-6+6n+6-4n-1$$

$= -n - 1$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1} F(z)$$

Therefore, $Z^{-1}\{F(z)\} = 2^{n+2} - n - 1$

(xxiv) $\dfrac{9z^3}{(3z-1)^2(z-2)}$

**Solution:** Let,

$$F(z) = \frac{9z^3}{(3z-1)^2(z-2)} = \frac{z^3}{((z-1/3)^2)(z-2)}$$

Then, $z^{n-1} F(z) = \dfrac{z^{n+2}}{(z-1/3)^2 (z-2)}$

Clearly, $z^{n-1} F(z)$ has simple poles at $z = 2$ and $z = \frac{1}{3}$ of degree 2.

Here, residue of $z^{n-1} F(z)$ at $z = 2$ is,

$$\underset{z=2}{\text{Res}}\, z^{n-1} F(z) = \lim_{z\to 2} \frac{z^{n+2}}{(z-1/3)^2} = \frac{2^{n+2}}{(5/3)^2} = \frac{9}{25}(2^{n+2}).$$

And, residue of $z^{n-1} F(z)$ at $z = 2$ is,

$$\underset{z=1/3}{\text{Res}}\, z^{n-1} F(z) = \underset{z=1/3}{\text{Res}} \frac{d}{dz}\left(\frac{z^{n+2}}{(z-2)}\right)$$

$$= \lim_{z\to 1/3}\left\{\frac{(n+2)z^{n+1}(z-2) - z^{n+2}}{(z-2)^2}\right\}$$

$$= \frac{(n+2)\left(\frac{1}{3}\right)^{n+1}\left(\frac{1}{3}-2\right) - \left(\frac{1}{3}\right)^{n+2}}{\left(\frac{1}{3}-2\right)^2}$$

$$= \frac{(n+2)\left(\frac{1}{3}\right)^{n+1}\left(-\frac{5}{3}\right) - \left(\frac{1}{3}\right)^{n+2}}{\frac{25}{9}}$$

$$= \left(\frac{1}{3}\right)^{n+2}\{(n+2)\times(-5) - 1\}\times\frac{9}{25}$$

$$= \left(\frac{1}{3}\right)^{n}\left\{\frac{(n+2)}{5} + \frac{1}{25}\right\}$$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1} F(z)$$

Therefore, $Z^{-1}\{F(z)\} = \dfrac{9}{25} 2^{n+2} - \left(\dfrac{1}{3}\right)^{n}\left\{\left(\dfrac{n+2}{5}\right) + \dfrac{1}{25}\right\}$

(xxv) $\dfrac{2z^2 + 3z}{z^2 + z + 1}$

**Solution:** Let,



$$F(z) = \frac{2z^2 + 3z}{z^2 + z + 1}$$

Then, $z^{n-1} F(z) = \dfrac{2z^{n+1} + 3z^n}{\left(z + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(z + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}$

$$= \frac{2z^{n+1} + 3z^n}{(z - e^{i(2\pi/3)})(z - e^{-i(2\pi/3)})}$$

Clearly $z^{n-1} F(z)$ has poles Here, residue of $z^{n-1} F(z)$ at $z = e^{i(2\pi/3)}$ and $e^{-i(2\pi/3)}$ which are simple.

Here, residue of $z^{n-1} F(z)$ at $z = e^{i(2\pi/3)}$ is,

$$\underset{z=e^{i2\pi/3}}{\text{Res}}\, z^{n-1} F(z) = \lim_{z\to e^{i2\pi/3}} \frac{2z^{n+1} + 3z^n}{z - e^{-i2\pi/3}}$$

$$= \frac{2e^{i(n+1)(2\pi/3)} + 3e^{i(2n\pi/3)}}{e^{i(2\pi/3)} - e^{i(2\pi/3)}}$$

$$= \frac{e^{i2n\pi/3}(2e^{i2\pi/3} + 3)}{\sqrt{3}\, i}$$

And, similarly the residue of $z^{n-1} F(z)$ at $z = e^{-i(2\pi/3)}$ is,

$$\underset{z=e^{-i2\pi/3}}{\text{Res}}\, z^{n-1} F(z) = \frac{e^{-i2n\pi/3}(2e^{-i2\pi/3} + 3)}{-\sqrt{3}\, i}$$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1} F(z)$$

Therefore, $Z^{-1}\{F(z)\} = \dfrac{e^{i2n\pi/3}(2e^{i2\pi/3} + 3)}{\sqrt{3}\, i} + \dfrac{e^{-i2n\pi/3}(2e^{-i2\pi/3} + 3)}{-\sqrt{3}\, i}$

$$= e^{i2n\pi/3}\frac{(2 + \sqrt{3}\, i)}{\sqrt{3}\, i} + e^{-i2n\pi/3}\left(\frac{2 - \sqrt{3}\, i}{-\sqrt{3}\, i}\right)$$

$$= \frac{2}{\sqrt{3}\, i}(e^{i2n\pi/3} - e^{-2n\pi/3}) + (e^{i2n\pi/3} + e^{-i2n\pi/3})$$

$$= \frac{4}{\sqrt{3}} \sin\frac{2n\pi}{3} + 2\cos\frac{2n\pi}{3}$$

(xxvi) $\dfrac{1 + z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$

**Solution:** Let, $F(z) = \dfrac{1 + z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} = \dfrac{z(z+1)}{z^2 - z + \frac{1}{2}}$

$$\Rightarrow F(z) = \frac{z^2 + z}{\left(z - \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{1}{2}} = \frac{z^2 + z}{\left(z - \frac{1}{2}\right)^2 - \frac{1}{4}}$$

$$\Rightarrow z^{n-1} F(z) = \frac{z^{n+1} + z^n}{\left(z - \frac{1}{2} + \frac{i}{2}\right)\left(z - \frac{1}{2} - \frac{i}{2}\right)}$$

Clearly $z^{n-1}F(z)$ has simple poles $\frac{1}{2}-\frac{i}{2}$ and $\frac{1}{2}+\frac{i}{2}$

Here, residue of $z^{n-1}F(z)$ at $z=\frac{1}{2}-\frac{i}{2}$ of order 1 is,

$$\underset{z=\frac{1}{2}-\frac{i}{2}}{\text{Res}} z^{n-1}F(z) = \lim_{z\to\frac{1}{2}-\frac{i}{2}}\left(\frac{z^{n-1}.(z^2+z)}{z-\frac{1}{2}-\frac{i}{2}}\right)$$

$$=\frac{\left(\frac{1}{2}-\frac{i}{2}\right)^{n+1}+\left(\frac{1}{2}-\frac{i}{2}\right)^{n}}{-i}$$

$$=\left(\frac{1}{2}\right)^{n+1} i(1-i)^{n+1}+\left(\frac{1}{2}\right)^{n}.i(1-i)^n$$

And, residue of $z^{n-1}F(z)$ at $z=\frac{1}{2}+\frac{i}{2}$ of order 1 is,

$$\underset{z=\frac{1}{2}+\frac{i}{2}}{\text{Res}} z^{n-1}F(z) = \lim_{z\to\frac{1}{2}+\frac{i}{2}}\left(\frac{z^{n-1}.(z^2+z)}{z-\frac{1}{2}+\frac{i}{2}}\right)$$

$$=\frac{\left(\frac{1}{2}+\frac{i}{2}\right)^{n-1}+\left[\left(\frac{1}{2}+\frac{i}{2}\right)^{n}+\left(\frac{1}{2}+\frac{i}{2}\right)\right]}{i}$$

$$=-\left(\frac{1}{2}\right)^{n+1} i(1+i)^{n+1}-\left(\frac{1}{2}\right)^{n}.i(1+i)^n$$

We know,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z)$$

Therefore,

$$Z^{-1}\{F(z)\}=\left(\frac{1}{2}\right)^{n+1} i[(1-i)^{n+1}-(1+i)^{n+1}]+\left(\frac{1}{2}\right)^{n}.i[(1-i)^n-(1+i)^n]$$

**(xxvii)** $\dfrac{1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$

**Solution:** Here,

$$F(z)=\frac{1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}=\frac{z^2}{z^2-\frac{3}{2}z+\frac{1}{2}}$$

Then, $$z^{n-1}F(z)=\frac{z^{n+1}}{z^2-\frac{3}{2}z+\frac{1}{2}}=\frac{z^{n+1}}{(z-1)\left(z-\frac{1}{2}\right)}$$

Clearly $z^{n-1}F(z)$ has simple poles at $z=1$ and $z=\frac{1}{2}$.

Here, residue of $z^{n-1}F(z)$ at $z=1$ of order 1 is,



$$\underset{z=1}{\text{Res}} z^{n-1}F(z)=\lim_{z\to1}(z-1)\frac{z^{n+1}}{(z-1)(z-1/2)}=\frac{1}{1/2}=2$$

And, residue of $z^{n-1}F(z)$ at $z=\frac{1}{2}$ of order 1 is,

$$\underset{z=1/2}{\text{Res}} z^{n-1}F(z)=\lim_{z\to1/2}\left(\frac{z^{n+1}}{z-1}\right)=\frac{(1/2)^{n+1}}{-1/2}=-\left(\frac{1}{2}\right)^n$$

We know,

$$Z^{-1}\{F(z)\}=\text{sum of residues of } z^{n-1}F(z)$$

Therefore, $$Z^{-1}\{F(z)\}=2-\left(\frac{1}{2}\right)^n$$

**(xxviii)** $\dfrac{16z^3}{(4z-1)^2(z-1)}$

**Solution:** Let,

$$F(z)=\frac{16z^3}{(4z-1)^2(z-1)}=\frac{z^3}{(z-1/4)^2(z-1)}$$

Then, $$z^{n-1}F(z)=\frac{z^{n+2}}{(z-1/4)^2(z-1)}$$

Clearly $z^{n-1}F(z)$ has simple poles at $z=1$ and at $z=\frac{1}{4}$ of order 2.

Here, residue of $z^{n-1}F(z)$ at $z=1$ is,

$$\underset{z=1}{\text{Res}} z^{n-1}F(z)=\lim_{z\to1}\frac{z^{n+2}}{(z-1/4)^2}=\frac{1}{(3/4)^2}=\frac{16}{9}$$

Here, residue of $z^{n-1}F(z)$ at $z=1/4$ is,

$$\underset{z=1/4}{\text{Res}} z^{n-1}F(z)=\lim_{z\to1/4}\frac{d}{dz}\left(\frac{z^{n+2}}{z-1}\right)$$

$$=\lim_{z\to1/4}\left\{\frac{(n+2)z^{n+1}(z-1)-z^{n+2}}{(z-1)^2}\right\}$$

$$=2\left(\frac{1}{4}\right)^3-3\left(\frac{1}{4}\right)^2$$

$$=\frac{(n+2)\left(\frac{1}{4}\right)^{n+1}\times\frac{-3}{4}-\left(\frac{1}{4}\right)^{n+2}}{\frac{9}{16}}$$

$$=\left(\frac{1}{4}\right)^{n+2}\{-3n-6-1\}\times\frac{16}{9}$$

$$=-\left(\frac{1}{4}\right)^n\frac{1}{9}(3n+7)$$

We know,

$$Z^{-1}\{F(z)\}=\text{sum of residues of } z^{n-1}F(z)$$

Therefore, $Z^{-1}\{F(z)\} = f(t) = \frac{16}{9} - \left(\frac{1}{4}\right)^n \frac{1}{9}(3n+7)$.

**(xxix)** $\dfrac{1+6z^{-2}+z^{-3}}{(1+z^{-1})\left(1+\frac{1}{5}z^{-1}\right)}$

**Solution:** Let,

$$F(z) = \frac{1+6z^{-2}+z^{-3}}{(1+z^{-1})\left(1+\frac{1}{5}z^{-1}\right)}$$

$$= \frac{z^{-3}(z^3+6z+1)}{z^{-2}(z+1)\left(z+\frac{1}{5}\right)} \quad = \frac{z^{-1}(z^3+6z+1)}{(z+1)\left(z+\frac{1}{5}\right)}$$

Then, $z^{n-1}F(z) = \dfrac{z^{n+1}+6z^{n-1}+z^{n-2}}{(z+1)\left(z+\frac{1}{5}\right)}$

Clearly $z^{n-1}F(z)$ has simple poles at $z = -1$ and $z = \frac{-1}{5}$.

Here, residue of $z^{n-1}F(z)$ at $z = -1$ is,

$$\underset{z=-1}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to(-1)} (z+1)\left(\frac{z^{n+1}+6z^{n-1}+z^{n-2}}{(z+1)\left(z+\frac{1}{5}\right)}\right)$$

$$= \frac{(-1)^{n+1}+6(-1)^{n-1}+(-1)^{n-2}}{-4/5}$$

$$= (-1)^n(-1-6+1)\times\frac{5}{-4} \quad = (-1)^n\frac{3\times 5}{2} \quad = \frac{15}{2}(-1)^n$$

And, residue of $z^{n-1}F(z)$ at $z = \frac{-1}{5}$ is,

$$\underset{z=-\frac{1}{5}}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to -\frac{1}{5}}\left(\frac{z^{n+1}+6z^{n-1}+z^{n-2}}{(z+1)}\right)$$

$$= \frac{\left(-\frac{1}{5}\right)^{n+1}+6\left(-\frac{1}{5}\right)^{n-1}+\left(-\frac{1}{5}\right)^{n-2}}{\left(-\frac{1}{5}+1\right)}$$

$$= \frac{\left(-\frac{1}{5}\right)^n\left(\left(-\frac{1}{5}\right)^1+6\left(-\frac{1}{5}\right)^{-1}+\left(-\frac{1}{5}\right)^{-2}\right)}{\frac{4}{5}}$$

$$= (-1)^n\left(\frac{1}{5}\right)^n\left(-\frac{1}{5}-30+25\right)\times\frac{5}{4}$$

$$= -(-1)^n\left(\frac{1}{5}\right)^n\frac{13}{2}$$



We know,

$Z^{-1}\{F(z)\}$ = sum of residues of $z^{n-1}F(z)$

Therefore, $Z^{-1}\{F(z)\} = f(t) = \frac{15}{2}(-1)^n - \frac{13}{2}(-1)^n\left(\frac{1}{5}\right)^n$

**(xxx)** $\dfrac{z-4}{(z-1)(z-2)^2}$

**Solution:** Let, $F(z) = \dfrac{z-4}{(z-1)(z-2)^2}$

Then, $z^{n-1}F(z) = \dfrac{z^n-4z^{n-1}}{(z-1)(z-2)^2}$

Clearly $z^{n-1}F(z)$ has simple poles at $z = 1$ and $z = 2$ of order 2.

Here, residue of $z^{n-1}F(z)$ at $z = 1$ is,

$$\underset{z=1}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to 1}\frac{z^n-4z^{n-1}}{(z-2)^2} = 1-4 = -3$$

And, residue of $z^{n-1}F(z)$ at $z = 2$ is,

$$\underset{z=2}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to 2}\frac{d}{dz}\left[\frac{z^n-4z^{n-1}}{z-1}\right]$$

$$= \lim_{z\to 2}\left\{\frac{\{n\,z^{n-1}\,4\,(n-1)\,z^{n-2}\}(z-1)-z^n+4z^{n-1}}{(z-1)^2}\right\}$$

$$= n\,.\,2^{n-1}-4n\,2^{n-2}+4\,.\,2^{n-2}-2^n+4\,.\,(2)^{n-1}$$

$$= 2^{n-1}\left\{n-4n\,.\,\frac{1}{2}+4\,.\,\frac{1}{2}-2+4\right\}$$

$$= 2^{n-1}\{n-2n+4\}$$

$$= -2^{n-1}(n-4)$$

We know,

$Z^{-1}\{F(z)\}$ = sum of residues of $z^{n-1}F(z)$

Therefore, $Z^{-1}\{F(z)\} = -3-2^{n-1}(n-4)$

**(xxxi)** $\dfrac{2z}{(2z-1)^2}$

**Solution:** Let,

$$F(z) = \frac{2z}{(2z-1)^2} = \frac{z}{2(z-1/2)^2}$$

Then, $z^{n-1}F(z) = \dfrac{z^n}{2(z-1/2)^2}$

Clearly $z^{n-1}F(z)$ has pole at $z = 1/2$ of order 2.

Here, residue of $z^{n-1}F(z)$ at $z = \frac{1}{2}$ is,

$$\underset{z=1/2}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to 1/2}\frac{d}{dz}\left[\frac{z^n}{2(z-1/2)^2}\times(z-1/2)^2\right]$$

$$= \lim_{z\to 1/2} \frac{n z^{n-1}}{2}$$

$$= \frac{n (1/2)^{n-1}}{2} = n\left(\frac{1}{2}\right)^n$$

We know,

$Z^{-1}\{F(z)\}$ = sum of residues of $z^{n-1} F(z)$

Therefore, $Z^{-1}\{F(z)\} = n\left(\frac{1}{2}\right)^n$

(xxxii) $\dfrac{z^3 - 20z}{(z-4)(z-2)^3}$

**Solution:** Process as in (xxiv).

(xxxiii) $\dfrac{20z + 5}{(z-2)(z-1/2)}$

**Solution:** Let,

$$F(z) \frac{20z+5}{(z-2)(z-1/2)}$$

Then, $z^{n-1} F(z) = \dfrac{20z^n + 5z^{n-1}}{(z-2)(z-1/2)}$

Clearly $z^{n-1} F(z)$ has simple poles at z = 2 and z = 1/2.

Here, residue of $z^{n-1} F(z)$ at z = 2 is,

$$\underset{z=2}{\text{Res}}\, z^{n-1} F(z) = \lim_{z\to 2}\left(\frac{20z^n + 5z^{n-1}}{z-1/2}\right)$$

$$= \frac{20(2)^n + 5(2)^{n-1}}{3/2} = 2^n\left(20 + \frac{5}{2}\right)\times\frac{2}{3} = 15(2)^n$$

And, residue of $z^{n-1} F(z)$ at z = 1/2 is,

$$\underset{z=1/2}{\text{Res}}\, z^{n-1} F(z) = \lim_{z\to 1/2}\left(\frac{20z^n + 5z^{n-1}}{z-2}\right)$$

$$= \frac{20(1/2)^n + 5(1/2)^{n-1}}{-3/2}$$

$$= \left(\frac{1}{2}\right)^n (20+10)\times\left(-\frac{2}{3}\right) = -20\left(\frac{1}{2}\right)^n$$

We know,

$Z^{-1}\{F(z)\}$ = sum of residues of $z^{n-1} F(z)$

Therefore, $Z^{-1}\{F(z)\} = 15(2)^n - 20\left(\frac{1}{2}\right)^n$



3. **Solve by using Z-Transform:**

(i) $y_{n+2} + y_n = 1$ **with** $y_0 = y_1 = 0$.

**Solution:** Given difference equation is

$$y_{n+2} + y_n = 1 \quad \text{......(i)}$$

with $y_0 = y_1 = 0$ .....(ii)

Since we have,

$$Z\{y_{n+k}\} = z^k\left[\bar{y} - y_0 - \frac{y_1}{z} - \frac{y_2}{z^2} - \ldots\ldots - \frac{y_{k-1}}{z^{k-1}}\right] \quad \text{where, } \bar{y} = Z\{y_n\}$$

Now, taking Z-Transform on (i) then,

$$Z\{y_{n+2}\} + Z\{y_n\} = Z\{1\}$$

$$\Rightarrow z^2\left[\bar{y} - y_0 - \frac{y_1}{z}\right] + \bar{y} = \frac{z}{z-1} \qquad \left[\because \bar{y} = Z\{y_n\} \text{ and } Z\{1\} = \frac{z}{z-1}\right]$$

Applying (ii) then,

$$z^2\bar{y} + \bar{y} = \frac{z}{z-1}$$

$$\Rightarrow \bar{y}(z^2+1) = \frac{z}{z-1}$$

$$\Rightarrow \bar{y} = \frac{z}{(z-1)(z^2+1)}$$

Now, taking inverse Z-Transform we get,

$$y_n = Z^{-1}\left\{\frac{z}{(z-1)(z^2+1)}\right\} \quad \text{........(iii)}$$

By Q.2(i) we have,

$$Z^{-1}\left\{\frac{z}{(z-1)(z^2+1)}\right\} = \frac{1}{2}\left[1 - \left(\cos\left(\frac{n\pi}{2}\right) + i\sin\left(\frac{n\pi}{2}\right)\right)\right]$$

Therefore (iii) becomes,

$$y_n = \frac{1}{2}\frac{1}{2}\left[1 - \left(\cos\left(\frac{n\pi}{2}\right) + i\sin\left(\frac{n\pi}{2}\right)\right)\right].$$

This is required solution of (i) satisfying (ii).

(ii) $y_{n+2} + 2y_{n+1} + y_n = n$ **with** $y_0 = y_1 = 0$.

**[2003 Fall Q. No. 5(a)] [2006 Spring Q. No. 5(a)]**

**Solution:** Given difference equation is,

$$y_{n+2} + 2y_{n+1} + y_n = n \quad \text{........(i)}$$

with $y_0 = y_1 = 0$ .......(ii)

Since we have

$$Z\{y_{n+k}\} = z^k\left[\bar{y} - y_0 - \frac{y_1}{z} - \frac{y_2}{z^2} - \ldots\ldots - \frac{y_{k-1}}{z^{k-1}}\right] \quad \text{where, } \bar{y} = Z\{y_n\}$$

Now, taking Z-Transform on (i) then

$$Z\{y_{n+2}\} + 2Z\{y_{n+1}\} + Z\{y_n\} = Z\{n\}$$

Then,

$$z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) + 2z(\bar{y} - y_0) + \bar{y} = \frac{z}{(z-1)^2} \quad \text{.......(iii)}$$

$$\left[\because \bar{y} = Z\{y_n\} \text{ and } Z\{n\} = \frac{z}{(z-1)^2}\right]$$

Applying (ii) on (iii) then

$$\bar{y}\,(z^2 + 2z + 1) = \frac{z}{(z-1)^2}$$

$$\Rightarrow \bar{y} = \frac{z}{(z-1)^2(z^2+2z+1)} = \frac{z}{(z-1)^2(z+1)^2}$$

Now, taking inverse Z–Transform then

$$y_n = Z^{-1}\left\{\frac{z}{(z-1)^2(z+1)^2}\right\} \quad \text{......(iv)}$$

Let, $F(z) = \dfrac{z}{(z-1)^2(z+1)^2}$

Then,

$$z^{n-1}F(z) = \frac{z^n}{(z-1)^2(z+1)^2}$$

Clearly $z^{n-1}F(z)$ has poles at $z = 1$ of order 2 and at $z = -1$ of order 2.

Since we have,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z) \quad \text{........(v)}$$

Here, residue of $z^{n-1}F(z)$ at $z = 1$ of order 2 is,

$$\underset{z=1}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2 z^{n-1}F(z)\right].$$

$$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{z^n}{(z+1)^2}\right]$$

$$= \lim_{z\to 1}\left[\frac{nz^{n-1}(z+1)^2 - 2z^n(z+1)}{(z+1)^4}\right]$$

$$= \frac{n(1)^{n-1}(2)^2 - 2(1)^n(2)}{(2)^4} = \frac{4n-4}{16} = \frac{n-1}{4}$$

And, residue of $z^{n-1}F(z)$ at $z = -1$ of order 2 is,

$$\underset{z=-1}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2 z^{n-1}F(z)\right]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[\frac{z^n}{(z-1)^2}\right]$$

$$= \lim_{z\to -1}\left[\frac{nz^{n-1}(z-1)^2 - 2z^n(z-1)}{(z-1)^4}\right]$$

$$= \frac{n(-1)^{n-1}(-2)^2 - 2(-1)^n(-2)}{(-2)^4}$$

$$= \frac{4n(-1)^{n-1} + 4(-1)^n}{16} = \frac{(-1)^{n-1}(n-1)}{4}$$



Thus, from (iv) and (v) we get,

$$y_n = \left(\frac{n-1}{4}\right)\{1 + (-1)^{n-1}\}$$

$$\Rightarrow y_n = \left(\frac{n-1}{4}\right)[1 - (-1)^n].$$

This is the required solution of (i) satisfying (ii).

**(iii) $U_{x+2} - 2\cos\alpha\, U_{x+1} + U_x = 0$ with $U_0 = 1, U_1 = \cos\alpha$**

**Solution:** Given difference equation is

$$U_{x+2} - 2\cos\alpha\, U_{x+1} + U_x = 0 \quad \text{.......(i)}$$

with $U_0 = 1,\ U_1 = \cos\alpha$

Now, taking Z–Transform on (i) then

$$Z\{U_{x+2}\} - 2\cos\alpha\, Z\{U_{x+1}\} + Z\{U_x\} = Z\{0\}$$

Then,

$$z^2\left(\bar{U} - U_0 - \frac{U_1}{z}\right) - 2z\cos\alpha\,(\bar{U} - U_0) + \bar{U} = 0 \qquad [\because Z\{U_x\} = \bar{U}]$$

Applying (ii) then,

$$\bar{U}\,[z^2 - 2z\cos\alpha + 1] - z^2 - z\cos\alpha + 2z\cos\alpha = 0$$

$$\Rightarrow \bar{U} = \frac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1}$$

Now taking inverse Z–Transform then,

$$U_x = Z^{-1}\left\{\frac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1}\right\} \quad \text{.......(iii)}$$

Let, $F(z) = \dfrac{z^2 - z\cos\alpha}{(z - e^{i\alpha})(z - e^{-i\alpha})}$

Then,

$$z^{x-1}F(z) = \frac{z^{x+1} - z^x\cos\alpha}{(z - e^{i\alpha})(z - e^{-i\alpha})}$$

Clearly, $z^{x-1}F(z)$ has poles at $z = e^{i\alpha}$ of order 1 and at $z = e^{-i\alpha}$ of order 1.

Since we have,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{x-1}F(z) \quad \text{.......(iv)}$$

Here, residue of $z^{x-1}F(z)$ at $z = e^{i\alpha}$ is

$$\underset{z=e^{i\alpha}}{\text{Res}}\, z^{x-1}F(z) = \lim_{z\to e^{i\alpha}}\left[(z - e^{i\alpha})\, z^{x-1}F(z)\right]$$

$$= \lim_{z\to e^{i\alpha}}\frac{z^{x+1} - z^x\cos\alpha}{z - e^{-i\alpha}}$$

$$= \frac{e^{i\alpha(x+1)} - e^{i\alpha x}\cos\alpha}{e^{i\alpha} - e^{-i\alpha}} = \frac{e^{-i\alpha(x+1)} - e^{-i\alpha x}\cos\alpha}{-2i\sin\alpha}$$

And the residue of $z^{x-1} F(z)$ at $z = e^{-i\alpha}$ is obtained from above with replacing i by $-i$ as,

$$\underset{z=e^{-i\alpha}}{\text{Res}} [z^{x-1} F(z)] = \frac{e^{-i\alpha(x+1)} - e^{-i\alpha x}\cos\alpha}{-2i\sin\alpha}$$

Thus, (iii) and (iv) gives,

$$U_x = \frac{e^{i\alpha(x+1)} - e^{i\alpha x}\cos\alpha}{2i\sin\alpha} + \frac{e^{-i\alpha(x+1)} - e^{-i\alpha x}\cos\alpha}{-2i\sin\alpha}$$

$$= \frac{e^{i\alpha(x+1)} - e^{-i\alpha(x+1)} - e^{i\alpha x}\cos\alpha + e^{-i\alpha x}\cos\alpha}{2i\sin\alpha}$$

$$= \frac{2i\sin(x+1)\alpha - 2i\sin\alpha x\cos\alpha}{2i\sin\alpha}$$

$$= \frac{\sin\alpha x\cos\alpha + \cos x\alpha\sin\alpha - \sin\alpha x\cos\alpha}{\sin\alpha} = \frac{\cos x\alpha\sin\alpha}{\sin\alpha} = \cos x\alpha$$

This is the required solution of (i) satisfying (ii).

**(iv)** $y_{n+2} - y_n = 2^n$ **with** $y_0 = 0, y_1 = 1.$

**[2005 Spring Q. No. 3(b)] [2009 Spring Q. No. 3(b)]**

**Solution:** Given difference equation is

$$y_{n+2} - y_n = 2^n \quad \text{.......(i)}$$

with $\quad y_0 = 0, y_1 = 1 \quad$ ........(ii)

Now, taking Z–Transform on (i) then

$$Z\{y_{n+2}\} - Z\{y_n\} = Z\{2^n\}$$

Then,

$$z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) - \bar{y} = \frac{z}{z-2} \qquad \left[\because \bar{y} = Z\{y_n\} \text{ and } Z\{a^n\} = \frac{z}{z-a}\right]$$

Applying (ii) then

$$z^2\left(\bar{y} - \frac{1}{z}\right) - \bar{y} = \frac{z}{z-2}$$

$$\Rightarrow \bar{y}(z^2 - 1) - z = \frac{z}{z-2}$$

$$\Rightarrow \bar{y} = \frac{1}{(z^2-1)}\left(\frac{z}{z-2} + z\right)$$

$$= \frac{z^2 - z}{(z-2)(z+1)(z-1)}$$

And, taking inverse Z–Transform then

$$y_n = Z^{-1}\left\{\frac{z^2 - z}{(z-2)(z+1)(z-1)}\right\} \quad \text{.......(iii)}$$

Let,

$$F(z) = \frac{z^2 - z}{(z-2)(z+1)(z-1)}.$$



Then, $\quad z^{n-1} F(z) = \frac{z^{n+1} - z^n}{(z-2)(z+1)(z-1)}$

Clearly $z^{n-1} F(z)$ has poles at $z = 2$, $z = -1$ and $z = 1$ of simple order.

Since we have,

$$Z^{-1}\{F(z)\} = \text{sum of residue of } z^{n-1} F(z) \quad \text{.....(iv)}$$

Here, residue of $z^{n-1} F(z)$ at $z = 2$ is

$$\underset{z=2}{\text{Res}} (z^{n-1} F(z)) = \lim_{z\to2} [(z-2) z^{n-1} F(z)]$$

$$= \lim_{z\to2}\left(\frac{z^{n+1} - z^n}{(z+1)(z-1)}\right) = \frac{2^{n+1} - 2^n}{(3)(1)} = \frac{2^n(2-1)}{3} = \frac{2^n}{3}$$

And, residue of $z^{n-1} F(z)$ at $z = -1$ is,

$$\underset{z=-1}{\text{Res}} (z^{n-1} F(z)) = \lim_{z\to-1} [(z+1) z^{n-1} F(z)]$$

$$= \lim_{z\to-1}\left(\frac{z^{n+1} - z^n}{(z-2)(z-1)}\right)$$

$$= \frac{(-1)^{n+1} - (-1)^n}{(-3)(-2)} = \frac{(-1)^n(-1-1)}{(-3)(-2)} = \frac{(-1)^{n+1}}{3}$$

Also, residue of $z^{n-1} F(z)$ at $z = 1$ is,

$$\underset{z=1}{\text{Res}} (z^{n-1} F(z)) = \lim_{z\to1} [(z-1) z^{n-1} F(z)]$$

$$= \lim_{z\to1}\left(\frac{z^{n+1} - z^n}{(z-2)(z+1)}\right) = \frac{1^{n+1} - 1^n}{(-1)(2)} = 0$$

Thus, with (iv), the equation (iii) becomes,

$$y_n = \frac{2^n}{3} + \frac{(-1)^{n+1}}{3} + 0$$

$$\Rightarrow y_n = \frac{2^n + (-1)^{n+1}}{3}$$

This is required solution of (i) satisfying (ii).

**(v)** $y_{x+2} - 3y_{x+1} + 2y_x = 0$ **with** $y_0 = 0, y_1 = 1.$ **[2006 Fall Q. No. 5(a)]**

**Solution:** Given difference equation is,

$$y_{x+2} - 3y_{x+1} + 2y_x = 0 \quad \text{.......(i)}$$

with $\quad y_0 = 0, y_1 = 1 \quad$ .......(ii)

Now, taking Z–Transform on (i) then,

$$Z\{y_{x+2}\} - 3Z\{y_{x+1}\} + 2Z\{y_x\} = Z\{0\}$$

Then,

$$z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) - 3z(\bar{y} - y_0) + 2\bar{y} = 0 \qquad \text{where, } \bar{y} = Z\{y_x\}$$

Applying (ii) then,

$$\bar{y}(z^2 - 3z + 2) - z = 0$$

$$\Rightarrow \quad \bar{y} = \frac{z}{z^2 - 3z + 2}$$

And, taking inverse Z–Transform we get,

$$y_x = Z^{-1}\left\{\frac{z}{z^2 - 3z + 2}\right\} \qquad \text{.........(iii)}$$

Let, $F(z) = \dfrac{z}{z^2 - 3z + 2}$

Then, [by Q. No. 2(iii)] $y_x = 2^x - 1$.

Thus, with (iv), the equation (iii) becomes,

$$y_x = 2^x - 1.$$

**(vi) $y_{x+2} - 4y_{x+1} + 4y_x = 0$ with $y_0 = 1$, $y_1 = 0$.**

**[2009 Fall Q. No. 6(b) OR][2008 Fall Q. No. 6(b) OR] [2004 Spring Q. No. 3(b)]**

**Solution:** Given difference equation is,

$$y_{x+2} - 4y_{x+1} + 4y_x = 0 \qquad \text{........ (1)}$$

with $y_0 = 1, y_1 = 0$ ........(2)

Now, taking Z–Transform on (i) then,

$$Z\{y_{x+2}\} - 4\,Z\{y_{x+1}\} + 4\,Z\{y_x\} = Z\{0\}$$

Then,

$$z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) - 4z(\bar{y} - y_0) + 4\bar{y} = 0 \qquad \text{where, } \bar{y} = Z\{y_x\}$$

Applying (ii) then

$$\bar{y}(z^2 - 4z + 4) - z^2 - 4z = 0$$

$$\Rightarrow \quad \bar{y} = \frac{z^2 - 4z}{(z-2)^2}$$

And taking inverse Z–Transform, we get

$$y_x = Z^{-1}\left\{\frac{z^2 - 4z}{(z-2)^2}\right\} \qquad \text{..........(3)}$$

Let, $F(z) = \dfrac{z^2 - 4z}{(z-2)^2}$

Thus, with [by Q. No. 2(iv)], the equation (3) gives,

$$y_x = 2^x(1 - x)$$

**(vii) $y_{k+2} - 4y_{k+1} + 4y_k = 0$ with $y_0 = 1$, $y_1 = 0$.**

**Solution:** Same question to (vi) with replacing x by k.

**(viii) $x_{n+1} - y_n = 1$, $y_{n+1} + x_n = 0$ where $x_0 = 0$, $y_0 = -1$.**

**Solution:** Given difference equation is

$$x_{n+1} - y_n = 1 \qquad \text{.....(1)}$$

$$y_{n+1} + x_n = 0 \qquad \text{.......(2)}$$

with $x_0 = 0, y_0 = -1$ ......(3)



Now, taking Z–Transform on (1) and (2) then,

$$Z\{x_{n+1}\} - Z\{y_n\} = Z\{1\}$$

$$Z\{y_{n+1}\} + Z\{x_n\} = Z\{0\}$$

Then,

$$z(\bar{x} - x_0) - \bar{y} = z/(z-1) \qquad \text{for } \bar{x} = Z\{x_n\},\ \bar{y} = Z\{y_n\}$$

$$z(\bar{y} - y_0) + \bar{x} = 0$$

Applying (3) then,

$$z\bar{x} - \bar{y} = z/(z-1) \qquad \text{.......(4)}$$

$$z\bar{y} + \bar{x} = -z \qquad \text{......(5)}$$

Substituting the value of $\bar{y}$ from (4) in (5) then,

$$z\left(z\bar{x} - \frac{z}{z-1}\right) + \bar{x} = -z$$

$$\Rightarrow \quad \bar{x}(z^2 + 1) = -z + \frac{z^2}{z-1} = \frac{z}{z-1}$$

$$\Rightarrow \quad \bar{x} = \frac{z}{(z-1)(z^2+1)} \qquad \text{.........(6)}$$

And taking inverse Z–Transform on (vi) then,

$$x_n = Z^{-1}\left\{\frac{z}{(z-1)(z^2+1)}\right\} \qquad \text{.......(7)}$$

Let, $F(z) = \dfrac{z}{(z-1)(z^2+1)}$

Thus, with [by Q. No. 2(i)], the equation (3) gives,

$$x_n = \frac{1}{2}\left[\frac{1}{2} - \left(\cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\right)\right]$$

Replacing n+1 by n then (ii) gives,

$$y_n = -\frac{1}{2}\left[1 - \left(\cos\left(\frac{(n-1)\pi}{2}\right) + \sin\left(\frac{(n-1)\pi}{2}\right)\right)\right]$$

$$= -\frac{1}{2}\left[1 - \left(\cos\frac{n\pi}{2}\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\sin\frac{\pi}{2} + \sin\frac{n\pi}{2}\cos\frac{\pi}{2} - \cos\frac{n\pi}{2}\sin\frac{\pi}{2}\right)\right]$$

$$= -\frac{1}{2}\left[1 - \left(\sin\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{2}\right)\right)\right]$$

Thus,

$$x_n = \frac{1}{2}\left[1 - \cos\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{2}\right)\right] \text{ and } y_n = \frac{1}{2}\left[-1 - \cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\right]$$

be solution of (i) and (ii) satisfying (iii).

**(ix) $y_{n+2} + 3y_{n+1} + 2y_n = 0$ where $y_0 = 0$, $y_1 = 1$.**

**Solution:** Given difference equation is

$$y_{n+2} + 3y_{n+1} + 2y_n = 0 \qquad \text{.......(i)}$$

with $y_0 = 0, y_1 = 1$ .......(ii)

Now, taking Z–Transform on (i) then,

$$Z\{y_{n+2}\} + 3Z\{y_{n+1}\} + 2Z\{y_n\} = Z\{0\}$$

Then,

$$z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) + 3z(\bar{y} - y_0) + 2\bar{y} = 0 \quad \text{for } \bar{y} = Z\{y_n\}$$

Applying (ii) then,

$$z^2\left(\bar{y} - \frac{1}{z}\right) + 3z\bar{y} + 2\bar{y} = 0$$

$$\Rightarrow \bar{y}(z^2 + 3z + 2) = z$$

$$\Rightarrow \bar{y} = \frac{z}{z^2 + 3z + 2}$$

Now, taking inverse Z–Transform we get,

$$y_n = Z^{-1}\left\{\frac{z}{z^2 + 3z + 2}\right\} \quad \text{.........(iii)}$$

Let, $F(z) = \frac{z}{z^2 + 3z + 2} = \frac{z}{(z+2)(z+1)}$

Then $z^{n-1}F(z) = \frac{z^n}{(z+1)(z+2)}$

Clearly, $z^{n-1}F(z)$ has poles at $z = -1$ and at $z = -2$ of simple order.

Since we have,

$$Z^{-1}\{F(z)\} = \text{sum of residue of } z^{n-1}F(z) \quad \text{.......(iv)}$$

Here, residue of $z^{n-1}F(z)$ at $z = -1$ is

$$\underset{z=-1}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to -1}(z+1)z^{n-1}F(z)$$

$$= \lim_{z\to -1}\left(\frac{z^n}{z+2}\right) = \frac{(-1)^n}{1} = (-1)^n$$

And residue of $z^{n-1}F(z)$ at $z = -2$ is

$$\underset{z=-2}{\text{Res}}\, z^{n-1}F(z) = \lim_{z\to -2}(z+2)z^{n-1}F(z)$$

$$= \lim_{z\to -2}\left(\frac{z^n}{z+1}\right) = \frac{(-2)^n}{-1} = -(-2)^n$$

Thus, (iii) becomes with (iv) as

$$y_n = (-1)^n - (-2)^n$$

This is the required solution of (i) satisfying (ii).

**(x)** $y_{n+2} - 3y_{n+1} + 2y_n = 4^n$ **where** $y_0 = 0, y_1 = 1.$ **[2008 Spring Q. No. 3(b)]**

**Solution:** Given difference equation is

$$y_{n+2} - 3y_{n+1} + 2y_n = 4^n \quad \text{......(i)}$$

with $\quad y_0 = 0$ and $y_1 = 1 \quad$ ........(ii)

Now, taking Z–Transform on (i) then,

$$Z\{y_{n+2}\} - 3Z\{y_{n+1}\} + 2Z\{y_n\} = Z\{4^n\}$$



Then,

$$z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) - 3z(\bar{y} - y_0) + 2\bar{y} = \frac{z}{z-4}$$

Applying (ii) then,

$$\bar{y}(z^2 - 3z + 2) - z = \frac{z}{z-4}$$

$$\Rightarrow \bar{y} = \frac{1}{z^2 - 3z + 2}\left(\frac{z}{z-4} + z\right)$$

$$= \frac{z^2 - 3z}{(z-1)(z-2)(z-4)}$$

And taking inverse Z–Transform we get,

$$y_n = Z^{-1}\left\{\frac{z^2 - 3z}{(z-1)(z-2)(z-4)}\right\} \quad \text{........(iii)}$$

Let,

$$F(z) = \frac{z^2 - 3z}{(z-1)(z-2)(z-4)}$$

Then,

$$z^{n-1}F(z) = \frac{z^{n+1} - 3z^n}{(z-1)(z-2)(z-4)}$$

Clearly, $z^{n-1}F(z)$ has poles at $z = 1, z = 2, z = 4$ of simple order.

Since we have,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z) \quad \text{.......(iv)}$$

Here, residue of $z^{n-1}F(z)$ at $z = 1$ is,

$$\underset{z=-1}{\text{Res}}\,(z^{n-1}F(z)) = \lim_{z\to 1}[(z-1)z^{n-1}F(z)]$$

$$= \lim_{z\to 1}\left(\frac{z^{n+1} - 3z^n}{(z-2)(z-4)}\right) = \frac{1-3}{(-1)(-3)} = \frac{-2}{3}$$

And residue of $z^{n-1}F(z)$ at $z = 2$ is,

$$\underset{z=2}{\text{Res}}\,(z^{n-1}F(z)) = \lim_{z\to 2}(z-2)z^{n-1}F(z)$$

$$= \lim_{z\to 2}\left(\frac{z^{n+1} - 3z^n}{(z-1)(z-4)}\right) = \frac{2^{n+1} - 3(2^n)}{(1)(-2)} = \frac{2^n(2-3)}{-2} = 2^{n-1}$$

Also, residue of $z^{n-1}F(z)$ at $z = 4$ is

$$\underset{z=4}{\text{Res}}\,(z^{n-1}F(z)) = \lim_{z\to 4}(z-4)z^{n-1}F(z)$$

$$= \lim_{z\to 4}\left(\frac{z^{n+1} - 3z^n}{(z-1)(z-2)}\right)$$

$$= \frac{4^{n+1} - 3(4^n)}{(3)(2)} = \frac{4^n(4-3)}{6} = \frac{4^n}{6} = \frac{2^{2n}}{6} = \frac{2^{2n-1}}{3}$$

Therefore, (iii) becomes with (iv) is

$$y_n = -\frac{2}{3} + 2^{n-1} + \frac{2^{2n-1}}{3}$$

This is the required solution of (i) satisfying (ii).

**(xi)** $y_{n+2} - 3y_{n+1} + \frac{1}{4}y_n = 1$ **where** $y_0 = 1, y_1 = 2$.

**(xii)** $y_{n+2} - 4y_{n+1} + 4y_n = 2^n$ **where** $y_0 = 0, y_1 = 1$.

**[2011 Spring Q. No. 3(b)] [2012 Fall Q. No. 3(b)]**

**Solution:** Given difference equation is

$$y_{n+2} - 4y_{n+1} + 4y_n = 2^n \quad \text{..........(i)}$$

with $\quad y_0 = 0, y_1 = 1 \quad$ ..........(ii)

Now, taking Z–Transform on (i) then

$$Z\{y_{n+2}\} - 4Z\{y_{n+1}\} + 4Z\{y_n\} = Z\{2^n\}$$

Then,

$$z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) - 4z(\bar{y} - y_0) + 4\bar{y} = \frac{z}{z-2} \quad \text{for } y_n = Z\{y_n\}$$

Applying (ii) then,

$$z^2\left(\bar{y} - \frac{1}{z}\right) - 4z\bar{y} + 4\bar{y} = \frac{z}{z-2}$$

$$\Rightarrow \bar{y}(z^2 - 4z + 4) = \frac{z}{z-2} + z$$

$$\Rightarrow \bar{y} = \frac{z^2 - z}{(z-2)(z-2)^2}$$

$$\Rightarrow \bar{y} = \frac{z^2 - z}{(z-2)^3}$$

And taking inverse Z–Transform we get,

$$y_n = z^{-1}\left\{\frac{z^2 - z}{(z-2)^3}\right\} \quad \text{.........(iii)}$$

Let, $\quad F(z) = \frac{z^2 - z}{(z-2)^3}$

Then, $\quad z^{n-1}F(z) = \frac{z^{n+1} - z^n}{(z-2)^3}$

Clearly $z^{n-1}F(z)$ has poles at $z = 2$ of order 3.

Since we have,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z) \quad \text{.........(iv)}$$

Here, residue of $z^{n-1}F(z)$ at $z = 2$ of order 3 is,

$$\operatorname*{Res}_{z=2} z^{n-1}F(z) = \lim_{z\to 2}\left(\frac{1}{2}\frac{d^2}{dz^2}[(z-2)^3 z^{n-1}F(z)]\right)$$

$$= \lim_{z\to 2}\left(\frac{1}{2}\frac{d^2}{dz^2}(z^{n+1} - z^n)\right)$$

$$= \lim_{z\to 2}\left(\frac{1}{2}\frac{d}{dz}[(n+1)z^n - nz^{n-1}]\right)$$

$$= \lim_{z\to 2}\left[\frac{(n+1)nz^{n-1} - n(n-1)z^{n-2}}{2}\right]$$

$$= \frac{1}{2}(n(n+1)2^{n-1} - n(n-1)2^{n-1})$$



$$= n2^{n-3}[2n + 2 - n + 1]$$

$$= n2^{n-3}(n+3)$$

$$= n(n+3)2^{n-3}$$

Thus, with (iv), the equation (iii) becomes,

$$y_n = n(n+3)2^{n-3}$$

This is the required solution of (i) satisfying (ii).

**(xiii)** $y_{n+2} + 4y_{n+1} + 3y_n = 2n$ **where** $y_0 = 0, y_1 = 1$.

**Solution:** Given difference equation is,

$$y_{n+2} + 4y_{n+1} + 3y_n = 2n \quad \text{......(i)}$$

with $\quad y_0 = 0, y_1 = 1 \quad$ .......(ii)

Now, taking Z–Transform on (i) then

$$Z\{y_{n+2}\} + 4Z\{y_{n+1}\} + 3Z\{y_n\} = 2Z\{n\}$$

Then, $\quad z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) + 4z(\bar{y} - y_0) + 3\bar{y} = 2\left(\frac{z}{(z-1)^2}\right)$

Applying (ii) then,

$$\bar{y}(z^2 + 4z + 3) - z = \frac{2z}{(z-1)^2}$$

$$\Rightarrow \bar{y} = \left(\frac{1}{z^2 + 4z + 3}\right)\left(\frac{2z}{(z-1)^2} + z\right)$$

$$= \frac{z^3 - 2z^2 + 3z}{(z-1)^2(z+3)(z+1)}$$

And taking inverse Z–Transform we get,

$$y_n = Z^{-1}\left\{\frac{z^3 - 2z + 3z}{(z-1)^2(z+1)(z+3)}\right\} \quad \text{.......(iii)}$$

Let,

$$F(z) = \frac{z^3 - 2z^2 + 3z}{(z-1)^2(z+1)(z+3)}$$

Then, $\quad z^{n-1}F(z) = \frac{z^{n+2} - 2z^{n+1} + 3z^n}{(z-1)^2(z+1)(z+3)}$

Clearly $z^{n-1}F(z)$ has poles at $z = 1$ of order 2 and at $z = -1, z = -3$ of simple order.

Since we have,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z) \quad \text{.......(iv)}$$

Here, residue of $z^{n-1}F(z)$ at $z = 1$ of order 2 is,

$$\operatorname*{Res}_{z=1} z^{n-1}F(z) = \lim_{z\to 1}\frac{d}{dz}\left(\frac{z^{n+2} - 2z^{n+1} + 3z^n}{(z+1)(z+3)}\right)$$

$$= \lim_{z\to 1}\frac{d}{dz}\left(\frac{z^{n+2} - 2z^{n+1} + 3z^n}{z^2 + 4z + 3}\right)$$

$$= \lim_{z\to 1}\frac{(z^2 + 4z + 3)[(n+2)(z)^{n+1} - (n+1)(z)^n + 3nz^{n-1}] - (z^{n+2} - 2z^{n+1} + 3z^n)(2z+4)}{(z^2 + 4z + 3)^2}$$

$$= \frac{(1+4+3)[(n+2) - 2(n+1) + 3n] - (1-2+3)(2+4)}{(1+4+3)^2}$$

$$= \frac{8(2n) - 12}{64}$$

$$= \frac{4n - 3}{16}$$

And, residue of $z^{n-1}F(z)$ at $z = -1$ is,

$$\underset{z=-1}{\text{Res}}\ z^{n-1}F(z) = \lim_{z\to -1}(z+1)z^{n-1}F(z)$$

$$= \lim_{z\to -1}\left(\frac{z^{n+2} - 2z^{n+1} + 3z^n}{(z-1)^2(z+3)}\right)$$

$$= \frac{(-1)^{n+2} - 2(-1)^{n+1} + 3(-1)^n}{(-2)^2(2)}$$

$$= \frac{(-1)^n(1+2+3)}{8}$$

$$= \frac{3(-1)^n}{4}$$

Also, residue of $z^{n-1}F(z)$ at $z = -3$ is,

$$\underset{z=-3}{\text{Res}}\ z^{n-1}F(z) = \lim_{z\to -3}(z+3)z^{n-1}F(z)$$

$$= \lim_{z\to -3}\left(\frac{z^{n+2} - 2z^{n+1} + 3z^n}{(z-1)^2(z+1)}\right)$$

$$= \frac{(-3)^{n+2} - 2(-3)^{n+1} + 3(-3)^n}{(-4)^2(-2)}$$

$$= \frac{(-3)^n[9+6+3]}{-32} = \frac{(-3)^n(18)}{-32} = -\frac{(-3)^n.9}{16}$$

Thus, with (iv), equation (iii) becomes,

$$y_n = Z^{-1}\{F(z)\} = \frac{4n-3}{16} + \frac{3(-1)^n}{4} - \frac{9(-3)^n}{16}$$

This is the required solution of (i) satisfying (ii).

**(xiv) $2y_n - 2y_{n-1} + y_{n-2} = u(n)$ where $y_n = 0$ for $n < 0$ and $u(n) = 1$ for $n \geq 0$, $u(n) = 0$ for $n < 0$.**

**Solution:** Given difference equation is,

$$2y_n - 2y_{n-1} + y_{n-2} = u(n) \quad \text{.......(i)}$$

Now, taking Z–Transform on (i) then

$$2\bar{y} - 2z^{-1}\bar{y} + z^{-2}\bar{y} = Z\{u(n)\}$$

$$\Rightarrow \bar{y}(2 - 2z^{-1} + z^{-2}) = \frac{z}{z-1}$$

$$\Rightarrow \bar{y}\left(\frac{2z^2 - 2z + 1}{z^2}\right) = \frac{z}{z-1}$$

$$\Rightarrow \bar{y} = \frac{z^3}{(z-1)(2z^2 - 2z + 1)}$$

And taking inverse Z–Transform we get,



$$y_n = Z^{-1}\left\{\frac{z^3}{(z-1)(2z^2-2z+1)}\right\} \quad \text{.......(ii)}$$

Let,

$$F(z) = \frac{1}{2}\,\frac{z^3}{(z-1)(z^2 - z + 1/2)}$$

Then,

$$z^{n-1}F(z) = \frac{z^{n+2}}{2(z-1)(z^2 - z + 1/2)}$$

$$= \frac{z^{n+2}}{2(z-1)\left(z - \left(\frac{1}{2} + \frac{i}{2}\right)\right)\left(z - \left(\frac{1}{2} - \frac{i}{2}\right)\right)}$$

Clearly, $z^{n-1}F(z)$ has poles at $z = 1$, $z = (1+i)/2$, $z = (1-i)/2$.

Since we have,

$$Z^{-1}\{F(z)\} = \text{sum of residue of } z^{n-1}F(z) \quad \text{.......(iii)}$$

Here, residue of $z^{n-1}F(z)$ at $z = 1$ is,

$$\underset{z=1}{\text{Res}}\ z^{n-1}F(z) = \lim_{z\to 1}(z-1)z^{n-1}F(z)$$

$$= \lim_{z\to 1}\left(\frac{1}{2}\right)\left(\frac{z^{n+2}}{z^2 - z + 1/2}\right)$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{1 - 1 + 1/2}\right)$$

$$= 1$$

And, residue of $z^{n-1}F(z)$ at $z = (1+i)/2$ is,

$$\underset{z=(1+i)/2}{\text{Res}}\ z^{n-1}F(z) = \lim_{z\to(1+i)/2}\left(z - \frac{1+i}{2}\right)z^{n-1}F(z)$$

$$= \lim_{z\to(1+i)/2}\frac{z^{n+2}}{2(z-1)\left(z - \left(\frac{1}{2} - \frac{i}{2}\right)\right)}$$

$$= \left(\frac{1}{2}\right)\frac{(1+i)^{n+2}}{2^{n+2}\left(\frac{1+i}{2} - 1\right)\left(\frac{1+i}{2} - \left(\frac{1}{2} - \frac{i}{2}\right)\right)}$$

$$= \left(\frac{1}{2}\right)\frac{(1+i)^{n+2}}{2^{n+2}\left(\frac{i-1}{2}\right)i}$$

Also, residue of $z^{n-1}F(z)$ at $z = (1-i)/2$ is,

$$\underset{z=(1-i)/2}{\text{Res}}\ z^{n-1}F(z) = \left(\frac{1}{2}\right)\frac{(1-i)^{n+2}}{2^{n+2}\left(\frac{-i-1}{2}\right)(-i)} \quad \text{[as above]}$$

$$= \left(\frac{1}{2}\right)\frac{(1-i)^{n+2}}{2^{n+2}\left(\frac{i+1}{2}\right)(i)}$$

Thus,

$$y_n = Z^{-1}\{F(z)\} = 2\pi i\left(1 + \left(\frac{1}{2}\right)\frac{(1+i)^{n+2}}{2^{n+2}\left(\frac{i-1}{2}\right)i} + \left(\frac{1}{2}\right)\frac{(1-i)^{n+2}}{2^{n+2}\left(\frac{i+1}{2}\right)(i)}\right)$$

**(xv)** $y_n + \frac{y_{n-1}}{4} = \delta(n) + \frac{\delta(n-1)}{3}$ **for** $n \geq 0$, $y_0 = 0$.

**Solution:** Given difference equation is,

$$y_n + \frac{y_{n-1}}{4} = \delta(n) + \frac{\delta(n-1)}{3} \qquad \text{.........(i)}$$

Now, taking Z–Transform on (i) then

$$Z\{y_n\} + \frac{1}{4}Z\{y_{n-1}\} = Z\{\delta(n)\} + \frac{1}{3}Z\{\delta(n-1)\}$$

Then,

$$\bar{y} + \frac{1}{4}z^{-1}\bar{y} = 1 + \frac{1}{3}z^{-1}$$

$$\Rightarrow \bar{y}\left(\frac{4z+1}{4z}\right) = \left(\frac{3z+1}{3z}\right)$$

$$\Rightarrow \bar{y} = \frac{4}{3}\left(\frac{3z+1}{4z+1}\right) = \frac{z+1/3}{z+1/4}$$

And taking inverse Z–Transform we get,

$$y_n = Z^{-1}\left\{\frac{z+1/3}{z+1/4}\right\} \qquad \text{.........(ii)}$$

Let,

$$F(z) = \frac{z+1/3}{z+1/4}$$

Then, $$z^{n-1}F(z) = \frac{z^{n-1}(z+1/3)}{z+1/4}$$

Clearly, $z^{n-1}F(z)$ has poles at $z = -\frac{1}{4}$

Since we have,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z) \qquad \text{.......(iii)}$$

Here, residue of $z^{n-1}F(z)$ at $z = -\frac{1}{4}$ is,

$$\underset{z=-1/4}{\text{Res}}\ z^{n-1}F(z) = \lim_{z\to -1/4}(z+1/4)z^{n-1}F(z)$$

$$= \lim_{z\to -1/4}[z^{n-1}(z+1/3)]$$

$$= \left(-\frac{1}{4}\right)^{n-1}\left(-\frac{1}{4}+\frac{1}{3}\right) = \left(-\frac{1}{4}\right)^{n-1}\left(\frac{1}{12}\right)$$

Thus, $y_n = \frac{1}{12}\left(-\frac{1}{4}\right)^{n-1}$.

This is the required solution of (i) satisfying (ii).



**(xvi)** $y_{n+1} + \frac{y_n}{4} = \left(\frac{1}{4}\right)^{-n}$ **where** $y_0 = 0$.

**Solution:** Given difference equation is,

$$y_{n+1} + \frac{y_n}{4} = \left(\frac{1}{4}\right)^{-n} \qquad \text{......(i)}$$

with $\quad y_0 = 0 \qquad$ ..........(ii)

Now, taking Z–Transform on (i) then,

$$Z\{y_{n+1}\} + \frac{1}{4}Z\{y_n\} = Z\{4^n\}$$

Then, $$z(\bar{y} - y_0) + \frac{1}{4}\bar{y} = \frac{z}{z-4}$$

Applying (ii) then

$$\bar{y}\left(z + \frac{1}{4}\right) = \frac{z}{z-4} \Rightarrow \bar{y} = \frac{4z}{(z-4)(4z+1)}$$

$$\Rightarrow \bar{y} = \frac{z}{(z-4)(z+1/4)}$$

And taking inverse Z–Transform we get,

$$y_n = Z^{-1}\left\{\frac{z}{(z-4)(z+1/4)}\right\} \qquad \text{........(iii)}$$

Let,

$$F(z) = \frac{z}{(z-4)(z+1/4)}$$

Then, $$z^{n-1}F(z) = \frac{z^n}{(z-4)(z+1/4)}$$

Clearly $z^{n-1}F(z)$ has simple poles at $z = 4$, $z = -\frac{1}{4}$.

Since we have,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z) \qquad \text{.......(iv)}$$

Here, residue of $z^{n-1}F(z)$ at $z = 4$ is,

$$\underset{z=4}{\text{Res}}\ z^{n-1}F(z) = \lim_{z\to 4}(z-4)z^{n-1}F(z)$$

$$= \lim_{z\to 4}\left(\frac{z^n}{z+1/4}\right) = \frac{4^n}{4+1/4} = \frac{4^{n+1}}{17}$$

And, residue of $z^{n-1}F(z)$ at $z = -¼$ is,

$$\underset{z=-1/4}{\text{Res}}\ z^{n-1}F(z) = \lim_{z\to -1/4}(z+¼)z^{n-1}F(z)$$

$$= \lim_{z\to -1/4}\left(\frac{z^n}{z-4}\right)$$

$$= \left[\frac{(-1/4)^n}{-1/4-4}\right]$$

$$= (-1)^{n-1}\left(\frac{1/4}{1/4+4}\right) = (-1)^{n-1}\left(\frac{1}{17}\right)$$

Thus, with (iv), equation (iii) gives,

$$y_n = \frac{4^{n+1}}{17} + \frac{(-1)^{n-1}}{17}$$

***Note:** If we reform the question as* $y_{n+1} + \frac{y_n}{4} = \left(\frac{1}{4}\right)^n$ *with* $y_0 = 0$ *then we obtain* $y_n = 2\left(\frac{1}{4}\right)^n - 2\left(-\frac{1}{4}\right)^n$.

**4. Solve the difference equation**

$$y_{n+2} + (a + b)\, y_{n+1} + ab\, y_n = 0,$$ **where a and b are constants.**

**Solution:** Given difference equation is,

$$y_{n+2} + (a + b)\, y_{n+1} + ab y_n = 0, \quad \text{where a and b are constants}$$

Now, taking Z–Transform on (i) then,

$$Z\{y_{n+2}\} + (a + b)\, Z\{y_{n+1}\} + ab\, Z\{y_n\} = 0$$

$$\Rightarrow \; z^2\left[\bar{y} - y_0 - \frac{y_1}{z}\right] + (a + b)\, z[\bar{y} - y_0] + ab\, \bar{y} = 0$$

$$\Rightarrow \; \bar{y}\,(z^2 + (a + b)z + ab) = z^2 y_0 + zy_1 + (a + b)\, zy_0$$

$$\Rightarrow \; \bar{y} = \frac{z^2 y_0 + zy_1 + (a + b)\, z\, y_0}{z^2 + (a + b)\, z + ab} = F(z) \quad (\text{let})$$

Then we have, $y_n = Z^{-1}\,(F(z))$.

And, we have,

$$Z^{-1}\,\{F(z)\} = \text{sum of residues of } z^{n-1}\, F(z).$$

Now,

$$z^{n-1}\, F(z) = \frac{z^{n+1}\, y_0 + z^n y_1 + (a + b)\, z^n\, y_0}{(z + a)\,(z + b)}$$

Here poles of $z^{n-1}\, F(z)$ are at $z = -a, -b$ which are simple.

So,

$$\underset{z=-a}{\text{Res}}\; z^{n-1}\, F(z) = \lim_{z\to(-a)} (z + a)\, z^{n-1}\, F(z)$$

$$= \lim_{z\to(-a)} \frac{z^{n+1}\, y_0 + z^n y_1 + (a + b)\, z^n\, y_0}{(z + b)}$$

$$= \frac{(-a)^{n+1}\, y_0 + (-a)^n\, y_1 + (a + b)\, (-a)^n\, y_0}{b - a}$$

$$= \frac{(-a)^{n+1}\, y_0 + (-a)^n\, y_1 - (a)^{n+1}\, y_0 + b(a)^n\, y_0}{b - a}$$

$$= \frac{by_0 + y_1}{b - a}\,(-a)^n$$

And,

$$\underset{z=-b}{\text{Res}}\; z^{n-1}\, F(z) = \frac{(-b)^{n+1}\, y_0 + (-b)^n\, y_1 + (a + b)\, (-b)^n\, y_0}{a - b}$$

$$= \frac{(-b)^{n+1}\, y_0 + (-b)^n\, y_1 - a\,(-b^n)\, y_0 - (b)^{n+1}\, y_0}{a - b}$$

$$= \frac{ay_0 + y_1}{a - b}\,(-b)^n$$



Now,

$$y_n = z^{-1}\{F(z)\}$$

$$\Rightarrow \; y_n = \frac{by_0 + y_1}{b - a}\,(-a)^n + \frac{ay_0 + y_1}{a - b}\,(-b)^n$$

**5. Solve the difference equation**

$$y_{n+2} + 2ay_{n+1} + a^2 y_n = 0$$

**Solution:** Given difference equation is.

$$y_{n+2} + 2ay_{n+1} + a^2 y_n = 0.$$

Now, taking Z–Transform on (i) then,

$$Z\{y_{n+2}\} + 2a\, Z\{y_{n+1}\} + a^2\, Z\{y_n\} = 0$$

$$\Rightarrow \; z^2\left[\bar{y} - y_0 - \frac{y_1}{z}\right] + 2az\,(\bar{y} - y_0) + a^2\bar{y} = 0$$

$$\Rightarrow \; \bar{y}\,(z^2 + 2az + a^2) = z^2 y_0 + zy_1 + 2azy_0$$

$$\Rightarrow \; \bar{y} = (z^2 y_0 + zy_1 + 2azy_0)/(z + a)^2 = F(z)$$

Then we have, $y_n = z^{-1}\,(F(z))$.

And we have,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}\, F(z).$$

Now,

$$z^{n-1}\, F(z) = \frac{(z^{n+1}\, y_0 + z^n y_1 + 2az^n y_0)}{(z + a)^2}$$

Here $z^{n-1}\, F(z)$ has poles at $z = -a$ of degree 2.

$$\underset{z=-a}{\text{Res}}\; z^{n-1}\, F(z) = \lim_{z\to -a} \frac{d}{dz}\left((z + a)^2\left(\frac{z^{n+1}\, y_0 + z^n y_1 + 2az^n}{(z + a)^2}\right) y_0\right)$$

$$= \lim_{z\to -a} [(n + 1)\, z^n\, y_0 + nz^{n-1}\, y_1 + 2anz^{n-1}\, y_0]$$

$$= (n + 1)\,(-a)^n\, y_0 + n(-a)^{n-1}\, y_1 + 2an\,(-a)^{n-1}\, y_0$$

$$= (-a)^n\, y_0 + n(-a)^{n-1}\, [y_1 + 2ay_0 - ay_0]$$

$$= (-a)^n\, y_0 + [ay_0 + y_1]\, n\,(-a)^{n-1}$$

Noω,

$$y_n = Z^{-1}\,(F(z))$$

$$\Rightarrow \; y_n = (-a)^n\, y_0 + [ay_0 + y_1]\, n\,(-a)^{n-1}$$

**6. Consider the difference equation**

$$y_{n+2} = y_{n+1} + y_n, \qquad \text{where } y_0 = 0,\; y_1 = 1$$

**Find the solution of $y_n$. Show that the limiting value of $\frac{y_{n+1}}{y_n} = \frac{1 + \sqrt{5}}{2}$, when $n \to \infty$.**

**Solution:** Now, given difference equation is

$$y_{n+2} - y_{n+1} - y_n = 0$$

Taking Z–Transform both sides.

$$z^2\left[\bar{y} - y_0 - \frac{y_1}{z}\right] - z(\bar{y} - y_0) - \bar{y} = 0$$

$$\Rightarrow \quad z^2\bar{y} - z - z\bar{y} - \bar{y} = 0$$

$$\Rightarrow \quad \bar{y} = \frac{z}{z^2 - z - 1} = F(z)$$

Then we have, $y_n = z^{-1}(F(z))$.

And we have,

$$Z^{-1}\{F(z)\} = \text{sum of residues of } z^{n-1}F(z).$$

Now,

$$y_n = Z^{-1}\{F(z)\} = Z^{-1}\left\{\frac{z}{z^2 - z - 1}\right\}$$

Here, $\quad z^{n-1}F(z) = \dfrac{z^n}{z^2 - z - 1}$

Here poles of $z^{n-1}F(z)$ are at $z = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

Here residue of $z^{n-1}F(z)$ at the poles,

$$\underset{z = \frac{1+\sqrt{5}}{2}}{\text{Res}} z^{n-1}F(z) = \lim_{z \to \frac{1+\sqrt{5}}{2}} \frac{z^n}{\left(z - \left(\frac{1-\sqrt{5}}{2}\right)\right)} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

And,

$$\underset{z = \frac{1-\sqrt{5}}{2}}{\text{Res}} z^{n-1}F(z) = \lim_{z \to \left(\frac{1-\sqrt{5}}{2}\right)} \frac{z^n}{\left(z - \left(\frac{1+\sqrt{5}}{2}\right)\right)} = \frac{\left(\frac{1-\sqrt{5}}{2}\right)^n}{-\sqrt{5}}$$

Here, $\quad y_n = Z^{-1}\{F(z)\}$.

$$\Rightarrow \quad y_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Again, we know

$$y_{n+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Now,

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]}{\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right]}$$

Here the term $\dfrac{1-\sqrt{5}}{2} = -0.618 < 1$

So when $n \to \infty$, $(-0.618)^n \to 0$

So, we get



$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \lim_{n\to\infty} \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{1+\sqrt{5}}{2}\right)^{n}} = \lim_{n\to\infty}\left(\frac{1+\sqrt{5}}{2}\right)$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \frac{1+\sqrt{5}}{2}$$

## OTHER IMPORTANT QUESTION FROM FINAL EXAM FOR PRACTICE

**2002 Q. No. 3(b) OR**

Find Z–Transform of cosh at sinbt.

**2002 Q. No. 2(b)**

Find Z[sin at] and Z[cos at]

**2003 Fall Q. No. 4(a); 2007 Spring Q. No. 2(b)**

Define Z–Transform of a function. Find Z–Transform of $e^{-at}$ and hence deduce the values of Z(cos at) and Z(sin at)

**2004 Spring Q. No. 2(b)**

Define Z–Transform of the function. Find (i) Z(1) (ii) $Z(na^n)$ (iii) $Z(e^{-at})$

**2004 Fall Q. No. 2(b)**

Find Z–Transform of $\cos\frac{n\pi}{2}$ and $\sin\frac{n\pi}{2}$

**2005 Spring Q. No. 2(b)**

Define Z–Transform. State and prove linearity property of Z–Transform. Evaluate $Z(t\,e^{2t})$.

**2005 Spring Q. No. 3(a)**

Show that $Z[n\,f(t)] = -z\frac{d}{dz}[Z(f(t))]$. Using it evaluate Z(nt) and $Z(n^2t)$.

**2005 Fall Q. No. 2(b)**

Define Z–Transform. State and prove first shifting theorem of Z–Transform; and then evaluate $Z(e^{-t})$, $Z(te^{-at})$ and $Z(e^{-at}/n!)$.

**2006 Spring Q. No. 2(b)**

Define Z–Transform. State and prove first shifting theorem of Z–Transform. Using it evaluate $Z(te^{2t})$ and $Z(ne^{-2t})$.

**2006 Fall Q. No. 4(a)**

State and prove linearity property of Z–Transform. Find $Z(t^k)$, where k is a positive integer.

**2006 Fall Q. No. 4(b)**

Define convolution of function in Z–Transform. State and prove convolution theorem in Z–Transform.

**2007 Spring Q. No. 3(a); 2008 Fall Q. No. 6(a); 2012 Fall Q. No. 3(a)**

State the first shifting theorem for Z transform an hence find

(i) Z(cos at) and (ii) Z(sin at)

**2007 Fall Q. No. 2(b)**

Define Z–Transform of the function f(t). Find Z–Transform of $f(t) = n^2a^n$ and $f(t) = \cos n\theta$.

**2008 Spring Q. No. 2(b)**

Define Z transform. State and prove first shifting theorem of Z transform. Using it evaluate Z transform of $a^n \cos bt$ and $a^n$ sinbt.

**2008 Fall Q. No. 5(b)**

Define Z–Transform of a function f(t) and by using the definition find the Z-Transform of: (i) $na^n$ and (ii) $\frac{1}{n+1}$

**2009 Spring Q. No. 2(b)**

Define Z transform. State and prove linearity property of z transform. Using it find Z transform of cos at and then find Z transform of $e^{bt}$ cos bt.

**2009 Fall Q. No. 5(b)**

Define Z–Transform of a function f(t) and by using the definition find the Z-Transform of (i) $(-1)^n$ and (ii) n

**2009 Fall Q. No. 6(a)**

State the first shifting theorem for Z transformation and hence find $Z[e^{-at}t]$.

**2011 Spring Q. No. 3.a.**

Define Z–Transform. State and prove linearity property of Z transform. Find $Z(e^{-iat})$ and hence deduce Z transform of sinat and cosat.

**2011 Fall Q. No. 6(a)**

Find the Z–Transform of the function $f(t) = e^{-iat}$ and hence deduce the value of Z(cosat) and Z(sinat).

**2012 Fall Q. No. 6(b)**

Define Z–Transform of a function f(t) and by using the definition find the Z–Transform of (i) $(-1)^n$ (ii) $\frac{1}{n!}$

**2015 Fall Q. No. 3(a)**

**Find the z-transform of $f(t) = a^n$ and hence find $Z\left\{\sin\left(\frac{n\pi}{2}\right)\right\}$ and $Z\left\{\cos\left(\frac{n\pi}{2}\right)\right\}$.**



**2016 Fall Q. No. 3(a)**

**Find Z-transform of the function $f(t) = e^{-iat}$ and hence deduce the value of Z(cosat) and Z(sinat).**

**2016 Fall Q. No. 4(a)**

**Define Z-transform. State and prove Second shifting theorem of Z-transform, Evaluate $Z(t^2 e^{-bt})$**

**2016 Spring Q. No. 6(a)**

**State and prove first and second shifting theorems in Z-transform. Find the value of $Z(a^n\cos bt)$ and $Z(a^n \sin bt)$.**

**2017 Fall Q. No. 2(b)**

**State and prove first shifting theorem for z-transform using it to find z (coshat sinbt)**

## Z transform Inverse

**2002 Q. No. 3(a)**

Find $Z^{-1}\left[\frac{z}{z^2+7z+10}\right]$

**2004 Spring Q. No. 3(a); 2007 Spring Q. No. 3(b)**

Find the inverse Z–Transform of the function $\frac{2z}{(z-1)(z^2+1)}$

**2004 Fall Q. No. 3(a); 2009 Fall Q. No. 6(b)**

Find the inverse Z–Transform of the function $\frac{z+2}{z^2-5z+6}$

**2005 Fall Q. No. 3(a)**

Show that $Z[nf(t)] = -z\frac{d}{dz}[F(z)]$ where $F(z) = Z[f(t)]$, and find $Z^{-1}\left[\frac{z}{(z+1)^2(z-1)}\right]$

**2006 Spring Q. No. 2(b) OR**

Find the inverse Z–Transform of $\frac{z}{z^2-2z+2}$

**2007 Fall Q. No. 3(a)**

Find $z^{-1}\left[\frac{z^2-3}{(z+2)(z^2+1)}\right]$

**2008 Spring Q. No. 3(a)**

State and prove initial and final value theorem on Z transform. Evaluate $Z^{-1}\left(\frac{z}{(z-1)(z-2)}\right)$.

**2008 Fall Q. No. 6(b)**

Find inverse Z transform of $\frac{z^2+1}{z^2-2z+2}$

**2009 Spring Q. No. 3(a)**

If $Z[f(t)] = f(z)$, derive the expression of $Z[f(t+kT)]$. Evaluate $Z^{-1}\left[\frac{z+3}{(z+1)(z+2)}\right]$.

**2011 Fall Q. No. 4(a)**

Define Z–Transform? Find the inverse of the Z–Transform of the function: $\frac{2z}{(z-1)(z^2+1)}$

**2015 Fall Q. No. 3(b)**

Find the inverse of z-transform of $\frac{3z^2+2z}{(z-3)^2(z-2)}$

**2016 Fall Q. No. 4(b) OR**

Find $Z^{-1}\frac{z^2+1}{z^2-2z+2}$.

**2017 Fall Q. No. 3(a)**

Find $Z^{-1}\left[\frac{2z^2+3z}{(z+2)(z-4)}\right]$

## Application of Z transform

**2002 Q. No. 3(b)**

Solve the difference equation $y_{n+1}+y_n=1$ given $y_0=0$

**2004 Fall Q. No. 3(b)**

Solve the difference equation $y_{n+1}+3y_n=5^n$ given $y_0=0$.

**2005 Fall Q. No. 3(b)**

Show that $Z(y_{n+k}) = z^k\left[\bar{y}-y_0-\frac{y_1}{z}-\frac{y_2}{z^2}-\ldots-\frac{y_{k-1}}{z^{k-1}}\right]$ where $\bar{y}=Z(y_n)$. and then solve the differential equation $y_{n+2}=1$, where $y_0=0=y_1$.

**2007 Fall Q. No. 3(b)**

Solve the difference equation $y_{n+2}-8y_{n+1}+16y_n=4^n$, where $y_0=0$ and $y_1=1$; by unit Z–Transform.

**2011 Fall Q. No. 5(a)**

Using Z–Transform, solve the difference equation $y_{n+2}+6y_{n+1}+9y_n=2^n$ when $y_0=y_1=0$.



**2015 Fall Q. No. 4(a)**

Solve the difference equation: $y_{n+2}+6y_{n+1}+9y_n=4^n$, where $y_0=0$ and $y_1=0$.

**2016 Fall Q. No. 3(b)**

Using Z-transform solve the difference equation

$y_{n+1}+6y_{n+1}+9y_n=2^n$ when $y_0=y_1=0$

**2016 Fall Q. No. 6(b)**

Using Z-transform, solve the difference equation

$y_{n+2}+6y_{n+1}+9y_n=2^n$ when $y_0=y_1=0$

**2017 Fall Q. No. 3(b)**

Solve the difference equation $y_{n+2}-3y_{n+1}+2y_n=0$, where $y_0=0$ and $y_1=1$; by using z-transform.

## SHORT QUESTIONS

**2003 Fall Q. No. 7(d)**

Find the Z–Transform of $f(n)=na^n$.

**2004 Fall Q. No. 7(e)**

Find the Z–Transform of $\frac{1}{n!}$

**2005 Fall Q. No. 7(d)**

Find $Z(1)$ & $Z(-1)^n$

**2005 Spring Q. No. 7(iv)**

Find $Z^{-1}\left[\frac{z}{(z-2)(z-3)}\right]$

**2006 Spring Q. No. 7(c)**

Find Z–Transform of $\sin n\pi/2$ and $\cos n\pi/2$

**2006 Fall Q. No. 7(b)**

Find Z–Transform of $\frac{1}{n+1}$

**2007 Fall Q. No. 7(d)**

Find $Z(t^2)$.

**2009 Spring Q. No. 7(e)**

Prove $Z(a^n)=\frac{z}{z-a}$

**2009 Spring Q. No. 7(d)**

State and prove linearly property of Z–Transform i.e.
Z[af(t) + bg(t)] = a Zf(t)} + b Z{g(t)}.

**2011 Spring Q. No. 7(a)**

Find Z(1).

**2011 Spring Q. No. 7(d)**

Find $Z^{-1}$ [z/(z – 3)]

**2015 Fall Q. No. 7(d)**

**Derive Z inverse of** $f(z) = \frac{z}{(z+1)(z-3)}$.

**2016 Spring Q. No. 7(c)**

**State and prove the linear property on Z-transform.**

❑❑❑

*Unit 10*

# ANALYTICAL SOLID GEOMETRY

## Exercise 10.1

**A. What curves are given by the following parametric representation?**

**1. $(t, t^2 + 2, 0)$**

**Solution:** Given that,

$$(x, y, z) = (t, t^2 + 2, 0)$$

$$\Rightarrow \quad x = t,\ y = t^2 + 2,\ z = 0$$

This gives that $y = x^2 + 2$, $z = 0$

**2. (0, 5 cost, 5 sint)**

**Solution:** Given that,

$$(x, y, z) = (0, 5\cos t, 5\sin t)$$

Clearly, the curve lies in yz-plane

So the given curve is a circle having radius r = 5.

Thus, $x = 0,\ y^2 + z^2 = r^2$

$$\Rightarrow \quad x = 0,\ y^2 + z^2 = 25$$

**3. (cosht, sinht, 0)**

**Solution:** Given that,

$$(x, y, z) = (\cosh t, \sinh t, 0)$$

$$\Rightarrow \quad x = \cosh t,\ y = \sinh t,\ z = 0$$

Clearly the given curve is hyperbolic and it lies on xy-plane.

Here, $x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1$

Thus, $x^2 - y^2 = 1$, $z = 0$

**B. Represents the following curves parametrically.**

**1. $y^2 + (z-3)^2 = 9,\ x = 0$**

**Solution:** Given that

$$y^2 + (z-3)^2 = 9,\ x = 0$$

Clearly, the given curve lies on yz-plane. And, the curve is a circle with center $(y_1, z_1) = (0, 3)$ and having radius 3. So,

$$y = r\cos t,\ z - 3 = r\sin t,\ x = 0$$

$$\Rightarrow \quad y = 3\cos t,\ z = 3 + 3\sin t,\ x = 0$$

$$\Rightarrow \quad (x, y, z) = (0, 3\cos t, 3 + 3\sin t)$$

2. $x^2 + y^2 = 1, y = z$

OR 2005 Spring Q. No. 7(v)

Find the position vector of the curve $x^2 + y^2 = 1$, $y = z$ in parametrical representation.

**Solution:** Given that,

$$x^2 + y^2 = 1, y = z$$

Clearly, the given curve is ellipse in xyz-plane.

Put $x = r\cos t$, $y = r\sin t$, and $y = z$ with $r = 1$.

So, $x = \cos t$, $y = \sin t$, $z = \sin t$

Thus, $(x, y, z) = (\cos t, \sin t, \sin t)$

**C. Find a tangent vector and the corresponding unit tangent vector $\vec{u}(t)$ at a given point.**

1. $\vec{r}(t) = t\vec{i} + t^3\vec{j}$ at $P(1, 1, 0)$

**Solution:** Given that,

$$\vec{r}(t) = t\vec{i} + t^3\vec{j} + 0.\vec{k}$$

So, $\frac{d\vec{r}}{dt} = \vec{r}'(t) = \vec{i} + 3t^2\vec{j} + 0.\vec{k}$

Therefore, the tangent vector be,

$$\frac{d\vec{r}}{dt} = (1, 3t^2, 0) \text{ at } (1, 1, 0)$$

Since, $\vec{r}(t) = (t, t^3, 0) = (x, y, z)$ at P.

$\Rightarrow x = t, y = t^3, z = 0$ at P

$\Rightarrow x = 1, y = 1, z = 0$ at P.

Thus, $t = 1$. So that, $\frac{d\vec{r}}{dt} = (1, 3, 0)$

Also, the unit tangent vector be

$$\hat{r}'(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{(1, 3, 0)}{\sqrt{1^2 + 3^2 + 0^2}} = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}, 0\right)$$

2. $\vec{r}(t) = \cos t\,\vec{i} + 2\sin t\,\vec{j}$ at $P\left(\frac{1}{2}, \sqrt{3}, 0\right)$. [2008 Spring Q. No. 7(b)]

**Solution:** Given that,

$$\vec{r}(t) = \cos t\,\vec{i} + 2\sin t\,\vec{j} + 0.\vec{k} = (x, y, z)$$

So, $x = \cos t$, $y = 2\sin t$, $z = 0$.

And given point be, $P\left(\frac{1}{2}, \sqrt{3}, 0\right)$.



So. $\frac{1}{2} = \cos t$, $\sqrt{3} = 2\sin t$, $0 = 0$

$\Rightarrow \cos 60° = \cos t$, $\frac{\sqrt{3}}{2} = \sin t$ i.e. $\sin 60° = \sin t$.

Then, $t = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sin 60°}{\cos 60°}\right) = 60°$

Since, $\vec{r}(t) = \cos t\,\vec{i} + 2\sin t\,\vec{j}$

So, $\vec{r}'(t) = -\sin t\,\vec{i} + 2\cos t\,\vec{j}$

$= -\sin 60°\,\vec{i} + 2\cos 60°\,\vec{j}$ at P

$= -\frac{\sqrt{3}}{2}\vec{i} + 2.\frac{1}{2}\vec{j}$ at P

$= \left(-\frac{\sqrt{3}}{2}, 1, 0\right)$

Thus, tangent vector at P is, $\left(-\frac{\sqrt{3}}{2}, 1, 0\right)$

And, unit tangent vector at P is

$$\hat{r}'(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\left(-\frac{\sqrt{3}}{2}, 1, 0\right)}{\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + 1^2 + 0^2}}$$

$$= \frac{1}{\sqrt{\frac{3}{4} + 1}}\left(-\frac{\sqrt{3}}{2}, 1, 0\right)$$

$$= \frac{2}{\sqrt{7}}\left(-\frac{\sqrt{3}}{2}, 1, 0\right)$$

$$= \left(\frac{-\sqrt{3}}{\sqrt{7}}, \frac{2}{\sqrt{7}}, 0\right)$$

3. $\vec{r}(t) = \cosh t\,\vec{i} + \sinh t\,\vec{j}$ at $P\left(\frac{5}{3}, \frac{4}{3}, 0\right)$ [2017 Fall Q. No. 7(b)]

**Solution:** Given that,

$$\vec{r}(t) = \cosh t\,\vec{i} + \sinh t\,\vec{j}$$

So, $\vec{r}'(t) = \sinh t\,\vec{i} + \cosh t\,\vec{j}$

Given point be, $P(x, y, z) = P\left(\frac{5}{3}, \frac{4}{3}, 0\right)$

$\Rightarrow x = \frac{5}{3}, y = \frac{4}{3}, z = 0$

i.e. $\cosh t = \frac{5}{3}$, $\sinh t = \frac{4}{3}$

Thus, the tangent vector at P be

$$\vec{r}'(t)_{\text{at } P} = \frac{4}{3}\vec{i} + \frac{5}{3}\vec{j} + 0.\vec{k} = \left(\frac{4}{3}, \frac{5}{3}, 0\right)$$

And, unit tangent vector at P be,

$$\hat{r}'(t)_{\text{at } P} = \frac{\left(\frac{4}{3}, \frac{5}{3}, 0\right)}{\sqrt{\left(\frac{4}{3}\right)^2 + \left(\frac{5}{3}\right)^2 + 0^2}}$$

$$= \frac{\left(\frac{4}{3}, \frac{5}{3}, 0\right)}{\sqrt{\frac{16}{9} + \frac{25}{9} + 0}} = \frac{3}{\sqrt{41}}\left(\frac{4}{3}, \frac{5}{3}, 0\right) = \left(\frac{4}{\sqrt{41}}, \frac{5}{\sqrt{41}}, 0\right).$$

## OTHER IMPORTANT QUESTION FROM FINAL EXAM

## SHORT QUESTIONS

**2002 Q. No. 7(d)**

**Write the equation of hyperboloid and paraboloid.**

**Solution: Equation of Hyperboloid:**

The equation of hyperboloid of one sheet is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ and of two sheet is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$.

**Equation of Paraboloid:**

The equation of paraboloid is $x^2 + y^2 = z$.

**2003 Fall Q. No. 7(b)**

**Write the equation of hyperboloid of two sheet and then sketch.**

**Solution:** The equation of hyperboloid of two sheet is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ ... (i)

**For sketch:**

(i) **Centre:** Clearly (i) has centre at (0, 0, 0).

(ii) **Symmetry:** Since all three variables have same degree, so it is symmetrical about all three axes.

(iii) **Intercept:** When y = 0 = z then $x^2 = -a^2$. So, x has only imaginary value which is non-acceptable for sketch.
When x = 0 = z then $y^2 = -b^2$. So, y has only imaginary value which is also non-acceptable for sketch.
When x = 0 = y then $z^2 = c^2$. So, $z = \pm c$. Therefore, the figure cuts z-axis at $z = \pm c$.

(iv) **Plane section:**

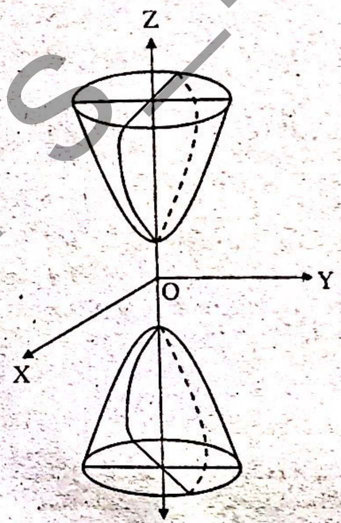


(a) **In xy-plane:** When z = 0 then (i) gives, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$. This is an imaginary ellipse.

(b) **In yz-plane:** When x = 0 then (i) gives, $\frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 \Rightarrow \frac{z^2}{c^2} - \frac{y^2}{b^2} = 1$.
This is a hyperbola.

(c) **In zx-plane:** When y = 0 then (i) gives, $\frac{x^2}{a^2} - \frac{z^2}{c^2} = -1 \Rightarrow \frac{z^2}{c^2} - \frac{x^2}{a^2} = 1$.
This is a hyperbola.

With the information the sketch of the hyperboloid is as in figure.




**2003 Fall Q. No. 7(a)**

**What do you mean by tangent to a curve?**

**Solution: Tangent to a Curve:**

Let C be a curve in space. The tangent on C at a point P of C is the limiting position of a straight line that through P and a point Q of C as Q tends to P along C.

**2005 Fall Q. No. 7(b); 2016 Fall Q. No. 7(d)**

**Sketch the paraboloid $z = x^2 + y^2$.**

**Solution:** The equation of paraboloid is $z = x^2 + y^2$ ... (i)

**For sketch:**

(i) **Symmetry:** Since the paraboloid has x and y with degree 2, so it is symmetrical about x-axis and y-axis.

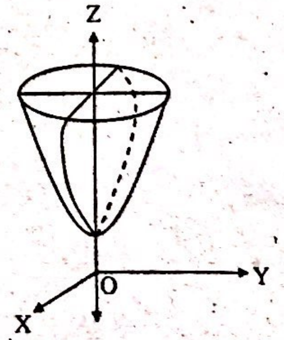
(ii) **Intercept:** Clearly the paraboloid has vertex at origin. So, it does not intersect the x-axis and y-axis and z-axis except at origin.

(iii) **Plane section:**

**(a) In yz-plane:** When x = 0 then (i) gives, $z = y^2$. This is a parabola having openward toward the positive z-axis.

**(b) In zx-plane:** When y = 0 then (i) gives, $z = x^2$. This is a parabola having openward toward the positive z-axis.

With the information the sketch of the hyperboloid is as in figure.




**2005 Fall Q. No. 7(a)**

**Define tangent and tangent plane of a curve at a point.**

**Solution: Tangent to a Curve:**

Let C be a curve in space. The tangent on C at a point P of C is the limiting position of a straight line that through P and a point Q of C as Q tends to P along C.

**Tangent Plane to a Curve:**

Let C be a curve in space. The tangents on C at a point P of C, is called tangent plane.

**2006 Spring Q. No. 7(d); 2006 Fall Q. No. 7(e); 2007 Spring Q. No. 7(c)**

**Write equation of an ellipsoid. Sketch it with centre and axis of symmetry.**

**Solution:** The equation of hyperboloid of two sheet is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ... (i)

**For sketch:**

**(i) Centre:** Clearly (i) has centre at (0, 0, 0).

**(ii) Symmetry:** Since all three variables have same degree, so it is symmetrical about all three axes.

**(iii) Intercept:**

When y = 0 = z then $x^2 = a^2$. So, $x = \pm a$. Therefore, the figure cuts x-axis at $x = \pm a$.

When x = 0 = z then $y^2 = b^2$. So, $y = \pm b$. Therefore, the figure cuts y-axis at $y = \pm b$.

When x = 0 = y then $z^2 = c^2$. So, $z = \pm c$. Therefore, the figure cuts z-axis at $z = \pm c$.

**(iv) Plane section:**

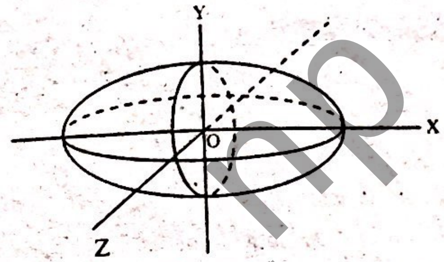
**(a) In xy-plane:** When z = 0 then (i) gives, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is an ellipse.

**(b) In yz-plane:** When x = 0 then (i) gives, $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. This is an ellipse.



**(c) In zx-plane:** When y = 0 then (i) gives, $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$. This is an ellipse.

With the information, the sketch of the ellipsoid is as in figure.




**2015 Fall Q. No. 7(b)**

**Write equation of an ellipsoid. Sketch it with centre and axis of symmetry.**

**2007 Fall Q. No. 7(e); 2012 Fall Q. No. 7(d); 2017 Fall Q. No. 7(a)**

**Find the parametric representation of the surface $x^2 + 4y^2 = 9, z = 3$.**

**Solution:** Given that,

$$x^2 + 4y^2 = 9, z = 3.$$

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{9/4} = 1, z = 3.$$

Clearly, the given curve is ellipsoid in xyz-plane.

Put y = t then $x = \pm\sqrt{9 - 4t^2}$ and z = 3.

Thus, $(x, y, z) = (\pm\sqrt{9 - 4t^2}, t, 3)$.

**2011 Fall Q. No. 7(a)**

**Represent the equation $x^2 + y^2 = 9, z = 5\tan^{-1}(y/x)$ parametrically.**

**Solution:** Given that,

$$x^2 + y^2 = 9, z = 5\tan^{-1}(y/x).$$

Clearly, the given curve is ellipsoid in xyz-plane.

Put, x = r cost, y = r sint with r = 3.

Therefore, $z = 5\tan^{-1}(\sin t/\cos t) = 5t$.

So, x = 3cost, y = 3sint, z = 5t

Thus, (x, y, z) = (3cost, 3sint, 5t).

**2011 Fall Q. No. 7(b)**

**Find the unit tangent vector to the curve $\vec{r}(t) = 2\cos t\,\vec{i} + \sin t\,\vec{j}$ at $(\sqrt{2}, \sqrt{2}, 0)$.**

**Hint: Similar to exercise 9.1 Q. No. C(ii).**

**2016 Fall Q. No. 7(c)**

Find the unit tangent vector to the curve $\vec{r}(t) = 2\cosh t\,\vec{i} + \sinh t\,\vec{j}$ at $(\sqrt{2}, \sqrt{2}, 0)$.

**2016 Spring Q. No. 7(b)**

Represent the curve $y^2 - (z-3)^2 = 9, x = 0$ parametrically.

## OTHER SOLVED PROBLEMS

**1. Write the equation of hyperboloid of one sheet and then sketch.**

**Solution:** The equation of hyperboloid of two sheet is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ ... (i)

**For sketch:**

(i) **Centre:** Clearly (i) has centre at (0, 0, 0).

(ii) **Symmetry:** Since all three variables have same degree, so it is symmetrical about all three axes.

(iii) **Intercept:**

When $y = 0 = z$ then $x^2 = a^2$. So, $x = \pm a$. Therefore, the figure cuts x-axis at $x = \pm a$.

When $x = 0 = z$ then $y^2 = b^2$. So, $y = \pm b$. Therefore, the figure cuts y-axis at $y = \pm b$.

When $x = 0 = y$ then $z^2 = -c^2$. So, z has only imaginary value which is non-acceptable for sketch.

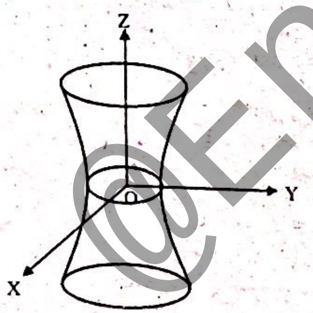
(iv) **Plane section:**

(a) **In xy-plane:** When z = 0 then (i) gives, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is an ellipse.

(b) **In yz-plane:** When x = 0 then (i) gives, $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. This is a hyperbola.

(c) **In zx-plane:** When y = 0 then (i) gives, $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$. This is a hyperbola.

With the information the sketch of the hyperboloid is as in figure.




**2. Write the equation of cone and then sketch.**

**Solution:** The equation of hyperboloid of two sheet is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ ... (i)



**For sketch:**

(i) **Centre:** Clearly (i) has centre at (0, 0, 0).

(ii) **Symmetry:** Since all three variables have same degree, so it is symmetrical about all three axes.

(iii) **Intercept:**

When $y = 0 = z$ then $x^2 = 0$. So, $x = 0$. Therefore, the figure cuts x-axis at origin. And, other two gives same value.

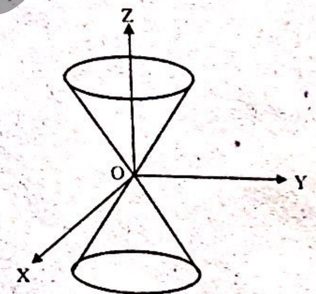
(iv) **Plane section:**

(a) **In xy-plane:** When z = 0 then (i) gives, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$. This is a point ellipse.

(b) **In yz-plane:** When x = 0 then (i) gives, $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \Rightarrow y = \pm\frac{bz}{c}$

(c) **In zx-plane:** When y = 0 then (i) gives, $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \Rightarrow x = \pm\frac{az}{c}$

With the information the sketch of the hyperboloid is as in figure.




**3. Write the equation of elliptic paraboloid and then sketch.**

**Solution:** The equation of hyperboloid of two sheet is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}$ ... (i)

**For sketch:**

(i) **Centre:** Clearly (i) has vertex at (0, 0, 0).

(ii) **Symmetry:** Since both x and y have same degree, so it is symmetrical about both x-axis and y-axis.

(iii) **Intercept:**

When $y = 0 = z$ then $x^2 = 0$. So, $x = 0$. Therefore, the figure cuts x-axis at origin. And, other two gives same value. Thus, the figure cuts the axes only at origin.

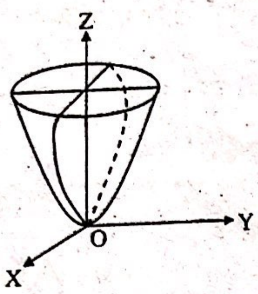
(iv) **Plane section:**

(a) **In xy-plane:** When z = 0 then (i) gives, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$. This is a point ellipse.

(b) **In yz-plane:** When x = 0 then (i) gives, $y^2 = \frac{2b^2 z}{c}$.

(c) **In zx-plane:** When y = 0 then (i) gives, $x^2 = \frac{2a^2 z}{c}$.

With the information the sketch of the hyperboloid is as in figure.

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